Periodic solutions of second order nonautonomous differential equations in Hilbert spaces

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Abstract. We study the abstract linear T-periodic problem

$$A(t)u(t) + [B(t)u'(t)]' + C(t)u'(t) = f(t), \quad t \in \mathbb{R},$$

Under appropriate hypotheses on the linear operators A, B, C, we show the existence of a unique T-periodic solution

$$u \in L^2_{\mathrm{loc}}(\mathbb{R}, V) \cap H^1_{\mathrm{loc}}(\mathbb{R}, H), \quad \text{with} \quad B(\cdot)u'(\cdot) \in H^1_{\mathrm{loc}}(\mathbb{R}, V^*),$$

where $f \in L^2_{\mathrm{loc}}(H)$ is T-periodic and V, H are complex, separable Hilbert spaces, such that $V \subseteq H \subseteq V^*$ with dense and continuous inclusions.

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0. Introduction

We consider here the problem of existence of periodic solutions to second order abstract linear equations in Hilbert spaces of the form

$$A(t)u(t) + [B(t)u'(t)]' + C(t)u'(t) = f(t). \tag{0.1}$$

Here $A(t)$ is the "elliptic part" of the equation, $(B(t)u'(t))'$ is a generalization of the second time derivative of the unknown u, and $C(t)u'(t)$ is a dissipative term.

This problem has been considered and studied by several authors. To our knowledge, however, there are no papers where this subject appears in the general form we propose here, namely the case in which all operators are time dependent: see [5], [7], [8], [9]. An extended and rational bibliography on periodic problems, until 1982, can be found in [12].

Concerning the structure of the operators, the simple Example 0.1, at the end of this Section, shows that the presence of the dissipative term

$C(t)u'(t)$ is essential in (0.1), in order to guarantee the uniqueness of the solution. Moreover, we also assume on the “elliptic part” $A(t)$ more general hypotheses, which allow in concrete cases to add terms containing lower order “space” derivatives.

We must enphasise that our results are important in many applications: as an example we quote [4].

We describe our problem in Section 1: we list our assumptions, we specify what kind of solutions we are looking for, and we state our main theorem (Theorem 1.3).

Section2 is dedicated to auxiliary results: an equivalent notion of solution, two easy “a priori” regularity results for our solutions, and two well known theorems due to C. Baiocchi [1] and J.-L. Lions [10], which are essential ingredients of our proof.

In Section 3, we prove that the solution of our problem is unique and depends continuously on the datuum $f(t)$. Here Baiocchi's result, which is a sort of “energy relation”, plays a central role.

The last two Sections concern the existence of solutions. In Section 4, we give a first result, under more restrictive assumptions on the operators, using Lions' result, which is a generalization of Lax-Milgram theorem. This is however the starting point for the proof of the general case: in the final Section 5, using the continuity method and the theory of “near operators” due to Campanato (see [3]) , we are able to give a general existence result.

We conclude this introductory Section by showing the counterexample announced above:

Example 0.1. Consider the one-dimensional wave equation in $[0, \pi] \times \mathbb{R}$ with periodicity conditions:

$$\begin{cases} u_{xx}(x,t) - u_{tt}(x,t) = 0, \qquad 0 < x < \pi, \quad t \in \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, \qquad t \in \mathbb{R}, \\ u(x,t+2\pi) = u(x,t), \qquad 0 < x < \pi, \quad t \in \mathbb{R}. \end{cases}$$

We have $A = \frac{d^2}{dx^2}$, $B = -I$ and $C = 0$ here. It is clear that $v(x,t) = \sin x \sin t$ and $w(x,t) = 0$ both solve the equation, so that uniqueness does not hold.

1. Assumptions and main result

If X is a generic Banach space, and $T > 0$, we denote by $L^2_T(X)$, $H^1_T(X)$, and so on, the set of T-periodic functions belonging to $L^2_{\text{loc}}(\mathbb{R}, X)$, $H^1_{\text{loc}}(\mathbb{R}, X)$, and so on.

We assume that V, H are complex, separable Hilbert spaces, such that $V \subseteq H \subseteq V^*$ with dense and continuous inclusions. For fixed $T > 0$, we consider the abstract T-periodic problem

$$\begin{cases} A(t)u(t) + [B(t)u'(t)]' + C(t)u'(t) = f(t), \quad t \in \mathbb{R}, \\ u(t+T) = u(t), \quad t \in \mathbb{R}, \end{cases} \tag{1.1}$$

where the unknown u satisfies

$$u \in L^2_T(V) \cap H^1_T(H); \tag{1.2}$$

in order to be a solution of our problem, the function u must satisfy the equation in (1.1) pointwise in the space V^*.
Since the inclusions $V \subseteq H \subseteq V^*$ are continuous, there is $\beta > 0$ such that

$$\|x\|_H \le \beta \|x\|_V \quad \forall x \in V, \qquad \|z\|_{V^*} \le \beta \|z\|_H \quad \forall z \in H. \tag{1.3}$$

Our assumptions are divided in two groups: the first one concerns the hypotheses which will assure our main result; the second one consists of stronger hypotheses, which lead to an auxiliary general existence result. Finally, by a density argument we will obtain our main theorem going back to the first group of assumptions.

Hypothesis 1.1. *We assume:*

- $f \in L^2_T(H)$*;*
- $A(t) = Q_1(t) + Q_2(t) + R(t)$ *for every* $t \in \mathbb{R}$*, and*

$$Q_1 \in H^{1,\infty}_T(\mathcal{L}(V,V^*)), \quad Q_2 \in H^{1,\infty}_T(\mathcal{L}(H)), \quad R \in L^\infty_T(\mathcal{L}(V,H)); \tag{1.4}$$

in addition

$$\begin{aligned}
&\langle Q_1(t)u, v\rangle_{V^*,V} = \overline{\langle Q_1(t)v, u\rangle_{V^*,V}} \quad \forall u,v \in V, \quad \forall t \in \mathbb{R},\\
&(Q_2(t)u, v)_H = \overline{(Q_2(t)v, u)_H} \quad \forall u,v \in H, \quad \forall t \in \mathbb{R},\\
&\langle Q_1(t)u, u\rangle_{V^*,V} \ge \nu_{Q_1}\|u\|_V^2 \quad \forall u \in V, \quad \forall t \in \mathbb{R},\\
&(Q_2(t)u, u)_H \ge 0 \quad \forall u \in H, \quad \forall t \in \mathbb{R},\\
&\|Q_1(\cdot)\|_{\mathcal{L}(V,V^*)} \le M_{Q_1} \text{ in } \mathbb{R}, \quad \|Q_1'(\cdot)\|_{\mathcal{L}(V,V^*)} \le M_{Q_1'} \quad \text{a.e. in } \mathbb{R},\\
&\|Q_2(\cdot)\|_{\mathcal{L}(V,H)} \le M_{Q_2}, \quad \|Q_2'(\cdot)\|_{\mathcal{L}(V,H)} \le M_{Q_2'} \quad \text{a.e. in } \mathbb{R},\\
&\|R(\cdot)\|_{\mathcal{L}(V,H)} \le M_R \text{ a.e. in } \mathbb{R},
\end{aligned} \tag{1.5}$$

where $\nu_{Q_1} > 0$*;*

- $B \in H^{1,\infty}_T(\mathcal{L}(H))$*,* $C \in L^\infty_T(\mathcal{L}(H))$*; in addition*

$$\begin{aligned}
&(B(t)x, y)_H = \overline{(B(t)y, x)_H} \quad \forall x,y \in H, \quad \forall t \in \mathbb{R},\\
&(B(t)x, x)_H \ge \nu_B\|x\|_H^2 \quad \forall x \in H, \quad \forall t \in \mathbb{R},\\
&(B'(t)x, x)_H \ge \nu_{B'}\|x\|_H^2 \quad \forall x \in H, \quad \text{for a.e. } t \in \mathbb{R},\\
&(C(t)x, x)_H \ge \nu_C\|x\|_H^2 \quad \forall x \in H, \quad \text{for a.e. } t \in \mathbb{R},\\
&\|B(\cdot)\|_{\mathcal{L}(H)} \le M_B \text{ in } \mathbb{R}, \quad \|B'(\cdot)\|_{\mathcal{L}(H)} \le M_{B'}, \text{ a.e. in } \mathbb{R},\\
&\|C(\cdot)\|_{\mathcal{L}(H)} \le M_C \quad \text{a.e. in } \mathbb{R},
\end{aligned} \tag{1.6}$$

where $\nu_B, \nu_{B'}, \nu_C > 0$.

We note that a necessary condition in order that a function u satisfies problem (1.1)–(1.2) is that $B(\cdot)u'(\cdot) \in H^1_T(V^*)$, since all other terms in the equation belong to this space.
We also observe that $A \in L^\infty_T(\mathcal{L}(V,V^*))$: indeed, for $u, v \in V$ we may write

$$\langle A(t)u, v\rangle_{V^*,V} = \langle Q_1(t)u, v\rangle_{V^*,V} + (Q_2(t)u, v)_H + (R(t)u, v)_H\,;$$

now we have

$$\begin{aligned}|(Q_2(t)u,v)_H| &\le \|Q_2(t)u\|_H\|v\|_H\\ &\le \|Q_2(t)\|_{\mathcal{L}(H)}\|u\|_H\|v\|_H \le \beta^2\|Q_2(t)\|_{\mathcal{L}(H)}\|u\|_V\|v\|_V\,,\end{aligned}$$

and similarly

$$\begin{aligned}|(R(t)u,v)_H| &\le \|R(t)u\|_H\|v\|_H\\ &\le \|R(t)\|_{\mathcal{L}(V,H)}\|u\|_V\|v\|_H \le \beta\|R(t)\|_{\mathcal{L}(V,H)}\|u\|_V\|v\|_V\,,\end{aligned}$$

so that our claim follows by (1.4).

Hypothesis 1.2. *We assume Hypothesis* 1.1 *and, in addition,*

- $f \in H^1_T(H)$;
- $A(\cdot) \in \mathcal{L}(\mathcal{D}(A), L^2_T(H))$, *where*

$$\mathcal{D}(A) = \{u \in L^2_T(V) : \; A(\cdot)u(\cdot) \in L^2_T(H)\}; \tag{1.7}$$

- $Q_1 \in H^{2,\infty}_T(\mathcal{L}(V,V^*))$, $Q_2 \in H^{2,\infty}_T(\mathcal{L}(H))$, $R \in H^{1,\infty}_T(\mathcal{L}(V,H))$;
- $B \in H^{2,\infty}_T(\mathcal{L}(H))$, $C \in H^{1,\infty}_T(\mathcal{L}(V,H))$.

We conclude this section by stating our main result:

Theorem 1.3. *Under Hypothesis* 1.1*, problem* (1.1)–(1.2) *has a unique T-periodic solution u, satisfying*

$$\int_0^T \|u(t)\|_V^2\,dt + \int_0^T \|u'(t)\|_H^2\,dt \le K\int_0^T \|f(t)\|_H^2\,dt.$$

The constant K depends only on the constants associated to A, B, C; in particular it is independent of T.

The proof is exposed in the last three sections.

2. Auxiliary results

First of all, we introduce an equivalent notion of solution.

Proposition 2.1. *Assume Hypothesis* 1.1*. If $u : \mathbb{R} \to V$ is a T-periodic function, then u satisfies* (1.1)–(1.2) *if and only if $u \in L^2_T(V) \cap H^1_T(H)$ and*

$$\begin{aligned}\int_0^T \big[\langle A(t)u(t),\varphi(t)\rangle_{V^*,V} &- B(t)u'(t),\varphi'(t))_H\\ &+ (C(t)u'(t),\varphi(t))_H\big]dt = \int_0^T (f(t),\varphi(t))_H\,dt\end{aligned} \tag{2.1}$$

for every $\varphi \in L^2_T(V) \cap H^1_T(H)$.

Proof. Assume that u satisfies (1.1)–(1.2), and fix $\varphi \in L^2_T(V) \cap H^1_T(H)$. There exists a sequence of functions $\{\varphi_n\} \subset H^1_T(V)$, such that $\varphi_n \to \varphi$ in $L^2(0,T;V) \cap H^1(0,T;H)$. We observe that

$$\begin{aligned}\int_0^T \langle [B(t)u'(t)]', \varphi_n(t)\rangle_{V^*,V}\, dt &= -\int_0^T \langle B(t)u'(t), \varphi_n'(t)\rangle_{V^*,V}\, dt\\ &= -\int_0^T (B(t)u'(t), \varphi_n'(t))_H\, dt;\end{aligned}$$

hence, multiplying by $\varphi_n(t)$ the equation in (1.1) and integrating in $[0,T]$, we deduce

$$\begin{aligned}0 &= \int_0^T [\langle A(t)u(t) + [B(t)u'(t)]' + C(t)u'(t), \varphi_n(t)\rangle_{V^*,V}]\, dt\\ &\quad - \int_0^T \langle f(t), \varphi_n(t)\rangle_{V^*,V}\, dt\\ &= \int_0^T [A(t)u(t), \varphi_n(t)\rangle_{V^*,V} - (B(t)u'(t), \varphi_n'(t))_H + (C(t)u'(t), \varphi_n(t))_H]\, dt\\ &\quad - \int_0^T (f(t), \varphi_n(t))_H\, dt.\end{aligned}$$

As $n \to \infty$ we immediately obtain (2.1).

