Asymptotic behavior of the condition number of two-level Toeplitz matrix sequences

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The aim of this paper

Estimation of the Euclidean condition number $\kappa(\cdot)$ of two-level Toeplitz matrices $T_{nm}(f)$ where n and m are large and where the symbol f is assumed to be L^∞ over $I^2 = (-\pi, \pi]^2$ and weakly sectorial.

A function is weakly sectorial if and only if there exists a straight line s passing from the complex zero such that its essential range is all contained in one of the two closed half planes and it is not all contained in s.

$$T_{n,m}(f) = \begin{pmatrix} T_0 & T_{-1} & \cdots & T_{1-n} \\ T_1 & T_0 & \ddots & T_{2-n} \\ \vdots & \ddots & \ddots & \vdots \\ T_{n-1} & T_{n-2} & \cdots & T_0 \end{pmatrix}$$

with

$$T_j = \begin{pmatrix} t_{j,0} & t_{j,-1} & \cdots & t_{j,1-m} \\ t_{j,1} & t_{j,0} & \ddots & t_{j,2-m} \\ \vdots & \ddots & \ddots & \vdots \\ t_{j,m-1} & t_{j,m-2} & \cdots & t_{j,0} \end{pmatrix},$$

$$t_{r,s} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) e^{-\mathrm{i}(rx+sy)} dx dy, \quad \mathrm{i}^2 = -1.$$

Applications of Two-level Toeplitz systems

- Image Restoration
- Image Processing
- Tomography
- Partial Differential Equations
- Integral Equations
- Markov Chains

History

S. Serra Capizzano, "On the extreme eigenvalues of Hermitian (block) Toeplitz matrices", *Linear Algebra Appl.*, **270** (1997), pp. 109–129.

S. Serra Capizzano, "How bad can positive definite Toeplitz matrices be?", *Numer. Funct. Anal. Optim.*, **21,1-2** (2000), pp. 255–261.

P. Tilli, "Universal bounds on the convergence rate of extreme Toeplitz eigenvalues", *Linear Algebra Appl.*, **366** (2003), pp. 403–416.

A. Böttcher and S. Grudsky, "Estimates for the condition numbers of large semidefinite Toeplitz matrices", *Preprint TU Chemnitz-Zwickau*, **97-13**, 1997.

A. Böttcher and S. Grudsky, "On the condition numbers of large semidefinite Toeplitz matrices", *Linear Algebra Appl.*, **279** (1998), pp. 285–301.

Definitions

- $S(Y,\delta) \quad = \quad \cup_{(\bar{x},\bar{y})\in Y}\{(x,y) : \|(x,y)-(\bar{x},\bar{y})\|_\infty < \delta\}.$

- The function $\omega_{(\bar{x},\bar{y})}$ is defined as

$$\frac{1}{\omega_{(\bar{x},\bar{y})}(\nu)} = \inf_{\substack{\{R_k\}_{k\in K} \in \\ Z((\bar{x},\bar{y}),\delta)}} \min_{k\in K} \sup_{\substack{c(t)\subset R_k \\ t\in[0,1]}} \left\{ \begin{array}{c} \operatorname{ess\,inf}\{u(x,y) : \\ (x,y)\notin S(\{(\bar{x},\bar{y})\},1/\nu), \\ (x,y)\in c(t)\} \end{array} \right\}$$

where $Z((\bar{x},\bar{y}),\delta)$ be the collection of all the sets $\{R_k\}_{k\in K}$ such that $\#K < \infty$, R_k circular sector centered on $(\bar{x},\bar{y})$ with positive angle independent of δ with $\cup_{k\in K} R_k = S(\{(\bar{x},\bar{y})\},\delta)$.