Assume conversely that $u \in L^2_T(V) \cap H^1_T(H)$ and that (2.1) holds for all $\varphi \in L^2_T(V) \cap H^1_T(H)$. Let us choose in (2.1) $\varphi(t) = v \cdot \psi(t)$, where $v \in V$ and $\psi \in H^1_T(\mathbb{R})$. Then

$$\begin{aligned}\int_0^T [\langle A(t)u(t), v\rangle_{V^*,V} + (C(t)u'(t), v)_H - (f(t), v)_H]\, \psi(t)\, dt\\ = \int_0^T (B(t)u'(t), v)_H\, \psi'(t)\, dt,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\int_0^T \langle A(t)u(t) + C(t)u'(t) - f(t), v\rangle_{V^*,V}\, \psi(t)\, dt\\ = \int_0^T \langle B(t)u'(t), v\rangle_{V^*,V}\, \psi'(t)\, dt.\end{aligned}$$

As a consequence, we have $\langle B(\cdot)u'(\cdot), v\rangle_{V^*,V} \in H^1(0,T)$ and

$$\frac{d}{dt}\langle B(t)u'(t), v\rangle_{V^*,V} = -\langle A(t)u(t) + C(t)u'(t) - f(t), v\rangle_{V^*,V}\,, \qquad t \in [0,T].$$

Since $v \in V$ is arbitrary, we deduce that $B(\cdot)u'(\cdot) \in H^1(0,T;V^*)$ and

$$[B(t)u'(t)]' = -A(t)u(t) - C(t)u'(t) + f(t), \qquad t \in [0,T].$$

By periodicity, this relation extends to $\mathbb{R}$, so that u satisfies (1.1)–(1.2). $\square$

We need now a regularity result for solutions of equation (1.1).

Proposition 2.2. *Under Hypothesis* 1.1*, let u be a solution of problem (*1.1*)–*(1.2)*. Then $u \in C^0_T(V) \cap C^1_T(H)$.*

Proof. Let u be a solution of (1.1)–(1.2). We consider a cut-off function $\vartheta \in C^\infty(\mathbb{R})$ such that $0 \le \vartheta(t) \le 1$ for every $t \in \mathbb{R}$, and

$$\vartheta(t) = \begin{cases} 1 \text{ if } & t \ge T, \\ 0 \text{ if } & t \le 0. \end{cases}$$

We multiply equation (1.1) by ϑ: writing

$$\begin{aligned}(Bu')'\vartheta = (Bu'\vartheta)' - Bu'\vartheta' &= (B(u\vartheta)')' - (Bu\vartheta')' - Bu'\vartheta' \\ &= (B(u\vartheta)')' - B'u\vartheta' - 2Bu'\vartheta' - Bu\vartheta''\end{aligned}$$

and

$$Cu'\vartheta = C(u\vartheta)' - Cu\vartheta',$$

we easily deduce that ϑu solves the Cauchy problem

$$\begin{cases} A(t)[\vartheta(t)u(t)] + [B(t)[\vartheta(t)u(t)]']' + C(t)[\vartheta(t)u(t)]' \\ \qquad = \vartheta(t)f(t) + 2\vartheta'(t)B(t)u'(t) + \vartheta''(t)B(t)u(t) & \quad t \in [0,2T]. \\ \qquad + \vartheta'(t)B'(t)u(t) + C(t)\vartheta'(t)u(t), \\ (\vartheta\, u)(0) = 0, \quad (\vartheta\, u)'(0) = 0. \end{cases} \tag{2.2}$$

Now, note that

$$f\,\vartheta + 2Bu'\,\vartheta' + Bu\,\vartheta'' + B'u\,\vartheta' + Cu\,\vartheta' \in L^2(0,2T;H);$$

thus, using Proposition 2.1 and writing problem (2.2) in the form (2.1), it follows by [1, Teorema 4.4] that the unique solution ϑu is in $C^0([0,2T],V) \cap C^1([0,2T],H)$. Hence, in particular, $u \in C^0([T,2T],V) \cap C^1([T,2T],H)$, and the result follows by T-periodicity. □

We also need a further regularity result for the solutions of problem (1.1)-(1.2): it was proved by Herrmann [7] in the case of constant coefficients. We prove it here in a more general form.

Theorem 2.3. *Assume Hypothesis* 1.2*. If u is a solution of problem* (1.1)–(1.2)*, then $u \in \mathcal{D}(A) \cap C^1_T(V) \cap C^2_T(H)$.*

Proof. Let us analyze the Cauchy problem (2.2): we see that the right member F has the form

$$\begin{aligned}F(t) &= \vartheta(t)f(t) + 2\vartheta'(t)B(t)u'(t) \\ &\quad + \vartheta''(t)B(t)u(t) + \vartheta'(t)B'(t)u(t) + C(t)\vartheta'(t)u(t).\end{aligned}$$

It follows from our assumptions that $F \in H^1([0,2T],H)$. Writing again (2.2) in the form (2.1), we may apply [1, Teorema 5.1] in the case $h = 1$ with $f_1 = f$ and $f_2 = 0$, obtaining $\vartheta u \in C^1([0,2T],V) \cap C^2([0,2T],H)$. Finally, since $\vartheta = 1$ in $[T,2T]$, we deduce that $u \in C^1([T,2T],V) \cap C^2([T,2T],H)$; using the equation and T-periodicity, it follows that $u \in \mathcal{D}(A)$. □

We recall now a well known result due to C. Baiocchi, which will be used in Section 3.

Theorem 2.4 ([1, Teorema 4.5])**.** *Assume Hypothesis* 1.1*, with the exception of T-periodicity, and replacing* $[0,T]$ *by* $[a,b]$*,* f *by* g*,* R *by* Q_2+R*. If* $g \in L^2(a,b;H)$*, and* $w \in C^0([a,b],V)\cap C^1([a,b],H)$ *is such that*

$$\int_a^b [\langle A(t)w(t),\varphi(t)\rangle_{V^*,V} - (B(t)w'(t),\varphi'(t))_H + (C(t)w'(t),\varphi(t))_H]\,dt$$

$$= \int_a^b g(t),\varphi(t))_H\,dt$$

for every $\varphi \in L^2(a,b;V)\cap H^1(a,b;H)$*, then for every* $t_0, t \in [a,b]$ *we have the "energy relation"*

$$\begin{aligned}
&\langle Q_1(t)w(t),w(t)\rangle_{V^*,V}X + (Q_2(t)w(t),w(t))_H + (B(t)w'(t),w'(t))_H\\
&\qquad + \int_{t_0}^t \Big[(B'(r)w'(r),w'(r))_H - \langle Q_1'(r)w(r),w(r)\rangle_{V^*,V}\\
&\qquad - (Q_2'(r)w(r),w(r))_H\Big]\,dr\\
&\qquad + 2\mathrm{Re}\int_{t_0}^t \Big[(R(r)w(r),w'(r))_H + (C(r)w'(r),w'(r))_H\Big]\,dr\\
&\qquad = \langle Q_1(t_0)w(t_0),w(t_0)\rangle_{V^*,V} + (Q_2(t_0)w(t_0),w(t_0))_H\\
&\qquad + (B(t_0)w'(t_0),w'(t_0))_H - 2\mathrm{Re}\int_{t_0}^t (g(r),w'(r))_H\,dr.
\end{aligned}\tag{2.3}$$

We end this section with an important result due to J.-L. Lions: it will be used in Section 4.

Theorem 2.5 ([10, Chap. III, Sect. 3.1])**.** *Let* F *be a complex Hilbert space and let* Φ *be a subspace of* F*, endowed with its own Hilbert norm* $\|\cdot\|_\Phi$*, such that*

$$\|\varphi\|_F \le \delta\|\varphi\|_\Phi \qquad \forall\varphi\in\Phi. \tag{2.4}$$

Let $E : F\times\Phi \to \mathbb{C}$ *a sesquilinear form, satisfying*

$$v \mapsto E(v,\varphi) \text{ is continuous on } F \quad \forall\varphi\in\Phi, \tag{2.5}$$

and

$$\exists\nu>0: \quad |E(\varphi,\varphi)| \ge \nu\|\varphi\|_\Phi^2 \quad \forall\varphi\in\Phi. \tag{2.6}$$

If $L:\Phi\to\mathbb{C}$ *is a continuous functional, then there is* $v\in F$ *such that*

$$E(v,\varphi) = L\varphi \qquad \forall\varphi\in\Phi.$$

3. Bound and uniqueness of the solution

This section is devoted to the proof of an "a priori" estimate for solutions of (1.2), (1.1): an obvious consequence of this estimate is uniqueness for such solutions.

Theorem 3.1. *Under Hypothesis* 1.1, *there is a constant* $K > 0$, *depending on the quantities* $\nu_{Q_1}, \nu_B, M_{Q_1'}, M_{Q_2'},, M_R, M_C, \beta, M_{B'}, M_B$ *but independent of* T, *such that for every* T*-periodic function* u, *satisfying* (1.2) *and* (1.1), *the estimate*

$$\int_0^T \left[\|u(r)\|_V^2 + \|u'(r)\|_H^2\right] dr \le K \int_0^T \|f(r)\|_H^2 \, dr$$

holds.

Proof. Assume that u is a T-periodic function satisfying (1.2) and (1.1). We multiply the equation in (1.1) by $e^{\lambda t}$, for a fixed $\lambda \in]0,1]$:

$$A(t)u(t)e^{\lambda t} + [B(t)u'(t)]'e^{\lambda t} + C(t)u'(t)e^{\lambda t} = f(t)e^{\lambda t}. \tag{3.1}$$

Define

$$w(t) = u(t)e^{\lambda t}, \qquad t \in [0, 2T]. \tag{3.2}$$

In order to replace u by w in (3.1), the following remarks are useful: first of all,

$$w'(t) = [u'(t) + \lambda u(t)]e^{\lambda t} = u'(t)e^{\lambda t} + \lambda w(t), \quad t \in [0, 2T]; \tag{3.3}$$

next, using (3.3),

$$\begin{aligned} [B(t)u'(t)]'e^{\lambda t} &= \frac{d}{dt}\left(B(t)u'(t)e^{\lambda t}\right) - \lambda B(t)u'(t)\, e^{\lambda t} \\ &= \frac{d}{dt}\left(B(t)[w'(t) - \lambda w(t)]\right) - \lambda B(t)[w'(t) - \lambda w(t)], \end{aligned}$$

so that

$$\begin{aligned} [B(t)u'(t)]'e^{\lambda t} &= [B(t)w'(t)]' - \lambda [B(t)w(t)]' \\ &\quad - \lambda B(t)w'(t) + \lambda^2 B(t)w(t). \end{aligned} \tag{3.4}$$

Hence (3.1) becomes, using (3.2) and (3.4),

$$\begin{aligned} &A(t)w(t) + [B(t)w'(t)]' + C(t)w'(t) \\ &\quad = f(t)e^{\lambda t} + \lambda C(t)w(t) + \lambda [B(t)w(t)]' + \lambda B(t)w'(t) - \lambda^2 B(t)w(t). \end{aligned} \tag{3.5}$$

We now apply Theorem 2.4, choosing $[a,b] = [0, 2T]$, $t_0 = T$, $t \in [0,T]$ and

$$\begin{aligned} g(t) = f(t)e^{\lambda t} &+ \lambda C(t)w(t) + \lambda B'(t)w(t) \\ &+ 2\lambda B(t)w'(t) - \lambda^2 B(t)w(t) : \end{aligned}$$

we get

$$
\begin{aligned}
&\langle Q_1(t)w(t), w(t)\rangle_{V^*,V} + (Q_2(t)w(t), w(t))_H + (B(t)w'(t), w'(t))_H \\
&+ \int_T^t (B'(r)w'(r), w'(r))_H \, dr - \int_T^t \langle Q_1'(r)w(r), w(r)\rangle_{V^*,V} \, dr \\
&- \int_T^t (Q_2'(r)w(r), w(r))_H \, dt + 2\mathrm{Re} \int_T^t (R(r)w(r), w'(r))_H \, dr \\
&+ 2 \int_T^t (C(r)w'(r), w'(r))_H \, dr \\
&= \langle Q_1(T)w(T), w(T)\rangle_{V^*,V} + (Q_2(T)w(T), w(T))_H \\
&+ (B(T)w'(T), w'(T))_H + 2\mathrm{Re} \int_T^t (f(r), w'(r))_H \, e^{\lambda r} \, dr \\
&+ 2\lambda \, \mathrm{Re} \int_T^t (C(r)w(r), w'(r))_H \, dr + \lambda \, \mathrm{Re} \int_T^t (B'(r)w(r), w'(r))_H \, dr \\
&+ 2\lambda \int_T^t (B(r)w'(r), w'(r))_H \, dr - 2\lambda^2 \, \mathrm{Re} \int_T^t (B(r)w(r), w'(r))_H \, dr.
\end{aligned}
\tag{3.6}
$$

Similarly, by Theorem 2.4, with $[a, b] = [0, 2T]$, $t_0 = T$, t replaced by $t + T$ and g as before:

$$
\begin{aligned}
&\langle Q_1(t)w(t+T), w(t+T)\rangle_{V^*,V} + (Q_2(t)w(t+T), w(t+T))_H \\
&+ (B(t)w'(t+T), w'(t+T))_H + \int_T^{t+T} (B'(r)w'(r), w'(r))_H \, dr \\
&- \int_T^{t+T} \langle Q_1'(r)w(r), w(r)\rangle_{V^*,V} \, dr - \int_T^{t+T} (Q_2'(r)w(r), w(r))_H \, dr \\
&+ 2\mathrm{Re} \int_T^{t+T} (R(r)w(r), w'(r))_H \, dr + 2 \int_T^{t+T} (C(r)w'(r), w'(r))_H \, dr \\
&= \langle Q_1(T)w(T), w(T)\rangle_{V^*,V} + (Q_2(T)w(T), w(T))_H \\
&+ (B(T)w'(T), w'(T))_H + 2\mathrm{Re} \int_T^{t+T} (f(r), w'(r))_H \, e^{\lambda r} \, dr \\
&+ 2\lambda \, \mathrm{Re} \int_T^{t+T} (C(r)w(r), w'(r))_H \, dr \\
&+ \lambda \, \mathrm{Re} \int_T^{t+T} (B'(r)w(r), w'(r))_H \, dr + 2 \int_T^{t+T} (B(r)w'(r), w'(r))_H \, dr \\
&- 2\lambda^2 \, \mathrm{Re} \int_t^{t+T} (B(r)w(r), w'(r))_H \, dr.
\end{aligned}
\tag{3.7}
$$

We now subtract (3.6) from (3.7), noticing that from (3.2) it follows that

$$
w'(t+T) = w'(t)e^{\lambda T}. \tag{3.8}
$$

We get

$$
\begin{aligned}
&\big[\langle Q_1(t)w(t), w(t)\rangle_{V^*,V} + (Q_2(t)w(t), w(t))_H \\
&\qquad\qquad + (B(t)w'(t), w'(t))_H\big]\big(e^{2\lambda T}-1\big) \\
&+ \int_t^{t+T} (B'(r)w'(r), w'(r))_H\,dr - \int_t^{t+T} \langle Q_1'(r)w(r), w(r)\rangle_{V^*,V}\,dr \\
&- \int_t^{t+T} (Q_2'(r)w(r), w(r))_H\,dr + 2\mathrm{Re}\int_t^{t+T} (R(r)w(r), w'(r))_H\,dr \\
&+ 2\int_t^{t+T} (C(r)w'(r), w'(r))_H\,dr \\
&= 2\mathrm{Re}\int_t^{t+T} (f(r), w'(r))_H\, e^{\lambda r}\,dr \\
&+ 2\lambda\,\mathrm{Re}\int_t^{t+T} (C(r)w(r), w'(r))_H\,dr \\
&+ \lambda\,\mathrm{Re}\int_t^{t+T} (B'(r)w(r), w'(r))_H\,dr \\
&+ 2\lambda\int_t^{t+T} (B(r)w'(r), w'(r))_H\,dr \\
&- 2\lambda^2\,\mathrm{Re}\int_t^{t+T} (B(r)w(r), w'(r))_H\,dr.
\end{aligned}
\tag{3.9}
$$

We rewrite (3.9) as

$$
\begin{aligned}
&\big(e^{2\lambda T}-1\big)\big[\langle Q_1(t)w(t), w(t)\rangle_{V^*,V} \\
&\qquad\qquad + (Q_2(t)w(t), w(t))_H + (B(t)w'(t), w'(t))_H\big] \\
&+ \int_t^{t+T} ([B'(r)+2C(r)]w'(r), w'(r))_H\,dr \\
&= 2\mathrm{Re}\int_t^{t+T} (f(r), w'(r))_H\, e^{\lambda r}\,dr \\
&+ \int_t^{t+T} \langle Q_1'(r)w(r), w(r)\rangle_{V^*,V}\,dr \\
&+ \int_t^{t+T} (Q_2'(r)w(r), w(r))_H\,dr - 2\mathrm{Re}\int_t^{t+T} (R(r)w(r), w'(r))_H\,dr \\
&+ \lambda\,\mathrm{Re}\int_t^{t+T} \big[2(C(r)w(r), w'(r))_H + (B'(r)w(r), w'(r))_H\big]\,dr \\
&+ 2\lambda\int_t^{t+T} (B(r)w'(r), w'(r))_H\,dr \\
&- 2\lambda^2\,\mathrm{Re}\int_t^{t+T} (B(r)w(r), w'(r))_H\,dr =: \sum_{j=1}^{8} T_j.
\end{aligned}
\tag{3.10}
$$

We now easily estimate each term T_j:

$$\begin{aligned}
&T_1 \leq \int_t^{t+T} \|f(r)\|_H^2 \, e^{2\gamma r} \, dr + \int_t^{t+T} \|w'(r)\|_H^2 \, dr, \\
&T_2 + T_3 \leq M_{Q_1'} \int_t^{t+T} \|w(r)\|_V^2 \, dr + M_{Q_2'} \int_t^{t+T} \|w(r)\|_H^2 \, dr, \\
&T_4 \leq M_R \left[\int_t^{t+T} \|w(r)\|_V^2 \, dr + \int_t^{t+T} \|w'(r)\|_H^2 \, dr \right], \\
&T_5 \leq M_C \left[\int_t^{t+T} \|w(r)\|_H^2 \, dr + \lambda^2 \int_t^{t+T} \|w'(r)\|_H^2 \, dr \right], \\
&T_6 \leq \frac{1}{2} \, M_{B'} \left[\int_t^{t+T} \|w(r)\|_H^2 \, dr + \lambda^2 \int_t^{t+T} \|w'(r)\|_H^2 \, dr \right], \\
&T_7 \leq 2\lambda M_B \int_t^{t+T} \|w'(r)\|_H^2 \, dr, \\
&T_8 \leq M_B \left[\int_t^{t+T} \|w(r)\|_H^2 \, dr + \lambda^4 \int_t^{t+T} \|w'(r)\|_H^2 \, dr \right].
\end{aligned}$$

Recalling (1.3), by (3.9) and the above estimates we deduce, using (1.5) and (1.6),

$$\begin{aligned}
&\big(e^{2\lambda T} - 1\big)(\nu_{Q_1} \wedge \nu_B) \left[\|w(t)\|_V^2 + \|w'(t)\|_H^2\right] \\
&\quad + (\nu_{B'} + 2\nu_C) \int_t^{t+T} \|w'(r)\|_H^2 \, dr \leq \int_t^{t+T} \|f(r)\|_H^2 \, e^{2\lambda r} \, dr \\
&\quad + C_1 \left[\int_t^{t+T} \|w(r)\|_V^2 \, dr + \int_t^{t+T} \|w'(r)\|_H^2 \, dr \right]
\end{aligned} \tag{3.11}$$

where, taking into account that $\lambda \in]0,1]$, we have set

$$\begin{aligned}
C_1 = \max \Big\{ 1, M_{Q_1'} \, , M_{Q_2'} \, \beta^2, M_R \, , M_C \, \beta^2, M_C \, , \\
\frac{1}{2} \, M_{B'} \beta^2, \, \frac{1}{2} \, M_{B'} \, , 2M_B \, , M_B \beta^2 \Big\}.
\end{aligned} \tag{3.12}$$

We now set, for simplicity,

$$\begin{aligned}
&W(r) = \|w(r)\|_V^2 + \|w'(r)\|_H^2 \, , \quad r \in [0, 2T]; \\
&U(t) = \int_t^{t+T} W(r) \, dr, \quad t \in [0, T],
\end{aligned} \tag{3.13}$$

and also

$$\begin{aligned}
&C_2 = \frac{1}{\nu_{Q_1} \wedge \nu_B} \, , \quad C_3 = C_1 C_2 \, , \\
&F(t) = C_2 \int_t^{t+T} \|f(r)\|_H^2 \, e^{2\lambda r} \, dr, \quad t \in [0, T].
\end{aligned} \tag{3.14}$$

By the definition (3.2) of w, for the function U defined by (3.13) we have, recalling (3.8),

$$\begin{aligned} U'(t) &= \|w(t+T)\|_V^2 + \|w'(t+T)\|_H^2 - \|w(t)\|_V^2 - \|w'(t)\|_H^2 \\ &= \|u(t+T)\|_V^2 \, e^{2\lambda(t+T)} + \|u'(t+T) + u(t+T)\|_H^2 \, e^{2\lambda(t+T)} \\ &\quad - \|u(t)\|_V^2 e^{2\lambda t} - \|u'(t) + u(t)\|_H^2 \, e^{2\lambda t} \\ &= \left[\|u(t)\|_V^2 + \|u'(t) + u(t)\|_H^2\right] e^{2\lambda t} \left(e^{2\lambda T} - 1\right), \qquad t \in [0,T], \end{aligned}$$

where in the last equality we used T-periodicity. Thus, using again (3.8), we conclude that

$$U'(t) = \left(e^{2\lambda T} - 1\right) \left[\|w(t)\|_V^2 + \|w'(t)\|_H^2\right], \qquad t \in [0,T]. \tag{3.15}$$

By (3.15) and (3.14), from (3.11) we deduce

$$U'(t) + C_2(\nu_{B'} + 2\nu_C) \int_t^{t+T} \|w'(r)\|_H^2 \, dr \le F(t) + C_3 U(t), \qquad t \in [0,T].$$

Hence, for every $h > 0$ we also have, for $0 \le t \le T$,

$$\begin{aligned} U'(t)e^{-ht} - hU(t)e^{-ht} + C_2(\nu_{B'} + 2\nu_C)e^{-ht} \int_t^{t+T} \|w'(r)\|_H^2 \, dr& \\ \le F(t)e^{-ht} + (C_3 - h)U(t)e^{-ht},& \end{aligned}$$

which means, for every $t \in [0,T]$,

$$\begin{aligned} \frac{d}{dt}\left[U(t)e^{-ht}\right] + C_2(\nu_{B'} + 2\nu_C)e^{-ht} \int_t^{t+T} \|w'(r)\|_H^2 \, dr& \\ \le F(t)e^{-ht} + (C_3 - h)U(t)e^{-ht}.& \end{aligned}$$

We integrate both members in $[0,T]$:

$$\begin{aligned} U(T)e^{-hT} - U(0) + C_2(\nu_{B'} + 2\nu_C) \int_0^T e^{-ht} \int_t^{t+T} \|w'(r)\|_H^2 \, dr dt& \\ \le \int_0^T F(t)e^{-ht} \, dt + (C_3 - h) \int_0^T U(t)e^{-ht} \, dt.& \end{aligned}$$

Recalling (3.13) and (3.14), we deduce

$$\begin{aligned} e^{-hT} \int_T^{2T} W(r) \, dr - \int_0^T W(r) \, dr& \\ + C_2(\nu_{B'} + 2\nu_C) \int_0^T e^{-ht} \int_t^{t+T} \|w'(r)\|_H^2 \, dr dt& \\ \le C_2 \int_0^T e^{-ht} \int_t^{t+T} \|f(r)\|_H^2 \, e^{2\lambda r} \, dr \, dt& \\ + (C_3 - h) \int_0^T e^{-ht} \int_t^{t+T} W(r) \, dr \, dt.& \end{aligned} \tag{3.16}$$

Let us analyze the terms appearing in (3.16): first, we observe that, by (3.13), (3.8) and T-periodicity,

$$\begin{aligned}\int_T^{2T} W(r)\,dr &= \int_T^{2T} \left[\|w(r)\|_V^2 + \|w'(r)\|_H^2\right] dr \\ &= \int_T^{2T} e^{2\lambda T}\left[\|w(r-T)\|_V^2 + \|w'(r-T)\|_H^2\right] dr \\ &= e^{2\lambda T}\int_0^T \left[\|w(s)\|_V^2 + \|w'(s)\|_H^2\right] ds = e^{2\lambda T}\int_0^T W(r)\,dr;\end{aligned} \tag{3.17}$$