Univariate case:

$$\frac{1}{\omega_{\bar{x}}(\nu)} := \operatorname{ess\,inf}\{u(x) : 1/\nu < |x-\bar{x}| < \delta\}$$

- The function $\omega_{\overline{\mathcal{C}}}$ is defined as

$$\omega_{\bar{\mathcal{C}}}(\nu) = \sup_{(\bar{x},\bar{y})\in\overline{\mathcal{C}}} \omega_{(\bar{x},\bar{y})}(\nu).$$

- The order of an isolated zero $(\bar{x},\bar{y})$ of u is defined as

$$\lim_{\nu\to\infty} \frac{\log(\omega_{(\bar{x},\bar{y})}(\nu))}{\log(\nu)}.$$

Examples

Zeros of finite order

$$i) f(x,y) = x^2 + (x+y^2)^4 : \qquad \omega_{(0,0)}(\nu) \sim \nu^2$$

$$ii) f(x,y) = |1-x^2-y^2| + (x^2+|y|)^4 \, e^{-\frac{1}{\sqrt{|1-x^2-y^2|}}} : \qquad \omega_C(\nu) \sim \nu$$

$$iii)\ f(x,y) = \begin{cases} e^{-\frac{1}{\sqrt{|x-1|+|y-1|}}} & x = y \\ (x-1)^2 + (y-x)^4 & \text{elsewhere} \end{cases} : \qquad \omega_{(1,1)}(\nu) \sim \nu^2$$

Zeros of logarithmic type

$$i)\ f(x,y) = \frac{1}{|\log(10^{-1}(|x|+|y|))|} : \qquad \omega_{(0,0)}(\nu) \sim \log(\nu)$$

Zeros of exponential type

$$i)\ f(x,y) = e^{-\frac{1}{x^2+|y|^3}} : \qquad \omega_{(0,0)}(\nu) \sim e^{\nu^2}$$

$$ii)\ f(x,y) = \begin{cases} e^{-\frac{1}{\sqrt{|x-1|+|y-1|}}} & x > 1, y > 1 \\ (x-1)^2 + (y-x)^4 & \text{elsewhere} \end{cases} : \qquad \omega_{(1,1)}(\nu) \sim e^{\sqrt{\nu}}$$

An estimate from above

The given Definitions, which characterize the order of zeros, are very important for studying the asymptotical ill-conditioning of matrices $T_{nm}(f)$ with f weakly sectorial and $n, m \to \infty$.

<u>**Lemma**</u> **1 (Dirichlet)** *Let* $\beta_1, \ldots, \beta_N$ *be real numbers and* $\mu > 0$. *Then there exists a number* $q \in \mathbf{N}$ *such that* $1 \leq q \leq (1/\mu + 1)^N$ *and* $q\beta_j \in \mathbf{Z} + (-\mu, \mu)$ *for all* $j = 1, \ldots, N$

<u>**Lemma**</u> **2 (Brown and Halmos)** *Let* $f \in L^\infty$ *and suppose that* $\mathrm{Re} f \geq d > 0$. *Then* $\|T_{nm}^{-1}(f)\| < 2/d$.

Theorem 1 *Let $f \in L^\infty$ be a 2-variate, 2π-periodic function. Suppose that $u := \mathrm{Re} f \geq 0$, has a finite number of isolated zeros $(x_j, y_j) \in I^2 = (-\pi, \pi]^2$, $j = 1, \ldots, k$, or has infinitely many zeros which form a finite set of disjoint curves $\mathcal{C}_j := \{(x, y) : \ C_j(x, y) = 0\}$, $j = 1, \ldots, q$, with $C_j(x, y)$ regular enough and 2π-periodic, and suppose that u is positive elsewhere. Define $\omega_i(n) = \omega_{(x_i, y_i)}(n)$, $i = 1, \ldots, k$, put*

$$\omega(n) := \max_{\substack{1 \leq i \leq k \\ 1 \leq j \leq q}} \{\omega_i(n), \omega_{\mathcal{C}_j}(n)\},$$

and let $v = \mathrm{Im} f$. If for every $j = 1, \ldots, q$ we have

$$\max\left\{\sup_{r \in \mathcal{H}} \#(\mathcal{C}_j \cap r), \sup_{r \in \mathcal{V}} \#(\mathcal{C}_j \cap r)\right\} < \infty,$$

with $\mathcal{H}$ denoting the set of all the horizontal lines and with $\mathcal{V}$ denoting the set of all the vertical lines, then

$$\kappa(T_{nm}(f)) \leq 12 \|f\|_\infty (\|v\|_\infty + 1) \omega(c(n + m)),$$

for all sufficiently large n, m, where $T_{nm}(f)$ is the $(nm \times nm)$ two-level block Toeplitz matrix generating by the function f and $c > 1$ is an absolute constant.

Sketch of the proof

We consider the discretization of the square $[-\pi, \pi]^2$ along the x axis and define the stripes $X_i, i = 1, \ldots, N$, along the y direction. We consider all the isolated zeros and just one point for every piece of the curve, to form a set of k_i points $p_i = (x_{i_j}, y_{i_j}), j = 1, \ldots, k_i$.

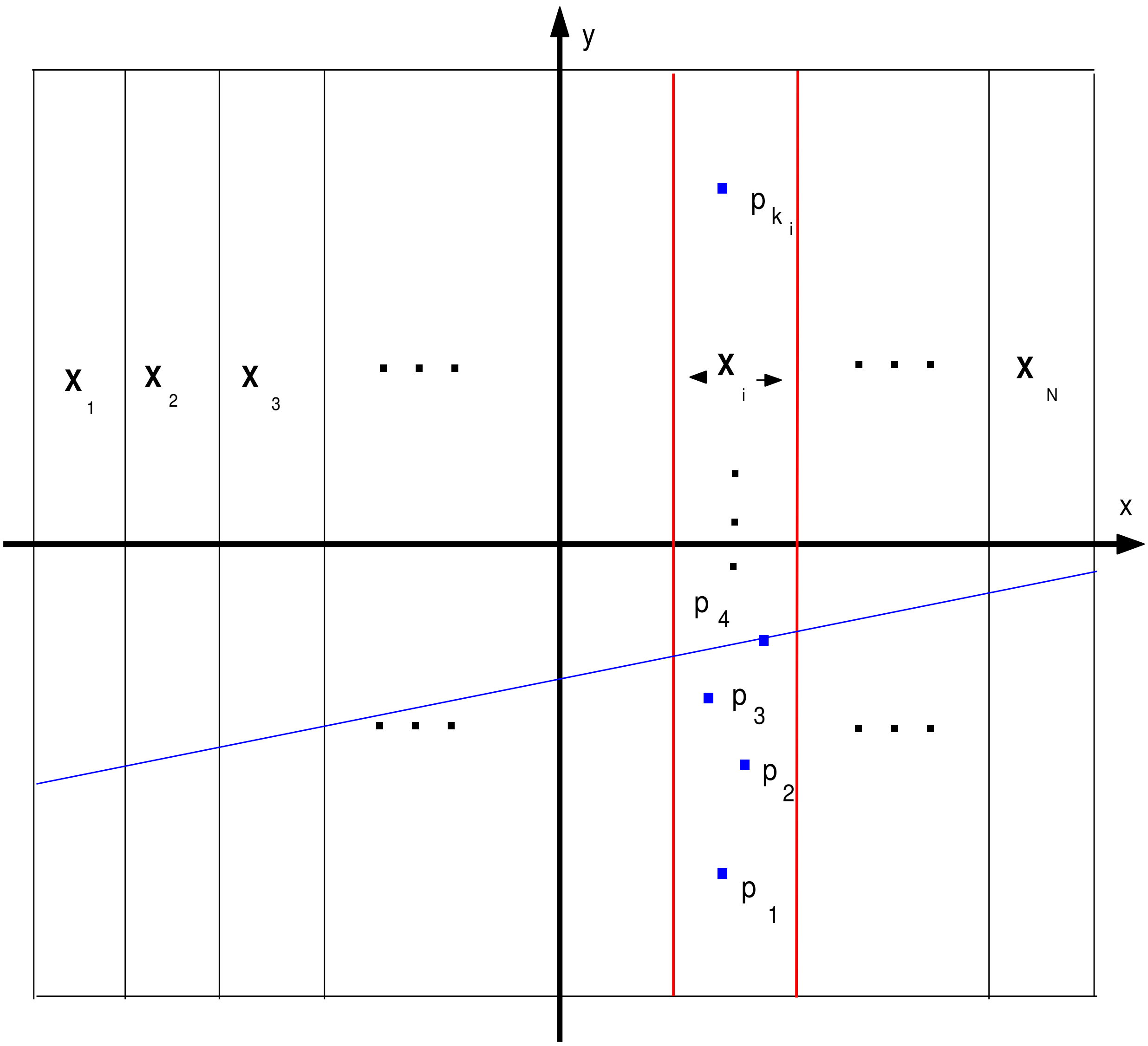

Figure 1: Discretization of the square $[-\pi, \pi]^2$

Lemma 1 (Dirichlet) with $\mu = 1/12$, $\beta_j = (nx_{i_j}+my_{i_j})/(2\pi)$, $j = 1, 2, \ldots, k_i$, for each strip X_i, gives:

$1 \le q_i \le 13^{k_i}$, $q_i(nx_{i_j}+my_{i_j}) \in 2\pi\mathbf{Z}+\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$, $j = 1, 2, \cdots, k_i$.

By similar manipulations as in the paper of Böttcher and Grudsky we get

$$\cos(q_i(nx+my)) > \frac{1}{2},$$
$$(x,y) \in X_i, \quad \|(x,y)-(x_{i_j}, y_{i_j})\|_\infty < \frac{\pi}{6q_i(n+m)}.$$

We consider the functions

$$g_{nm}(x,y) = \begin{cases} \cos(q_1(nx+my)), & (x,y) \in X_1, \\ \cos(q_2(nx+my)), & (x,y) \in X_2, \\ \vdots & \\ \cos(q_N(nx+my)), & (x,y) \in X_N. \end{cases},$$

$$b_{nm}(x,y) = f(x,y) + \mathrm{i}Mg_{nm}(x,y), \qquad M = 2(\|v\|_\infty + 1).$$