Moreover, by (3.14), Fubini-Tonelli's Theorem and T-periodicity,

$$\begin{aligned}&\int_0^T e^{-ht}\int_t^{t+T} \|f(r)\|_H^2\, e^{2\lambda r}\,dr\,dt \\ &= \int_0^T \|f(r\|_H^2\, e^{2\lambda r}\int_0^r e^{-ht}dt\,dr + \int_T^{2T} \|f(r\|_H^2\, e^{2\lambda r}\int_{r-T}^T e^{-ht}\,dt\,dr \\ &= \int_0^T \|f(r\|_H^2\, e^{2\lambda r}\,\frac{1-e^{-hr}}{h}\,dr \\ &+ \int_0^T \|f(s)\|_H^2\, e^{2\lambda s}\, e^{2\lambda T}\;\frac{e^{-hs}-e^{-hT}}{h}\,ds = \int_0^T \|f(r\|_H^2\, e^{2\lambda r}\,\Gamma(r)\,dr,\end{aligned} \tag{3.18}$$

where

$$\Gamma(r) = \frac{1}{h}\left[1 - e^{-hr} + e^{2\lambda T}(e^{-hr} - e^{-hT})\right]; \tag{3.19}$$

finally, by Fubini-Tonelli's Theorem and (3.17),

$$\begin{aligned}&\int_0^T e^{-ht}\int_t^{t+T} W(r)\,dr\,dt \\ &\qquad = \int_0^T W(r)\int_0^r e^{-ht}\,dt\,dr + \int_T^{2T} W(r)\int_{r-T}^T e^{-ht}dt\,dr \\ &\qquad = \int_0^T W(r)\,\frac{1-e^{-hr}}{h}\,dr + \int_0^T W(s)\,e^{2\lambda T}\;\frac{e^{-hs}-e^{-hT}}{h}\,ds \\ &\qquad = \int_0^T W(r)\,\Gamma(r)\,dr,\end{aligned} \tag{3.20}$$

with $\Gamma(r)$ given by (3.19). Note that, for $0 \le r \le T$,

$$0 < \Gamma(r) \le \frac{1}{h}(1 + e^{2\lambda T})(1 - e^{-hT}). \tag{3.21}$$

Consequently, neglecting the last term in the left member of (3.16), and using (3.17), (3.18), and (3.20) we deduce

$$\begin{aligned}&\left[e^{(2\lambda - h)T} - 1\right]\int_0^T W(r)\,dr \\ &\qquad\qquad \le C_2\int_0^T \|f(r\|_H^2\, e^{2\lambda r}\,\Gamma(r)\,dr + (C_3 - h)\int_0^T W(r)\,\Gamma(r)\,dr.\end{aligned}$$

and hence, by (3.21)

$$\begin{aligned}\Big[e^{(2\lambda-h)T}-1+(h-C_3)\frac{1-e^{-hT}}{h}\big(1+e^{2\lambda T}\big)\Big]\int_0^T W(r)\,dr \\ \le C_2\,\frac{1-e^{-hT}}{h}\,\big(1+e^{2\lambda T}\big)\,e^{2\lambda T}\int_0^T \|f(r)\|_H^2\,dr.\end{aligned} \tag{3.22}$$

It is not hard to verify that the coefficient in the right member of (3.22) is positive for large h: indeed, easy calculations lead to

$$\begin{aligned}&e^{(2\lambda-h)T}-1+(h-C_3)\frac{1-e^{-hT}}{h}\,\big(1+e^{2\lambda T}\big)\\ &\qquad=\frac{1}{h}\Big[h\left(e^{2\lambda T}-e^{-hT}\right)-C_3\Big(1+e^{2\lambda T}-e^{-hT}-e^{(2\lambda-h)T}\Big)\Big]\\ &\qquad>\frac{1}{h}\left[h\left(e^{2\lambda T}-1\right)-C_3\left(1+e^{2\lambda T}\right)\right],\end{aligned}$$

and the last member is positive, provided $h>\frac{C_3\left(e^{2\lambda T}+1\right)}{e^{2\lambda T}-1}$. More precisely, if we choose

$$h=\frac{2C_3\left(e^{2\lambda T}+1\right)}{e^{2\lambda T}-1},$$

then we get

$$\begin{aligned}&e^{(2\lambda-h)T}-1+(h-C_3)\frac{1-e^{-hT}}{h}\,\big(1+e^{2\lambda T}\big)\\ &\qquad>\frac{1}{h}\left[h\left(e^{2\lambda T}-1\right)-C_3\left(1+e^{2\lambda T}\right)\right]=\frac{C_3}{h}\,\left(e^{2\lambda T}+1\right)=\frac{e^{2\lambda T}-1}{2}\,.\end{aligned}$$

Hence by (3.22) we obtain

$$\int_0^T W(r)\,dr\le 2C_2\frac{1-e^{-hT}}{h}\,\frac{e^{2\lambda T}+1}{e^{2\lambda T}-1}\,e^{2\lambda T}\int_0^T \|f(r)\|_H^2\,dr.$$

Thus, recalling the definition of h, we conclude that

$$\int_0^T W(r)\,dr\le C_4e^{4\lambda T}\int_0^T \|f(r)\|_H^2\,dr,$$

where, by (3.14), $C_4=C_1^{-1}$.
We finally go back to u: by (3.13),

$$\int_0^T \left[\|w(r)\|_V^2+\|w'(r)\|_H^2\right]dr\le C_4\,e^{2\lambda T}\int_0^T \|f(r)\|_H^2\,dr,$$

so that

$$\begin{aligned}\int_0^T \|u(r)\|_V^2\,dr&\le\int_0^T \|u(r)\|_V^2\,e^{2\lambda r}\,dr\\ &=\int_0^T \|w(r)\|_V^2\,dr\le C_4\,e^{2\lambda T}\int_0^T \|f(r)\|_H^2\,dr,\end{aligned}$$

and

$$\begin{aligned}\int_0^T \|u'(r)\|_H^2\,dr &\le 2\int_0^T \left[\|u'(r)+u(r)\|_H^2 + \|u(r)\|_H^2\right] dr \le \\ &\le 2\int_0^T \|u'(r)+u(r)\|_H^2\, e^{2\lambda r}\,dr + 2\beta^2 \int_0^T \|u(r)\|_V^2\,dr \\ &= 2\int_0^T \|w'(r)\|_H^2\,dr + 2\beta^2\int_0^T \|u(r)\|_V^2\,dr \\ &\le 2C_4(1+\beta^2)e^{2\lambda T}\int_0^T \|f(r)\|_H^2\,dr.\end{aligned}$$

We can then conclude that

$$\int_0^T \left[\|u(r)\|_V^2 + \|u'(r)\|_H^2\right] dr \le K_0\, e^{2\lambda T}\int_0^T \|f(r)\|_H^2\,dr, \tag{3.23}$$

where $K_0 = C_1^{-1}(3+2\beta^2)$.

Finally we select γ: if $T \le 1$, we choose $\lambda = 1$, so that the constant in (3.23) becomes $K_0 e^{2T}$ and can be taken as $K_0 e^2$; if instead $T > 1$, we choose $\lambda = T^{-1} < 1$ and the final constant becomes again $K_0 e^2$. Thus, in any case, setting $K = K_0 e^2$, the constant in (3.23) does not depend on T. Theorem 3.1 is proved. ☐

Corollary 3.2. *Under Hypotheses* 1.1*, problem* (1.1)-(1.2) *has at most one* T*-periodic solution.*

4. Existence in a special case

In this Section, by appealing to Theorem 2.5, we prove the following theorem.

Theorem 4.1. *Under Hypothesis* 1.1*, if in addition the condition*

$$\begin{aligned}&\frac{\nu_{Q_1}^2(\nu_{B'}+2\nu_C)^2}{\beta^2(M_{B'}+M_C)^2+\nu_{Q_1}M_B+\sqrt{[\beta^2(M_{B'}+M_C)^2+\nu_{Q_1}M_B]^2+2\nu_{Q_1}^2 M_B^2}} \\ &\qquad > (\nu_{B'}+2\nu_C)\left(M_{Q_1'}+\beta^2 M_{Q_2'}\right)+2\beta^2 M_R^2\end{aligned} \tag{4.1}$$

holds, then problem (1.1)-(1.2) *has a unique* T*-periodic solution* u*, satisfying*

$$\int_0^T \|u(t)\|_V^2\,dt + \int_0^T \|u'(t)\|_H^2\,dt \le c\int_0^T \|f(t)\|_H^2\,dt,$$

where the constant c *depends only on* A, B, C*.*

Proof. First of all, by Proposition 2.1 we may consider, in place of (1.1), the equivalent equation (2.1). Next, we introduce, for fixed $\gamma > 0$, the spaces

$$F = \{v \in L^2_{\mathrm{loc}}(\mathbb{R},V)\cap H^1_{\mathrm{loc}}(\mathbb{R},H)) :\ v(t+T) = e^{\gamma T}v(t)\ \ \forall t\in\mathbb{R}\}, \tag{4.2}$$

$$\Phi = \{\varphi \in H^2_{\mathrm{loc}}(\mathbb{R},V) :\ \varphi(t+T) = e^{\gamma T}\varphi(t)\ \ \forall t\in\mathbb{R}\}. \tag{4.3}$$

We define on F and Φ the following inner products:

$$(v,u)_F = \int_0^T (v(r),u(r))_V \, e^{-2\gamma r}\, dr + \int_0^T (v'(r),u'(r))_H \, e^{-2\gamma r}\, dr, \tag{4.4}$$

$$(\varphi,\psi)_\Phi = \int_0^T (\varphi(r),\psi(r))_V \, e^{-2\gamma r}\, dr + \int_0^T (\varphi'(r),\psi'(r))_H \, e^{-2\gamma r}\, dr. \tag{4.5}$$

It is clear that F is a complex Hilbert space and that

$$\|\varphi\|_F = \|\varphi\|_\Phi \qquad \forall \varphi \in \Phi; \tag{4.6}$$

on the contrary, the space Φ is not complete with respect to the norm $\|\cdot\|_\Phi$. We introduce the sesqulinear form $E : F \times \Phi \to \mathbb{C}$ by setting, for $v \in F$ and $\varphi \in \Phi$,

$$\begin{aligned} E(v,\varphi) = \int_0^T \Big[& \langle A(t)v,\varphi'\rangle_{V^*,V}\, e^{-2\gamma t} - (B(t)v', [\varphi' e^{-2\gamma t}]')_H \\ & - \gamma (B(t)v',\varphi')_H \, e^{-2\gamma t} + \gamma (B(t)v,\varphi'')_H \, e^{-2\gamma t} \\ & - \gamma^2 (B(t)v,\varphi')_H \, e^{-2\gamma t} + (C(t)v',\varphi')_H \, e^{-2\gamma t} \\ & - \gamma (C(t)v,\varphi')_H \, e^{-2\gamma t} \Big] dt. \end{aligned} \tag{4.7}$$

We also introduce the linear functional $L : \Phi \to \mathbb{C}$ as

$$L\varphi = \int_0^T (g(t),\varphi'(t))_H \, e^{-2\gamma t}\, dt, \tag{4.8}$$

where $g \in L^2_{\mathrm{loc}}(\mathbb{R},H)$ with $g(t+T) = g(t)e^{\gamma T}$ for every $t \in \mathbb{R}$.
We will prove that the equation

$$E(v,\varphi) = L\varphi \qquad \forall \varphi \in \Phi \tag{4.9}$$

can be solved with the use of Theorem 2.5; to this purpose, we need some preliminary facts. The first one concerns a different description of the space Φ in (4.3).

Lemma 4.2. *It is $\varphi \in \Phi$ if and only if there exists a function $\vartheta \in H^1_T(V)$ such that*

$$\varphi(t) = \int_t^{t+T} \vartheta(s)\, e^{\gamma s}\, ds \qquad \forall t \in \mathbb{R}. \tag{4.10}$$

Proof. Let φ be given by (4.10), with $\vartheta \in H^1_T(V)$. Then we have on one hand

$$\varphi'(t) = \vartheta(t+T)e^{\gamma(t+T)} - \vartheta(t)e^{\gamma t} = \vartheta(t)e^{\gamma t}(e^{\gamma T}-1) \in H^1_{\mathrm{loc}}(\mathbb{R},V);$$

on the other hand

$$\begin{aligned} \varphi(t+T) &= \int_{t+T}^{t+2T} \vartheta(s)e^{\gamma s}\, ds = \int_t^{t+T} \vartheta(q+T)e^{\gamma(q+T)}\, dq \\ &= e^{\gamma T} \int_t^{t+T} \vartheta(q)e^{\gamma q}\, dq = \varphi(t)e^{\gamma T}. \end{aligned}$$

Hence $\varphi \in \Phi$.
Assume conversely that $\varphi \in \Phi$. By (4.3) we have $\varphi'(t+T) = \varphi'(t)\, e^{\gamma T}$. Define

$$\vartheta(t) = e^{-\gamma t}\varphi'(t)[e^{\gamma T}-1]^{-1} \in H^1_{\mathrm{loc}}(\mathbb{R}, V);$$

then, for every $t \in \mathbb{R}$,

$$\vartheta(t+T) = e^{-\gamma(t+T)}\varphi'(t+T)[e^{\gamma T}-1]^{-1} = \varphi'(t)e^{-\gamma t}[e^{\gamma T}-1]^{-1} = \vartheta(t),$$

so that $\varphi \in H^1_T(V)$. In addition

$$\begin{aligned}\int_t^{t+T} \vartheta(s)e^{\gamma s}\, ds &= \int_t^{t+T} \varphi'(s)\, ds[e^{\gamma T}-1]^{-1} \\ &= [\varphi(t+T)-\varphi(t)][e^{\gamma T}-1]^{\vartheta-1} \\ &= [\varphi(t)e^{\gamma T}-\varphi(t)][e^{\gamma T}-1]^{-1} = \varphi(t).\end{aligned}$$

This proves the second inclusion. ☐

The second preliminary result gives the equivalence of equation (4.9) with the weak form of our problem (1.1)–(1.2), i.e. equation (2.1).

Proposition 4.3. *Let $v \in F$. Then v solves* (4.9) *if and only if $u(t) = v(t)e^{-\gamma t}$ (which belongs to $L^2_T(V) \cap H^1_T(H)$) satisfies* (2.1)*, i.e.*

$$\begin{aligned}\int_0^T &\big[\langle A(t)u(t),\vartheta(t)\rangle_{V^*,V} - (B(t)u'(t), \vartheta'(t))_H \\ &\qquad + (C(t)u'(t), \vartheta(t))_H\big]dt = \int_0^T (f(t), \vartheta(t))_H\, dt\end{aligned} \tag{4.11}$$

for all $\vartheta \in H^1_T(V)$, where $f(t) = g(t)e^{-\gamma t}$.

Proof. By Lemma 4.2, every $\varphi \in \Phi$ may be written in the form (4.10), with $\vartheta \in H^1_T(V)$. Then we know that

$$\varphi'(t) = \vartheta(t+T)e^{\gamma(t+T)} - \vartheta(t)e^{\gamma t} = \vartheta(t)e^{\gamma t}(e^{\gamma T}-1);$$

hence

$$\begin{aligned}E(v,\theta) = (e^{\gamma T}-1)\int_0^T &\Big[\langle A(t)v, \vartheta\rangle_{V^*,V}\, e^{-\gamma t} - (B(t)v', [\vartheta e^{-\gamma t}]')_H \\ &- \gamma(B(t)v', \vartheta)_H\, e^{-\gamma t} + \gamma(B(t)v, [\vartheta e^{\gamma t}]')_H\, e^{-2\gamma t} \\ &- \gamma^2(B(t)v, \vartheta)_H\, e^{-\gamma t} + (C(t)v', \vartheta)_H\, e^{-\gamma t} - \gamma(C(t)v, \vartheta)_H\, e^{-\gamma t}\Big]dt.\end{aligned}$$

Now we observe that

$$\begin{aligned}-(B(t)v', [\vartheta e^{-\gamma t}]')_H &= -(B(t)v', \vartheta')_H\, e^{-\gamma t} + \gamma(B(t)v', \vartheta)_H\, e^{-\gamma t} \\ &= -(B(t)[v\, e^{-\gamma t}]', \vartheta')_H - \gamma(B(t)v, \vartheta')_H\, e^{-\gamma t} + \gamma(B(t)v', \vartheta)_H\, e^{-\gamma t};\end{aligned}$$

next,

$$\gamma(B(t)v, [\vartheta e^{\gamma t}]')_H\, e^{-2\gamma t} = \gamma(B(t)v, \vartheta')_H\, e^{-\gamma t} + \gamma^2(B(t)v, \vartheta)_H\, e^{-\gamma t},$$

and finally

$$(C(t)v', \vartheta)_H\, e^{-\gamma t} - \gamma(C(t)v, \vartheta)_H\, e^{-\gamma t} = (C(t)[v\, e^{-\gamma t}]', \vartheta)_H\, .$$

Thus we may rewrite $E(v,\varphi)$, after some cancellations, as

$$\begin{aligned} E(v,\varphi) &= (e^{\gamma T}-1)\int_0^T \big[\langle A(t)v,\vartheta\rangle_{V^*,V}\, e^{-\gamma t} - (B(t)[v\,e^{-\gamma t}]',\vartheta')_H \\ &\qquad\qquad + (C(t)[v\,e^{-\gamma t}]',\vartheta)_H\big]\,dt \\ &= (e^{\gamma T}-1)\int_0^T \big[\langle A(t)u(t),\vartheta(t)\rangle_{V^*,V} - (B(t)u'(t),\vartheta'(t))_H \\ &\qquad\qquad + (C(t)u'(t),\vartheta(t))_H\big]\,dt. \end{aligned} \tag{4.12}$$

Inserting now the function φ, written in the form (4.10), into the functional L, we obtain

$$\begin{aligned} L\varphi &= (e^{\gamma T}-1)\int_0^T (g(t),\vartheta(t))_H\, e^{-\gamma t}\,dt \\ &= (e^{\gamma T}-1)\int_0^T (f(t),\vartheta(t))_H\,dt. \end{aligned}$$

So, if v solves (4.9), then u is a solution of (4.11). Conversely, assume that u solves (4.11) for every $\vartheta\in H^1_T(V)$: then, setting $v(t)=u(t)e^{\gamma t}$, we can go back and see that v solves (4.9) for every φ having the form (4.10). By Lemma 4.2, we conclude that v solves (4.9) for every $\varphi\in\Phi$. □

We now verify that the spaces F, Φ defined in (4.2) and (4.3), as well as the form (4.7) and the functional (4.8), satisfy the conditions required for the use of Theorem 2.5. As already remarked, F is a complex Hilbert space endowed with the inner product (4.4); moreover, Φ, endowed with the inner product (4.5), is a non-complete inner product space which is continuously embedded into F, in view of (4.6). The bilinear form $E(v,\varphi)$ is also bounded with respect to v for every fixed φ: indeed, by (4.7), (1.3) and easy estimates, we deduce

$$|E(v,\varphi)|\le K_0\,\|v\|_F\,\|\varphi\|_\Phi \qquad \forall v\in F,\quad \forall\varphi\in\Phi,$$

where K_0 is a constant depending on the quantities M_Q, M_R, M_B, M_C introduced in Section 1, as well as on γ.
We now prove (2.6). First of all we observe that (2.6) will follow by the stronger estimate

$$\mathrm{Re}\,(E(\varphi,\varphi)\ge \nu\|\varphi\|^2_\Phi \qquad \forall\varphi\in\Phi. \tag{4.13}$$

We rewrite (4.7) as

$$E(\varphi,\varphi)=\sum_{i=1}^{7} J_i, \tag{4.14}$$

where, of course, J_1, J_2, $\dots$, J_7 are the integral terms appearing on the right-hand side of (4.7). Concerning J_1 we have by (1.4)

$$\begin{aligned} J_1 &= \int_0^T \langle A(t)\varphi(t), \varphi'(t)\rangle_{V^*,V}\, e^{-2\gamma t}\, dt \\ &= \int_0^T \langle Q_1(t)\varphi(t), \varphi'(t)\rangle_{V^*,V}\, e^{-2\gamma t}\, dt \\ &+ \int_0^T (Q_2(t)\varphi(t), \varphi'(t))_H\, e^{-2\gamma t}\, dt + \int_0^T (R(t)\varphi(t), \varphi'(t))_H\, e^{-2\gamma t}\, dt. \end{aligned} \tag{4.15}$$

On the other hand

$$\begin{aligned} \int_0^T \langle Q_1(t)\varphi(t), \varphi'(t)\rangle_{V^*,V}\, e^{-2\gamma t}\, dt &= \int_0^T \frac{d}{dt}\left[\langle Q_1(t)\varphi(t), \varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\right] dt \\ &- \int_0^T \langle Q_1'(t)\varphi(t), \varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\, dt - \int_0^T \langle Q_1(t)\varphi'(t), \varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\, dt \\ &+ 2\gamma \int_0^T \langle Q_1(t)\varphi(t), \varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\, dt, \end{aligned}$$

and the first term on the right-hand side vanishes, since $\varphi \in \Phi$. Moreover, taking into account that

$$\begin{aligned} \operatorname{Re} \int_0^T \langle Q_1(t)\varphi(t), \varphi'(t)\rangle_{V^*,V}\, e^{-2\gamma t}\, dt \\ = \operatorname{Re} \int_0^T \langle Q_1(t)\varphi'(t), \varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\, dt \end{aligned}$$

by (1.5) we get

$$\begin{aligned} \operatorname{Re} \int_0^T \langle Q_1(t)\varphi(t), \varphi'(t)\rangle_{V^*,V}\, e^{-2\gamma t}\, dt &= -\frac{1}{2} \int_0^T \langle Q_1'(t)\varphi(t), \varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\, dt \\ &+ \gamma \int_0^T \langle Q_1(t)\varphi(t), \varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\, dt. \end{aligned}$$

Quite similarly we deduce

$$\begin{aligned} \operatorname{Re} \int_0^T (Q_2(t)\varphi(t), \varphi'(t))_H\, e^{-2\gamma t}\, dt \\ = -\frac{1}{2} \int_0^T (Q_2'(t)\varphi(t), \varphi(t))_H\, e^{-2\gamma t}\, dt \\ + \gamma \int_0^T (Q_2(t)\varphi(t), \varphi(t))_H\, e^{-2\gamma t}\, dt. \end{aligned}$$

Hence, by (4.15),

$$\begin{aligned}\operatorname{Re} J_1 = &-\frac{1}{2}\int_0^T \langle Q_1'(t)\varphi(t),\varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\,dt\\ &+\frac{1}{2}\int_0^T Q_2'(t)\varphi(t),\varphi(t))_H\, e^{-2\gamma t}\,dt\\ &+\gamma\int_0^T \langle Q_1(t)\varphi(t),\varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\,dt\\ &+\gamma\int_0^T (Q_2(t)\varphi(t),\varphi(t))_H\, e^{-2\gamma t}\,dt\\ &+\operatorname{Re}\int_0^T (R(t)\varphi(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt.\end{aligned} \tag{4.16}$$

Next, we consider $J_2 + J_3$. Computing the derivative, we get

$$\begin{aligned}J_2+J_3 = &-\int_0^T (B(t)\varphi'(t),[\varphi'(t)\,e^{-2\gamma t}]')_H\,dt\\ &-\gamma\int_0^T (B(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt = -\int_0^T (B(t)\varphi'(t),\varphi''(t))_H\, e^{-2\gamma t}\,dt\\ &+2\gamma\int_0^T (B(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt-\gamma\int_0^T (B(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt.\end{aligned}$$

which means

$$\begin{aligned}J_2+J_3 = &-\int_0^T (B(t)\varphi'(t),\varphi''(t))_H\, e^{-2\gamma t}\,dt\\ &+\gamma\int_0^T (B(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt.\end{aligned} \tag{4.17}$$

On the other hand

$$\begin{aligned}-\int_0^T (B(t)\varphi'(t),\varphi''(t))_H\, e^{-2\gamma t}\,dt = &-\int_0^T \frac{d}{dt}\left[(B(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\right]dt\\ &+\int_0^T (B'(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt+\int_0^T (B(t)\varphi''(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt\\ &-2\gamma\int_0^T (B(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt,\end{aligned}$$

and the first term on the right-hand side is zero, since $\varphi \in \Phi$. Moreover, since, by (1.6), $B(t)$ is selfadjoint, we have

$$\begin{aligned}-\operatorname{Re}&\int_0^T (B(t)\varphi'(t),\varphi''(t))_H\, e^{-2\gamma t}\,dt\\ &=\frac{1}{2}\int_0^T (B'(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt\\ &-\gamma\int_0^T (B(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt.\end{aligned} \tag{4.18}$$

Hence, by (4.17),

$$\mathrm{Re}\,(J_2+J_3)=J_2+J_3=\frac{1}{2}\int_0^T (B'(t)\varphi'(t),\varphi'(t))_H\,e^{-2\gamma t}\,dt. \tag{4.19}$$