It is proved that the Fourier coefficients c_{l_1,l_2} of $g_{nm}(x,y)$, $|l_1| \le n-1$ and $|l_2| \le m-1$, are all zero. Therefore

$$T_{nm}(b_{nm}) = T_{nm}(f). \tag{1}$$

Then, by following analogous topological analysis as Böttcher and Grudsky have done and by using Lemma [2] (Brown and Halmos) we get

$$\begin{aligned} \|T_{nm}^{-1}(f)\| &< 12(\|v\|_\infty + 1)\omega\left(\tfrac{6q(n+m)}{\pi}\right) \\ &= 12(\|v\|_\infty + 1)\omega(c(n+m)). \end{aligned}$$

An estimate from below

<u>Lemma</u> 3 (Böttcher and Grudsky) *For $j, m \in \mathbf{N}$, let*

$$P_m^j(\theta) = (1 + e^{\mathrm{i}\theta} + \cdots + e^{\mathrm{i}m\theta})^j = e^{\mathrm{i}j\theta/2}\left(\frac{\sin\frac{m+1}{2}\theta}{\sin\frac{\theta}{2}}\right)^j.$$

Then

$$\|P_m^j\|_2^2 > \frac{16}{9\pi}\frac{1}{\sqrt{j}}(m+1)^{2j-1}.$$

<u>Theorem</u> 2 *Let $\alpha_1, \alpha_2, \beta_1, \beta_2$, be positive numbers, let $(x_0, y_0) \in I^2$, and suppose $f \in L^\infty$ be a 2-variate, 2π-periodic function.*

(a) If $f(x, y) = \mathcal{O}(|x - x_0|^{\alpha_1} + |y - y_0|^{\alpha_2})$ as $(x, y) \to (x_0, y_0)$ then there is a constant $C \in (0, \infty)$ such that

$$\kappa(T_{nm}(f)) \geq C\frac{n^{\alpha_1}m^{\alpha_2}}{n^{\alpha_1} + m^{\alpha_2}} \text{ for all } n \geq 1, m \geq 1.$$