We now look at J_4+J_5. It holds

$$\begin{aligned}
&J_4+J_5=\gamma\int_0^T (B(t)\varphi(t),\varphi''(t))_H\,e^{-2\gamma t}\,dt\\
&-\gamma^2\int_0^T (B(t)\varphi(t),\varphi'(t))_H\,e^{-2\gamma t}\,dt=\gamma\int_0^T (B(t)\varphi(t)\,e^{-\gamma t},[\varphi'(t)e^{-\gamma t}]')_H\,dt\\
&+\gamma^2\int_0^T (B(t)\varphi(t)\,e^{-\gamma t},\varphi'(t))_H\,e^{-\gamma t}\,dt-\gamma^2\int_0^T (B(t)\varphi(t),\varphi'(t))_H\,e^{-2\gamma t}\,dt
\end{aligned}$$

so that

$$J_4+J_5=\gamma\int_0^T (B(t)\varphi(t)\,e^{-\gamma t},[\varphi'(t)\,e^{-\gamma t}]')_H\,dt. \tag{4.20}$$

On the other hand

$$\begin{aligned}
&\gamma\int_0^T (B(t)\varphi(t)\,e^{-\gamma t},[\varphi'(t)\,e^{-\gamma t}]')_H\,dt\\
&=\gamma\int_0^T \frac{d}{dt}(B(t)\varphi(t)\,e^{-\gamma t},\varphi'(t)\,e^{-\gamma t})_H\,dt-\gamma\int_0^T (B'(t)\varphi(t),\varphi'(t))_H\,e^{-2\gamma t}\,dt\\
&-\gamma\int_0^T (B(t)\varphi'(t),\varphi'(t))_H\,e^{-2\gamma t}\,dt+\gamma^2\int_0^T (B(t)\varphi(t),\varphi'(t))_H\,e^{-2\gamma t}\,dt.
\end{aligned}$$

Thus, by (4.20),

$$\begin{aligned}
\mathrm{Re}\,(J_4+J_5)=&-\gamma\mathrm{Re}\int_0^T B'(t)\varphi(t),\varphi'(t))_H\,e^{-2\gamma t}\,dt\\
&-\gamma\int_0^T (B(t)\varphi'(t),\varphi'(t))_H\,e^{-2\gamma t}\,dt\\
&+\gamma^2\mathrm{Re}\int_0^T (B(t)\varphi(t),\varphi'(t))_H\,e^{-2\gamma t}\,dt.
\end{aligned} \tag{4.21}$$

Finally, we leave the terms J_6+J_7 as they are:

$$\begin{aligned}
J_6+J_7=&\int_0^T (C(t)\varphi'(t),\varphi'(t))_H\,e^{-2\gamma t}\,dt\\
&-\gamma\int_0^T (C(t)\varphi(t),\varphi'(t))_H\,e^{-2\gamma t}\,dt.
\end{aligned} \tag{4.22}$$

Summing up, using (4.14) and (4.16), (4.19), (4.21), (4.22), we have

$$\begin{aligned}
\operatorname{Re} E(\varphi,\varphi) &= -\frac{1}{2}\int_0^T \langle Q_1'(t)\varphi(t),\varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\,dt \\
&+ \gamma\int_0^T \langle Q_1(t)\varphi(t),\varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\,dt - \frac{1}{2}\int_0^T (Q_2'(t)\varphi(t),\varphi(t))_H\, e^{-2\gamma t}\,dt \\
&+ \gamma\int_0^T (Q_2(t)\varphi(t),\varphi(t))_H\, e^{-2\gamma t}\,dt + \operatorname{Re}\int_0^T (R(t)\varphi(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt \\
&+ \frac{1}{2}\int_0^T (B'(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt - \gamma\operatorname{Re}\int_0^T B'(t)\varphi(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt \\
&- \gamma\int_0^T (B(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt + \gamma^2\operatorname{Re}\int_0^T (B(t)\varphi(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt \\
&+ \int_0^T (C(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt - \gamma\operatorname{Re}\int_0^T (C(t)\varphi(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt.
\end{aligned} \tag{4.23}$$

Before estimating $\operatorname{Re} E(\varphi,\varphi)$ from below, we need a useful lemma.

Lemma 4.4. *The following estimate holds:*

$$\gamma^2\int_0^T \|\varphi(t)\|_H^2\, e^{-2\gamma t}\,dt \le \int_0^T \|\varphi'(t)\|_H^2\, e^{-2\gamma t}\,dt \qquad \forall\varphi\in\Phi.$$

Proof. We have

$$\begin{aligned}
0 &\le \int_0^T \|(\varphi(t)\, e^{-\gamma t})'\|_H^2\,dt = \int_0^T \|\varphi'(t)\, e^{-\gamma t} - \gamma\,\varphi(t)\, e^{-\gamma t}\|_H^2\,dt \\
&= \int_0^T \|\varphi'(t)\|_H^2\, e^{-2\gamma t}\,dt + \gamma^2\int_0^T \|\varphi(t)\|_H^2\, e^{-2\gamma t}\,dt \\
&- 2\gamma\operatorname{Re}\int_0^T (\varphi'(t),\varphi(t))_H\, e^{-2\gamma t}\,dt.
\end{aligned} \tag{4.24}$$

On the other hand, since $\varphi\in\Phi$,

$$\begin{aligned}
\operatorname{Re}\int_0^T (\varphi'(t),\varphi(t))_H\, e^{-2\gamma t}\,dt &= \frac{1}{2}\int_0^T \left[\frac{d}{dt}\|\varphi(t)\|_H^2\right] e^{-2\gamma t}\,dt \\
&= \frac{1}{2}\int_0^T \frac{d}{dt}\left[\|\varphi(t)\|_H^2\, e^{-2\gamma t}\right]dt + \gamma\int_0^T \|\varphi(t)\|_H^2\, e^{-2\gamma t}\,dt \\
&= \gamma\int_0^T \|\varphi(t)\|_H^2\, e^{-2\gamma t}\,dt.
\end{aligned}$$

Thus, by (4.24),

$$\begin{aligned}
0 \le \int_0^T \|\varphi'(t)\|_H^2\, e^{-2\gamma t}\,dt &+ \gamma^2\int_0^T \|\varphi(t)\|_H^2\, e^{-2\gamma t}\,dt \\
&- 2\gamma^2\int_0^T \|\varphi(t)\|_H^2\, e^{-2\gamma t}\,dt,
\end{aligned}$$

and the result follows. $\square$

We now proceed with the estimate of $\operatorname{Re} E(\varphi,\varphi)$ from below. We write

$$\begin{aligned}
&\operatorname{Re} E(\varphi,\varphi) \ge -\frac{1}{2}\int_0^T \langle Q_1'(t)\varphi(t),\varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\,dt\\
&+\gamma\int_0^T \langle Q_1(t)\varphi(t),\varphi(t)\rangle_{V^*,V}\, e^{-2\gamma t}\,dt\\
&-\frac{1}{2}\int_0^T (Q_2'(t)\varphi(t),\varphi(t))_H\, e^{-2\gamma t}\,dt\\
&+\gamma\int_0^T (Q_2(t)\varphi(t),\varphi(t))_H\, e^{-2\gamma t}\,dt+\frac{1}{2}\int_0^T (B'(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt\\
&-\gamma\int_0^T (B(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt+\int_0^T (C(t)\varphi'(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt\\
&-\left|\int_0^T (R(t)\varphi(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt\right|-\gamma\left|\int_0^T B'(t)\varphi(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt\right|\\
&-\gamma^2\left|\int_0^T (B(t)\varphi(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt\right|\\
&-\gamma\left|\int_0^T (C(t)\varphi(t),\varphi'(t))_H\, e^{-2\gamma t}\,dt\right|.
\end{aligned}$$

As a consequence, using the assumptions of Section 1 and the inequality $pq \le \frac{\sigma}{2}\,p^2+\frac{1}{2\sigma}\,q^2$ for every $\sigma>0$, we get for $\vartheta,\delta,\varepsilon>0$

$$\begin{aligned}
\operatorname{Re} E(\varphi,\varphi) \ge{}& \int_0^T \|\varphi(t)\|_V^2\, e^{-2\gamma t}\,dt\left[\gamma\,\nu_{Q_1}-\frac{M_{Q_1'}}{2}\right]\\
&-\int_0^T \|\varphi(t)\|_H^2\, e^{-2\gamma t}\,dt\left[\gamma\,\frac{M_{Q_2'}}{2}\right]\\
&+\int_0^T \|\varphi'(t)\|_H^2\, e^{-2\gamma t}\,dt\left[\left(\frac{1}{2}\nu_{B'}+\nu_C\right)-\gamma\,M_B\right]\\
&-\int_0^T \|\varphi'(t)\|_H^2\, e^{-2\gamma t}\,dt\left[\frac{\vartheta\,M_R}{2}+\frac{\gamma\,M_{B'}}{2\delta}+\frac{\gamma^2\,M_B}{2\varepsilon}+\frac{\gamma\,M_C}{2\delta}\right]\\
&-\int_0^T \|\varphi(t)\|_H^2\, e^{-2\gamma t}\,dt\left[\frac{M_R}{2\vartheta}+\frac{\gamma\,\delta\,M_{B'}}{2}+\frac{\gamma^2\,\varepsilon\,M_B}{2}+\frac{\gamma\,\delta\,M_C}{2}\right].
\end{aligned}$$

We use here Lemma 4.4 to estimate from below the third term in the last line of the right member:

$$-\frac{1}{2}\,\gamma^2\,\varepsilon\,M_B\int_0^T \|\varphi(t)\|_H^2\, e^{-2\gamma t}\,dt \ge -\frac{1}{2}\,\varepsilon\,M_B\int_0^T \|\varphi'(t)\|_H^2\, e^{-2\gamma t}\,dt.$$

Recalling (1.3), we obtain the final estimate:

$$
\begin{aligned}
\operatorname{Re} E(\varphi,\varphi) \geq & \int_0^T \|\varphi(t)\|_V^2 \, e^{-\gamma t} \, dt \\
& \times \left[\gamma \, \nu_{Q_1} - \frac{M_{Q_1'}}{2} - \frac{\beta^2}{2} \left(M_{Q_2'} + \frac{M_R}{\vartheta} + \gamma \, \delta \, (M_{B'} + M_C) \right) \right] \\
& + \int_0^T \|\varphi'(t)\|_H^2 \, e^{-\gamma t} dt \times \left[\left(\frac{\nu_{B'}}{2} + \nu_C \right) \right. \\
& \left. - \gamma \, M_B - \frac{1}{2} \left(\vartheta \, M_R + \gamma \, \frac{M_{B'} + M_C}{\delta} + \frac{\gamma^2 \, M_B}{\varepsilon} + \varepsilon \, M_B \right) \right].
\end{aligned}
\tag{4.25}
$$

In order to obtain a good estimate we need that the coefficients of the right member of (4.25) are both positive for some $\gamma > 0$. We rewrite the two conditions as follows:

$$
\begin{cases}
\left(\nu_{Q_1} - \frac{\beta^2 \, \delta \, (M_{B'} + M_C)}{2} \right) \gamma - \frac{M_{Q_1'}}{2} - \frac{\beta^2 \, M_{Q_2'}}{2} - \frac{\beta^2 \, M_R}{2\vartheta} > 0 \\
- \frac{M_B}{2\varepsilon} \, \gamma^2 - \left(M_B + \frac{M_{B'} + M_C}{2\delta} \right) \gamma + \left(\frac{\nu_{B'}}{2} + \nu_C \right) - \frac{\vartheta \, M_R}{2} - \frac{\varepsilon \, M_B}{2} > 0.
\end{cases}
\tag{4.26}
$$

The first inequality in (4.26) is satisfied when

$$
\gamma > \frac{M_{Q_1'} + \beta^2 \, M_{Q_2'} + \frac{\beta^2 \, M_R}{\vartheta}}{2\nu_{Q_1} - \beta^2 \, \delta \, (M_{B'} + M_C)} =: \gamma_0 > 0,
\tag{4.27}
$$

provided δ is sufficiently small.
The second inequality in (4.26) has the form $a\gamma^2 + b\gamma + c > 0$, with $a < 0$, while $c > 0$ provided ϑ and ε are sufficiently small. Hence for small ϑ and ε the discriminant $\Delta = b^2 - 4ac$ satisfies $\Delta > b^2 \geq 0$. Thus in this case the corresponding polynomial in the variable γ has two real roots with different signs:

$$
\gamma_1 = \frac{b - \sqrt{\Delta}}{2|a|} < 0 < \frac{b + \sqrt{\Delta}}{2|a|} = \gamma_2 \, .
$$