(b) If $f(x, y) = \mathcal{O}(1/|\log|(x - x_0)/\pi||^{\alpha_1} + 1/|\log|(y - y_0)/\pi||^{\alpha_2})$ as $(x, y) \to (x_0, y_0)$ then there is a constant $C \in (0, \infty)$ such that

$$\kappa(T_{nm}(f)) \geq C\frac{(\log n)^{\alpha_1}(\log m)^{\alpha_2}}{(\log n)^{\alpha_1} + (\log m)^{\alpha_2}} \text{ for all } n \geq 1, m \geq 1.$$

(c) If $f(x, y) = \mathcal{O}(e^{-\beta_1|x - x_0|^{-\alpha_1}} + e^{-\beta_2|y - y_0|^{-\alpha_2}})$ as $(x, y) \to (x_0, y_0)$ then

$$\lim_{n,m\to\infty} n^{-k_1}m^{-k_2}\kappa(T_{nm}(f)) = \infty$$

for every $k_1, k_2 > 0$.

Sketch of the proof

Let that $|f(x,y)| \leq K(|x|^{\alpha_1} + |y|^{\alpha_2}),\ |x|,|y| < \delta$ and fix $n, m > 1/\delta$. We consider now the trigonometric polynomials

$$P^{j_1 j_2}_{m_1 m_2}(x,y) = P^{j_1}_{m_1}(x) P^{j_2}_{m_2}(y) = e^{\mathrm{i}\frac{1}{2}(j_1 x + j_2 y)} \left(\frac{\sin \frac{m_1+1}{2} x}{\sin \frac{x}{2}} \right)^{j_1} \left(\frac{\sin \frac{m_2+1}{2} y}{\sin \frac{y}{2}} \right)^{j_2}.$$

We follow now the same technique indicated by Böttcher and Grudsky. Consider

$$\begin{array}{rcl} 4\pi^2 \|f P^{j_1 j_2}_{m_1 m_2}\|_2^2 & = & \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Phi(x,y) dx dy \\ & = & \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} \int_{-\frac{1}{m_1}}^{\frac{1}{m_1}} \Phi dx dy + \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} \int_{\frac{1}{m_1} < |x| < \delta} \Phi dx dy \\ & + & \int_{\frac{1}{m_2} < |y| < \delta} \int_{-\frac{1}{m_1}}^{\frac{1}{m_1}} \Phi dx dy + \int_{\frac{1}{m_2} < |y| < \delta} \int_{\frac{1}{m_1} < |x| < \delta} \Phi dx dy \\ & + & \int_{-\frac{1}{m_2}}^{\frac{1}{m_2}} \int_{\delta < |x| < \pi} \Phi dx dy + \int_{\delta < |y| < \pi} \int_{-\frac{1}{m_1}}^{\frac{1}{m_1}} \Phi dx dy \\ & + & \int_{\frac{1}{m_2} < |y| < \delta} \int_{\delta < |x| < \pi} \Phi dx dy + \int_{\delta < |y| < \pi} \int_{\frac{1}{m_2} < |x| < \delta} \Phi dx dy \\ & + & \int_{\delta < |y| < \pi} \int_{\delta < |x| < \pi} \Phi dx dy. \end{array}$$

Then, we find upper bounds for the above integrals to obtain

$$\|f P^{j_1 j_2}_{m_1 m_2}\|_2^2 \leq C_1 \left(\frac{1}{m_1^{\alpha_1}} + \frac{1}{m_2^{\alpha_2}} \right)^2 \|P^{j_1 j_2}_{m_1 m_2}\|_2^2, \quad C_1 < \infty.$$