Thus, in order that the system (4.26) has a positive solution γ, we need that $\gamma_0 < \gamma_2$. Noticing that

$$
\gamma_2 = \frac{\sqrt{\Delta} - |b|}{2|a|} = \frac{1}{2|a|} \frac{4|a|c}{\sqrt{\Delta} + |b|} = \frac{2c}{\sqrt{\Delta} + |b|} \, ,
$$

we must require that

$$
\frac{\nu_{B'} + 2\nu_C - \vartheta \, M_R - \varepsilon \, M_B}{\frac{M_{B'} + M_C}{\delta} + M_B + \sqrt{\left[\frac{M_{B'} + M_C}{\delta} + M_B \right]^2 + \frac{M_B}{\varepsilon} \left[\nu_{B'} + 2\nu_C - \vartheta \, M_R - \varepsilon \, M_B \right]}}
> \frac{M_{Q_1'} + \beta^2 \, M_{Q_2'} + \frac{\beta^2 \, M_R}{\vartheta}}{2\nu_{Q_1} - \delta \, \beta^2 (M_{B'} + M_C)} \, ,
$$

a rather complicated condition. Let us choose now

$$\vartheta = \frac{\nu_{B'} + 2\nu_C}{2M_R}\,, \text{ so that } M_R\,\vartheta = \frac{\nu_{B'} + 2\nu_C}{2}\,,$$

$$\varepsilon = \frac{\nu_{B'} + 2\nu_C}{2M_B}\,, \text{ so that } \varepsilon\, M_B = \frac{\nu_{B'} + 2\nu_C}{2}\,,$$

and

$$\delta = \frac{\nu_{Q_1}}{\beta^2(M_{B'} + M_C)}\,, \text{ so that } 2\nu_{Q_1} - \delta\beta^2(M_{B'} + M_C) = \nu_{Q_1}\,.$$

The above condition becomes

$$\begin{aligned} &\frac{\nu_{B'} + 2\nu_C}{\frac{\beta^2(M_{B'}+M_C)^2}{\nu_{Q_1}} + M_B + \sqrt{\left[\frac{\beta^2(M_{B'}+M_C)^2}{\nu_{Q_1}} + M_B\right]^2 + 2\,M_B^2}} \\ &\qquad > \frac{M_{Q_1'} + \beta^2\,M_{Q_2'} + + \frac{2\beta^2\,M_R^2}{\nu_{B'}+2\nu_C}}{\nu_{Q_1}}, \end{aligned} \tag{4.28}$$

i.e., after some easy calculations,

$$\begin{aligned} &\frac{\nu_{Q_1}^2\,(\nu_{B'} + 2\nu_C)^2}{\beta^2(M_{B'} + M_C)^2 + \nu_{Q_1}M_B + \sqrt{[\beta^2(M_{B'} + M_C)^2 + \nu_{Q_1}\,M_B]^2 + 2\nu_{Q_1}^2\,M_B^2}} \\ &\qquad > (\nu_{B'} + 2\nu_C)\big(M_{Q_1'} + \beta^2\,M_{Q_2'}\big) + 2\beta^2\,M_R^2\,, \end{aligned}$$

which is precisely (4.1). In conclusion, if $M_{Q_1'}$, $M_{Q_2'}$ and M_R are sufficiently small, there is some positive γ such that (4.26) holds true with the chosen $\vartheta, \delta, \varepsilon$, and consequently there is a positive constant ν such that

$$|E(\varphi, \varphi)| \geq \nu \|\varphi\|_\Phi^2 \qquad \forall \varphi \in \Phi,$$

i.e. (2.6) holds. Thus Theorem 2.5 is applicable and (4.9) has a solution v; by Proposition 4.3 and Proposition 2.1, the function $u(t) = v(t)e^{-\gamma t}$ solves problem (1.1)–(1.2). The estimate follows by Theorem 3.1. Theorem 4.1 is completely proved. □

5. A general existence result

The goal of this section is the proof of Theorem 1.3. The result of the more restrictive Theorem 4.1 is the starting point of our argument, which consists of four intermediate steps.

Step 1: under Hypothesis 1.1, transformation of the problem.

We consider here problem (1.1)–(1.2) with the operator

$$\mathcal{A}u = A(\cdot)u(\cdot) + (B(\cdot)u'(\cdot))' + C(\cdot)u'(\cdot), \tag{5.1}$$

which satisfies Hypothesis 1.1. We will perform a change of variable which transforms the operator $\mathcal{A}$ into another, say $\mathcal{A}^0$, and the problem (1.1)–(1.2) into another, with a different period T^0. It will turn out that $\mathcal{A}^0$ will satisfy, in addition, condition (4.1), relative to the new problem and the new operator:

hence, by Theorem 4.1, the equation $\mathcal{A}^0 v = g$ will be uniquely solvable for all $g \in L^2_{T_0}(H)$.
To start with, we fix $\alpha \in]0,1[$ and change variable, setting $t = \alpha s$. If we define for $s \in \mathbb{R}$,

$$\begin{aligned} v(s) = u(\alpha s), \quad \tilde{A}(s) = A(\alpha s), \quad B_\alpha(s) = B(\alpha s), \\ C_\alpha(s) = C(\alpha s), \quad g(s) = f(\alpha s), \end{aligned}$$

we easily get

$$\begin{cases} (\tilde{\mathcal{A}} v)(s) = \tilde{A}(s) v(s) + \left[\dfrac{B_\alpha(s)}{\alpha} v'(s) \right]' + \dfrac{C_\alpha(s)}{\alpha} v'(s) = g(s), \quad s \in \mathbb{R}, \\ v\left(s + \dfrac{T}{\alpha} \right) = v(s), \end{cases}$$

Setting $T^0 = \frac{T}{\alpha}$, $\tilde{B}(s) = \frac{B_\alpha(s)}{\alpha}$, and $\tilde{C}(s) = \frac{C_\alpha(s)}{\alpha}$, we have

$$\tilde{\mathcal{A}} v = \tilde{A}(s) v(s) + [\tilde{B}(s) v'(s)]' + \tilde{C}(s) v'(s), \tag{5.2}$$

and the above problem is transformed into

$$\begin{cases} (\tilde{\mathcal{A}} v)(s) = g(s), \quad s \in \mathbb{R}, \\ v\left(s + T^0\right) = v(s). \end{cases} \tag{5.3}$$

Obviously, a T-periodic function $u(t)$ is a solution of (1.1)–(1.2) if and only if the T^0-periodic function $v(s)$ solves (5.3). The constants appearing in (1.5) and (1.6), relative to $\tilde{A} = \tilde{Q}_1 + \tilde{Q}_2 + \tilde{R}$, $\tilde{B}$, $\tilde{C}$, are

$$\begin{aligned} &\nu_{\tilde{B}'} = \nu_{B'}\,, \quad \nu_{\tilde{C}} = \frac{\nu_C}{\alpha}\,, \quad \nu_{\tilde{B}} = \frac{\nu_B}{\alpha}, \quad \nu_{\tilde{Q}_1} = \nu_{Q_1}\,, \\ &M_{\tilde{R}} = M_R\,, \quad M_{\tilde{C}} = \frac{M_C}{\alpha}\,, \quad M_{\tilde{B}'} = M_{B'}\,, \\ &M_{\tilde{Q_1}'} = \alpha M_{Q_1'}\,, \quad M_{\tilde{Q_2}'} = \alpha M_{Q_2'},\,, \quad M_{\tilde{B}} = \frac{M_B}{\alpha}\,. \end{aligned} \tag{5.4}$$

We now verify whether or not the operator $\tilde{\mathcal{A}}$ satisfies condition (4.1). According to (5.4) we obtain

$$\begin{aligned} &\frac{\nu_{Q_1}^2 \left(\nu_{B'} + \frac{2\nu_C}{\alpha}\right)^2}{\beta^2 \left(M_{B'} + \frac{M_C}{\alpha}\right)^2 + \nu_{Q_1} \frac{M_B}{\alpha} + \sqrt{\left[\beta^2 \left(M_{B'} + \frac{M_C}{\alpha}\right)^2 + \nu_{Q_1} \frac{M_B}{\alpha}\right]^2 + 2\nu_{Q_1}^2 \frac{M_B^2}{\alpha^2}}} \\ &\qquad > \left(\nu_{B'} + \tfrac{2\nu_C}{\alpha}\right) \left(\alpha\, M_{Q_1'} + \beta^2\, \alpha\, M_{Q_2'}\right) + 2\beta^2 M_R^2\,, \end{aligned} \tag{5.5}$$

which means

$$\begin{aligned} &\frac{\nu_{Q_1}^2 (\alpha\, \nu_{B'} + 2\nu_C)}{\beta^2 (\alpha\, M_{B'} + M_C)^2 + \nu_{Q_1}\, \alpha\, M_B + \sqrt{\left[\beta^2 (\alpha\, M_{B'} + M_C)^2 + \nu_{Q_1}\, \alpha\, M_B\right]^2 + 2\nu_{Q_1}^2\, \alpha^2\, M_B^2}} \\ &\qquad > M_{Q_1'} + \beta^2\, M_{Q_2'} + \frac{2\beta^2\, M_R^2}{\alpha \nu_{B'} + 2\nu_C}\,. \end{aligned} \tag{5.6}$$

This condition may or may not be true. So, we still modify, under Hypothesis 1.1, the operator $\tilde{\mathcal{A}}$: we fix $\sigma > 0$, replace $\tilde{C}$ by $\sigma \tilde{C}$, and set

$$(\tilde{\mathcal{A}}^0 v)(s) = \tilde{A}(s) v(s) + [\tilde{B}(s) v'(s)]' + \sigma \tilde{C}(s) v'(s). \tag{5.7}$$

Since $\nu_{\sigma\tilde{C}} = \sigma\nu_{\tilde{C}}$ and $M_{\sigma\tilde{C}} = M\nu_{\tilde{C}}$, condition (5.6) becomes

$$\begin{aligned}\frac{\nu_{Q_1}^2(\alpha\,\nu_{B'}+2\sigma\,\nu_C)}{\beta^2(\alpha\,M_{B'}+\sigma\,M_C)^2+\nu_{Q_1}\,\alpha\,M_B+\sqrt{\left[\beta^2(\alpha\,M_{B'}+\sigma\,M_C)^2+\nu_{Q_1}\,\alpha\,M_B\right]^2+2\nu_{Q_1}^2\,\alpha^2\,M_B^2}} \\ > M_{Q_1'} + \beta^2\,M_{Q_2'} + \frac{2\alpha\beta^2\,M_R^2}{\alpha\nu_{B'}+2\sigma\nu_C}\,.\end{aligned} \tag{5.8}$$

This inequality is satisfied if α and σ are sufficiently small: indeed, for $\alpha = 0$ it reduces to

$$\frac{\nu_{Q_1}^2\nu_C}{\beta^2\,\sigma\,M_C^2} > M_{Q_1'} + \beta^2\,M_{Q_2'},$$

which is true for sufficiently small σ; hence (5.8) is true for sufficiently small α and σ, as well. By Theorem 4.1, the problem

$$\begin{cases}(\tilde{\mathcal{A}}^0 v)(s) = g(s), \quad s \in \mathbb{R}, \\ v\left(s+T^0\right) = v(s),\end{cases} \tag{5.9}$$

has a unique solution in $H^1_{T^0}(H) \cap L^2_{T^0}(V)$ for every $g \in L^2_{T^0}(H)$, provided α and σ are positive and sufficiently small. So, ultimately, Step 1 is proved with the period $T^0 = T/\alpha$ and the operator $\tilde{\mathcal{A}}^0$.

Step 2: under Hypothesis 1.2, $\mathcal{R}(\tilde{\mathcal{A}})$ is dense in $L^2_{T^0}(H)$.

Let $\sigma > 0$ and $\alpha > 0$ be such that (5.8) is satisfied. Consider the operator $\tilde{\mathcal{A}}^0$ defined by (5.7), this time under Hypothesis 1.2; set, according to (1.7),

$$\mathcal{D}(\tilde{A}) = \left\{v \in L^2_{T^0}(H) : \ \tilde{A}v \in L^2_{T^0}(H)\right\},$$

and consider $\tilde{\mathcal{A}}^0$ as an operator from $\mathcal{D}(\tilde{A})$ to $L^2_{T^0}(H)$. We note that if $g \in H^1_{T^0}(H)$, by Theorem 2.3 (with T replaced by T^0) the solution of (5.9) is in $\mathcal{D}(\tilde{A}) \cap C^1_{T^0}(V) \cap C^2_{T^0}(H)$; hence, the range $\mathcal{R}(\tilde{\mathcal{A}}^0)$ is dense in $L^2_{T^0}(H)$, since it contains $H^1_{T^0}(H)$.
Always under Hypothesis 1.2, consider also the operator $\tilde{\mathcal{A}}$ defined in (5.2), as an operator from $\mathcal{D}(\tilde{A})$ to $L^2_{T^0}(H)$; note that for $\mathcal{A}^0$ and $\tilde{\mathcal{A}}$ the terms containing $\tilde{A}$ and $\tilde{B}$ coincide. In order to verify that the range of $\tilde{\mathcal{A}}$ is dense, too, we will use the continuity method and the results of the "near operators" theory (see [11]). Namely, for $0 \le \tau \le 1$ we define

$$\mathcal{A}_\tau = \tau\,\tilde{\mathcal{A}} + (1-\tau)\tilde{\mathcal{A}}^0, \qquad \tau \in [0,1].$$

Let us consider the set

$$I = \{\tau \in [0,1] : \ \mathcal{R}(\mathcal{A}_\tau) \text{ is dense in } L^2_{T^0}(H)\} :$$

we will now verify that

- **(i).** $I \neq \emptyset$;
- **(ii).** I is open;
- **(iii).** I is closed.