From this, by taking $n = m_1 j_1 + k_1$ and $m = m_2 j_2 + k_2$, we infer

$$\begin{array}{rcl} \|T_{nm}(f) P^{j_1 j_2}_{m_1 m_2}\|_2^2 & \leq & \|f P^{j_1 j_2}_{m_1 m_2}\|_2^2 \leq C_2 \left(\frac{1}{n^{\alpha_1}} + \frac{1}{m^{\alpha_2}} \right)^2 \|P^{j_1 j_2}_{m_1 m_2}\|_2^2 \\ & \leq & C_2 \left(\frac{1}{n^{\alpha_1}} + \frac{1}{m^{\alpha_2}} \right)^2 \|T^{-1}_{nm}(f)\|_2^2 \|T_{nm}(f) P^{j_1 j_2}_{m_1 m_2}\|_2^2, \end{array}$$

$C_2 < \infty$, which implies that

$$\|T^{-1}_{nm}(f)\|_2 \geq C \frac{n^{\alpha_1} m^{\alpha_2}}{n^{\alpha_1} + m^{\alpha_2}}, \quad 0 < C < \infty.$$

Combination of the above two theorems

We suppose that

$$n \sim m \sim \nu,$$

since it is the only case having practical importance.
Let that the hypotheses of Theorem 1 (where the roots would be roots of the whole function f and not only of u) are holding and, for every root of f, the hypotheses of part (a) of Theorem 2 are also holding.
Then we can obtain results of the order of the condition number:

From (a) of Theorem 2 we get that

$$\kappa(T_{nm}(f)) \geq C_1 \left(\frac{n^{\alpha_1} m^{\alpha_2}}{n^{\alpha_1} + m^{\alpha_2}} \right) \sim \nu^{\alpha},$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$, which corresponds to the maximum value of α's over all the roots.

On the other hand, by Theorem 1 we have

$$\|T_{nm}^{-1}(f)\| < 12(\|v\|_\infty + 1)\omega(c(n+m)) = C_2\omega(\nu) = \mathcal{O}(\nu^{\alpha}).$$

Therefore

$$\kappa(T_{nm}(f)) \sim \nu^{\alpha}.$$

Examples

1.

$$f(x,y)=x^2+(x+y^2)^4: \qquad \omega_{(0,0)}(\nu)\sim\nu^2$$

$$|f(x,y)|\le K(|x|^2+|y|^8),\ |x|,|y|<\delta: \qquad \frac{n^2m^8}{n^2+m^8}\sim\nu^2$$

$$\kappa(T_{nm}(f))\sim\nu^2$$

2.

$$f(x,y)=|1-x^2-y^2|+(x^2+|y|)^4\ e^{-\frac{1}{\sqrt{|1-x^2-y^2|}}}: \qquad \omega_C(\nu)\sim\nu$$

$$\begin{array}{c}|f(x,y)|\le K(|x-x_i|+|y-y_i|),\ |x-x_i|,|y-y_i|<\delta,\\ (x_i,y_i)\in\{(x,y):1-x^2-y^2=0\}\end{array} : \frac{nm}{n+m}\sim\nu$$

$$\kappa(T_{nm}(f))\sim\nu$$

3.

$$f(x,y)=\frac{1}{|\log(10^{-1}(|x|+|y|))|}: \qquad \omega_{(0,0)}(\nu)\sim\log(\nu)$$

$$\begin{array}{c}|f(x,y)|\le K(\frac{1}{|\log|x/\pi||}+\frac{1}{|\log|y/\pi||}),\\ |x|,|y|<\delta\end{array} : \ \frac{\log(n)\log(m)}{\log(n)+\log(m)}\sim\log(\nu)$$

$$\kappa(T_{nm}(f))\sim\log(\nu)$$

4.

$$f(x,y) = \begin{cases} e^{-\frac{1}{\sqrt{|x-1|+|y-1|}}} & x = y \\ (x-1)^2 + (y-1)^4 & \text{elsewhere} \end{cases} \quad : \qquad \omega_{(1,1)}(\nu) \sim \nu^2$$

$$|f(x,y)| \leq K(|x-1|^2 + |y-1|^4),\ |x-1|, |y-1| < \delta : \ \frac{n^2 m^4}{n^2 + m^4} \sim \nu^2$$

$$\kappa(T_{nm}(f)) \sim \nu^2$$

5.