This will imply that $I = [0,1]$, that is, $\mathcal{R}(\tilde{\mathcal{A}})$ is dense in $L^2_{T^0}(H)$.
Indeed, (i) is true because $0 \in I$, since, as already observed, $\mathcal{R}(\tilde{\mathcal{A}}^0)$ is dense in $L^2_{T^0}(H)$.

Le us prove (ii). Fix $\tau_0 \in I$ and $\delta > 0$; for $\tau \in [0,1]$ and $|\tau - \tau_0| \le \delta$, we have for every $v \in H^1_{T^0}(H) \cap L^2_{T^0}(V)$

$$\begin{aligned}\mathcal{A}_\tau v - \mathcal{A}_{\tau_0} v &= [\tau + (1-\tau)\sigma]\tilde{\mathcal{A}}v' - [\tau_0 + (1-\tau_0)\sigma]\tilde{\mathcal{A}}^0 v' \\ &= (\tau - \tau_0)[\tilde{\mathcal{A}}v - \tilde{\mathcal{A}}^0 v] = (\tau-\tau_0)[\tilde{C}v - \sigma\tilde{C}v] = (\tau-\tau_0)(1-\sigma)\tilde{C}v',\end{aligned}$$

and hence we obtain the estimate

$$\|\mathcal{A}_\tau v - \mathcal{A}_{\tau_0} v\|_{L^2_{T^0}(H)} \le \delta(1-\sigma)M_C\|v'\|_{L^2_{T^0}(H)} \le \delta(1-\sigma)M_C\, K\|\mathcal{A}_{\tau_0}v\|_{L^2_{T/k}(H)},$$

where in the last step we have used Theorem 3.1 (with T^0 in place of T: recall that K is independent of T). Let us choose δ such that $\delta(1-\sigma)M_C\, K < 1$: then the operator $\mathcal{A}_\tau$ is near $\mathcal{A}_{\tau_0}$, according to [11, Definition 0.1]. Thus, since $\mathcal{R}(\mathcal{A}_{\tau_0})$ is dense in $L^2_{T^0}(H)$, the same property holds for $\mathcal{A}_\tau$, thanks to [11, Theorem 2.1]. Consequently $\tau \in I$ for $|\tau - \tau_0| < \delta$, i.e. I is open.

Let us prove (iii). Let $\{\tau_n\} \subset I$ be such that $\tau_n \to \tau$: we must show that $\tau \in I$. Indeed, quite similarly to (ii), for every $v \in H^1_{T^0}(H) \cap L^2_{T^0}(V)$ we have the estimate

$$\begin{aligned}\|\mathcal{A}_{\tau_n} v - \mathcal{A}_\tau v\|_{L^2_{T^0}(H)} &\le |\tau_n - \tau|(1-\sigma)M_C\,\|v'\|_{L^2_{T^0}(H)} \\ &\le |\tau_n - \tau|(1-\sigma)M_C\, K\|\mathcal{A}_{\tau_n} v\|_{L^2_{T^0}(H)}\,.\end{aligned}$$

Hence, for sufficiently large n we deduce that the operator $\mathcal{A}_\tau$ is near $\mathcal{A}_{\tau_n}$. As $\mathcal{R}(\mathcal{A}_{\tau_n})$ is dense in $L^2_{T^0}(H)$, we have $\mathcal{R}(\mathcal{A}_\tau)$ dense in $L^2_{T^0}(H)$, too, i.e. $\tau \in I$ and I is closed.

As already observed, we conclude that $I =]0,1]$ and, consequently, $\mathcal{R}(\tilde{\mathcal{A}})$ is dense in $L^2_{T^0}(H)$.

Step 3: under Hypothesis 1.2, the operator $\mathcal{A}$ (see (5.1)), is surjective on $L^2_T(H)$.

Always under Hypothesis 1.2, we first show that the operator $\tilde{\mathcal{A}}$ defined in (5.2), whose range is dense in $L^2_{T^0}(H)$ by Step 2, is in fact surjective. Let $g \in L^2_{T^0}(H)$, take a sequence $\{g_n\} \subset H^1_{T^0}(H)$ such that $g_n \to g$ in $L^2_{T^0}(H)$ and let v_n be such that $\tilde{\mathcal{A}}v_n = g_n$. Then, by Theorem 3.1, recalling that K does not depend on T, we get

$$\|v_n - v_m\|^2_{L^2_{T^0}(V)} + \|v_n' - v_m'\|^2_{L^2_{T^0}(H)} \le K\|g_n - g_m\|^2_{L^2_{T^0}(H)}\,.$$

[Thus, $\{v_n\}$ is a Cauchy sequence in $L^2_{T^0}(V) \cap H^1_{T^0}(H)$, so that it converges in this space to a function $v \in L^2_{T^0}(V) \cap H^1_{T^0}(H)$. Passing to the limit in the equation $\tilde{\mathcal{A}}v_n = g_n$ (written in the form (2.1), which is allowed by Proposition 2.1, with T replaced by T^0) we immediately find $\tilde{\mathcal{A}}v = g$. This shows that $\mathcal{R}(\tilde{\mathcal{A}}) = L^2_{T^0}(H)$.

Now we consider the operator $\mathcal{A}$ defined in (5.1). We set $u(t) = v(t/\alpha)$, i.e $v(s) = u(\alpha s)$: as remarked in Step 1, $v(s)$ solves (5.3) if and only if $u(t)$ is a solution of (1.1)–(1.2), or, equivalently, equation (2.1) (by Proposition 2.1). Thus we conclude that, under Hypothesis 1.2, problem (1.1)-(1.2) has a unique solution for every $f \in L^2_T(H)$, i.e. $\mathcal{A}$ is surjective.

Step 4: under Hypothesis 1.1, conclusion of the proof.

Finally, we want to prove the statement of Theorem 1.3, i.e. the existence result for problem (1.1)-(1.2), under the only Hypothesis 1.1. To this end, we select sequences of operators $\{Q_{1n}(\cdot)\} \subset H_T^{2,\infty}(\mathcal{L}(V,V^*))$, $\{Q_{2n}(\cdot)\} \subset H_T^{2,\infty}(\mathcal{L}(H))$, $\{R_n(\cdot)\} \subset H_T^{1,\infty}(\mathcal{L}(V,H))$, $\{B_n(\cdot)\} \subset H_T^{2,\infty}(\mathcal{L}(H))$, $\{C_n(\cdot)\} \subset H_T^{1,\infty}(\mathcal{L}(V,H))$ such that the constants appearing in Hypothesis 1.1 relative to the operators $Q_{1n}, Q_{2n}, R_n, B_n, C_n$ are the same as those relative to Q_1, Q_2, R, B, C, and in addition

$$\begin{aligned} Q_{1n} \to Q_1 \text{ in } H_T^{1,\infty}(\mathcal{L}(V,V^*)), &\quad Q_{2n} \to Q_2 \text{ in } H_T^{1,\infty}(\mathcal{L}(H)),\\ R_n \to R \text{ in } L_T^{\infty}(\mathcal{L}(H)), &\quad B_n \to B \text{ in } H_T^{1,\infty}(\mathcal{L}(H)),\\ C_n \to C \text{ in } L_T^{\infty}(\mathcal{L}(H)). & \end{aligned}$$

For instance, we can approximate the above functions by the convolution with a sequence of real mollifiers $\{\varphi_n\}$.

We set $A_n = Q_{1n}+Q_{2n}+R_n$ and $\mathcal{A}_n u = A_n(\cdot)u(B_n(\cdot)u')'+C_n(\cdot)u'$ for every $u \in H_T^1(H) \cap L_T^2(V)$; evidently, each operator $\mathcal{A}_n$ satisfies Hypothesis 1.2. Thus, for fixed $f \in L_T^2(H)$ let u_n be the solution of $\mathcal{A}_n u_n = f$, according to Step 3. Then, by Theorem 3.1 we see that $\{u_n\}$ is bounded in $H_T^1(H) \cap L_T^2(V)$. Passing to appropriate subsequences, we may suppose that $u_n \rightharpoonup u$ in $H_T^1(H)$ and $u_n \rightharpoonup w$ in $L_T^2(V)$; it follows that $u = w \in H_T^1(H) \cap L_T^2(V)$. Then we can write, using Proposition 2.1,

$$\begin{aligned} &\int_0^T (f(r),\varphi(r))_H\,dr\\ &= \int_0^T \big[\langle A_n(t)u_n(t),\varphi(t)\rangle_{V^*,V} - (B_n(t)u_n'(t),\varphi'(t))_H\\ &+ (C_n(t)u_n'(t),\varphi(t))_H\big]\,dt\\ &= \int_0^T \big[\langle [A_n(t)-A(t)]u_n(t),\varphi(t)\rangle_{V^*,V} - ([B_n(t)-B(t)]u_n'(t),\varphi'(t))_H\\ &+ ([C_n(t)-C(t)]u_n'(t),\varphi(t))_H\big]\,dt\\ &+ \int_0^T \big[\langle A(t)[u_n(t)-u(t)],\varphi(t)\rangle_{V^*,V} - (B(t)[u_n'(t)-u'(t)],\varphi'(t))_H\\ &+ (C(t)[u_n'(t)-u'(t)],\varphi(t))_H\big]\,dt\\ &+ \int_0^T [\langle At)u(t),\varphi(t)\rangle_{V^*,V} - (B(t)u'(t),\varphi'(t))_H + (C(t)u'(t),\varphi(t))_H]\,dt. \end{aligned}$$

As $n \to \infty$, we easily get

$$\begin{aligned} \int_0^T (f(r),\varphi(r))_H\,dr = \int_0^T &\big[\langle At)u(t),\varphi(t)\rangle_{V^*,V} - (B(t)u'(t),\varphi'(t))_H\\ &+ (C(t)u'(t),\varphi(t))_H\big]\,dt, \end{aligned}$$

i.e. $\mathcal{A}u = f$, as desired. Theorem 1.3 is completely proved.

Declaration The authors declare that they do not have any conflict of interests with anyone.

References

[1] C. Baiocchi, Soluzioni ordinarie e generalizzate del problema di Cauchy per equazioni differenziali astratte lineari del secondo ordine in spazi di Hilbert, Ricerche Mat. **16** (1967), 27–95.

[2] C. Baiocchi, Sulle equazioni differenziali astratte lineari del primo e del secondo ordine negli spazi di Hilbert, Ann. Mat. Pura Appl. (4) **76** (1967), 233-304.

[3] S. Campanato, A Cordes type condition for non linear and variational system. Rend. Accad. Naz. Sci. XL Mem. Mat. 107^0 (1989), no. XIII, fasc. 20, 307-321

[4] D. Cassani, A.Tarsia, Periodic solutions to nonlocal MEMS equations, Discrete and Continuous Dynamical Systems - Series S, volume 9, issue 3, pages 631-642 (2016).

[5] E. H. De Brito, Time-periodic weak solutions. Int. J. Math. Math. Sci. 13, No. 1, 145-150 (1990).

[6] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Springer Verlag, Berlin 1977.

[7] L. Herrmann, Periodic solutions of abstract differential equations: the Fourier method. Czechoslovak, Math J. 30 (165)(1980) 177-206.

[8] V. Keyantuo, C. Lizama, Periodic solutions of second order differential equations in Banach spaces, Math. Z. **253** (2006), 489-514.

[9] N. T. Lan, On the periodic solutions of complete second order differential equations on Hilbert spaces, East-West J. Math. **9** (2007), 1-24.

[10] J. L. Lions, Équations différentielles opérationnelles et problèmes aux limites, Springer Verlag, Berlin/Göttingen/Heidelberg 1961.

[11] A. Tarsia, Some topological properties preserved by nearness between operators and applications to P.D.E., Czech. Math. J. **46** (1996) 607-624.

[12] O. Vejvoda, Partial differential equations: time-periodic solutions, Martinus Nijhoff Publisher, The Hague/Boston/London 1982.

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