$$f(x,y) = \begin{cases} e^{-\frac{1}{\sqrt{|x-1|+|y-1|}}} & x > 1, y > 1 \\ (x-1)^2 + (y-1)^4 & \text{elsewhere} \end{cases} \quad : \quad \omega_{(1,1)}(\nu) \sim e^{\sqrt{\nu}}$$

$$|f(x,y)| \leq K(|x-1|^2 + |y-1|^4),\ |x-1|, |y-1| < \delta : \ \frac{n^2 m^4}{n^2 + m^4} \sim \nu^2$$

$$C_1 \nu^2 \leq \kappa(T_{nm}(f)) \leq C_2 e^{\sqrt{\nu}}$$

Table 1: The minimum eigenvalue of $T_{nm}(f)$ with $f(x,y) = (2 - 2\cos(x)) + (2 - 2\cos(y))^2$

n	m	N	$\lambda_{m^3 m} \equiv \lambda_{\min}(T_{m^3 m}(f))$	$\log(\lambda_{m^3 m}/\lambda_{8m^3 2m})$
4^3	4	256	0.39679	-
8^3	8	4096	0.05057	2.97
16^3	16	65536	0.0047839	3.4
32^3	32	1048576	0.00028	3.83

Table 2: The minimum eigenvalue of $T_{nm}(f \circ U)$ with $(f \circ U)(x,y) = (2 - 2\cos(x-y)) + (2 - 2\cos(x+y))^2$

n	m	N	$\lambda_{m^3 m} \equiv \lambda_{\min}(T_{m^3 m}(f))$	$\log(\lambda_{m^3 m}/\lambda_{8m^3 2m})$
4^3	4	256	0.78799	-
8^3	8	4096	0.17679	2.15
16^3	16	65536	0.04082	2.11
32^3	32	1048576	0.01121	1.87

A negative results for the Hermitian case

We consider the functions

$$f(x,y) = (2 - 2\cos(x)) + (2 - 2\cos(y))^2,$$

$$f(x-y, x+y) = f \circ U(x,y), \quad U = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

We suppose that $n = m^3$. From Tables 1 and 2 we obtain

$$\kappa(T_{m^3 m}(f)) \sim m^4, \qquad \kappa(T_{m^3 m}(f \circ U)) \sim m^2,$$

respectively

A positive results for the Hermitian case

<u>Theorem</u> 3 *Let $f \in L^\infty$ be a 2-variate, 2π-periodic, weakly sectorial function of additively separable type i.e. $f(x,y) = g(x)+h(y)$ for g and h being L^∞ and 2π-periodic. If $g, h \geq 0$ then*

$$\begin{aligned} \max\{\lambda_{\min}(T_n(g)), \lambda_{\min}(T_m(h))\} &\leq \lambda_{\min}(T_{nm}(f)) \\ &\leq \lambda_{\min}(T_n(g)) + \lambda_{\min}(T_m(h)). \end{aligned}$$

Therefore, defining

$$[\omega_t(\nu)]^{-1} := \max_{\bar{x} \text{ zero of } t(x)} \operatorname{ess\,inf}\{t(x) : 1/\nu < |x-\bar{x}| < \delta\}$$

with $t(x)$ nonnegative univariate function with a finite number of zeros, there exist positive constants C_1, C_2, c_1, and c_2 such that

$$C_1 \frac{\omega_g(c_1 n)\omega_h(c_1 m)}{\omega_g(c_1 n) + \omega_h(c_1 m)} \leq \kappa(T_{nm}(f)) \leq C_2 \min\{\omega_g(c_2 n), \omega_h(c_2 m)\}$$

with

$$\frac{1}{2}\min\{\omega_g(cn), \omega_h(cm)\} \leq \frac{\omega_g(cn)\omega_h(cm)}{\omega_g(cn)+\omega_h(cm)} \leq \min\{\omega_g(cn), \omega_h(cm)\}.$$