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Deforming Triangulations of Hyperbolic Manifolds with Geodesic Boundary

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Introduction

Since Thurston's seminal work in the late seventies [Thu79], hyperbolic geometry has become one of the main research areas in 3-dimensional topology and geometry. Several new ideas have been developed in order to better understand how to decide whether a manifold can support a geometric structure, and to describe such a structure, when it exists. In this context, many efforts have been devoted to investigating the shape of the deformation space of hyperbolic structures on a given 3-manifold M. Among the old and new techniques employed to this end, the analysis of the space of representations of the fundamental group of M into the group of isometries of hyperbolic 3-space, and the description of the geodesic triangulations of M itself, have proved particularly powerful.

In this thesis we develop the triangulation approach to the study of deformations for hyperbolic manifolds having *geodesic boundary*, and we briefly outline the representation approach.

Deformation of hyperbolic structures The space of complete infinite-volume hyperbolic structures supported by a fixed 3-manifold, when non-empty, is quite big, and its description has provided, among other things, a key ingredient of the proof of Thurston's hyperbolization Theorem for Haken manifolds [Thu82, Ota96, Ota98, Kap01]. On the contrary, Mostow-Prasad's rigidity Theorem implies that the space (of homotopy classes) of complete finite-volume structures supported by a fixed 3-manifold reduces either to the empty set or to a single point. This means that in the finite-volume case non-trivial deformations of the complete structure can give rise only to incomplete metrics. It turns out that such deformations are closely related to the geometry of manifolds which can be obtained from the original one by means of natural topological constructions, as we are now going to explain.

Dehn filling Let M be a complete orientable finite-volume non-compact hyperbolic 3-manifold without boundary (in what follows we will summarize all this information just by saying that M is *cusped*). Then M admits a natural compactification $\overline{M}$ obtained by adding to M some boundary tori. We can choose some of these tori and glue a solid torus to each of them, thus obtaining a 3-manifold N. This operation is called *Dehn filling*, and we say that N is obtained by Dehn filling M (or simply that N is a Dehn filling of M). We point out that the manifold M can be filled in infinitely many different ways, according to the infinitely many possibilities for the choice of the isotopy class of the gluing homeomorphism between the chosen toric components of $\partial\overline{M}$ and the boundary of the added solid tori.

A crucial fact is that the metric completions of many small deformations of the complete metric of M actually define complete hyperbolic structures on manifolds obtained by Dehn filling M. This phenomenon is at the heart of the proof of Thurston's hyperbolic Dehn filling Theorem [Thu79], which states that "almost all" the Dehn fillings of a cusped 3-manifold also support a complete finite-volume hyperbolic metric. Thurston himself described in his notes two different strategies for proving this result: the first one makes use of representations (see [Hod86] or [BP01] for detailed accounts on this approach), the second one is based on the technology of geodesic ideal triangulations [NZ85, BP92, PP00].

As mentioned before, in this thesis we describe how the deformation theory for cusped 3-manifolds extends to cusped 3-manifolds with boundary (*i.e.* to complete finite-volume non-compact hyperbolic 3-manifolds with compact geodesic boundary), mainly focusing our attention on the approach via geometric triangulations. In the case of non-empty geodesic boundary, one has to consider partially truncated tetrahedra instead of ideal ones, computations get more complicated, but basically the whole theory extends. In particular, we decribe here how partially truncated triangulations can be employed in order to construct examples of cusped 3-manifolds with boundary and to study their Dehn fillings.

The approach via triangulations There are at least two facts which support the idea that geometric triangulations are effective instruments for studying complete hyperbolic 3-manifolds and their deformations. First of all the triangulation approach to deformations of a hyperbolic manifold often allows an explicit description of many geometric invariants of the manifold itself and of its Dehn fillings. In Chapter 3 we provide explicit examples of how this principle applies in practice. Moreover in the empty (resp. non-empty) boundary case, geodesic triangulations have proved very useful for constructing and studying large classes of cusped (resp. bounded) hyperbolic manifolds. The idea of constructing hyperbolic structures on manifolds by suitably gluing to each other geodesic polyhedra has been systematically exploited in the cusped case by the software SnapPea [Sna], and has lead to a census of the simplest (in a suitable sense) 6075 cusped 3-manifolds [CHW99]. In the bounded case, the same strategy has been implemented by a software developed by Bruno Martelli, Carlo Petronio and myself, and has provided a census of the simplest 5192 complete finite-volume hyperbolic 3-manifolds with compact geodesic boundary [FMP03b].

Partially flat triangulations Before going into a more detailed description of the results proved in this thesis, it is maybe worth mentioning at least one drawback of the triangulation approach to deformations. The fact is that when constructing the general theory of deformations of cusped (or cusped and bounded) 3-manifolds via triangulations, some troubles arise due to the still unsettled question whether these manifolds always admit genuine geodesic triangulations. Epstein and Penner [EP88] and Kojima [Koj90, Koj92] have shown that decompositions into convex geodesic *polyhedra* always exist, but in general, if one subdivides polyhedra into tetrahedra, some flat tetrahedra may be required to reconcile the subdivisions of adjacent polyhedra. Therefore cusped and bounded manifolds admit degenerate geodesic triangulations, in which some of the tetrahedra are flattened out into (ideal or truncated) quadrilaterals. However, this seems to be more a technical difficulty than a substantial obstruction: partially flat triangulations of cusped manifolds are studied in [PW00], and their deformations can be dealt with yielding a suitably modified version of the proof of Thurston's hyperbolic Dehn filling Theorem (see [PP00]). Moreover, we expect the strategy described in [PP00] to easily translate into the context of degenerated partially truncated triangulations of cusped manifolds with boundary. This would straighten the idea that geodesic triangulations can be successfully employed to analyze small deformations of complete finite-volume hyperbolic 3-manifolds in full generality.

Below is a brief description of the contents of each chapter.

In **Chapter 1** we give a quick review of the deformation theory for cusped hyperbolic 3-manifolds, also sketching the proof of Thurston's hyperbolic Dehn filling Theorem via representations. We introduce the technology of (X, G)-structures and holonomy representations, also recalling some basic results in deformation theory of (X, G)-structures on compact manifolds. In order to better understand the hyperbolic case, we describe in detail the character variety associated to the space of $\mathrm{SL}(2, \mathbb{C})$-representations of the fundamental group of a hyperbolic 3-manifold. Most results are essentially taken from Thurston's notes [Thu79], notes on notes of Thurston [CEG87] and Culler and Shalen's paper [CS83], and should be considered classical by now. Our main effort here is aimed at giving a complete proof of the fact that the Teichmüller space $\mathcal{T}(M)$ of a hyperbolic 3-manifold M is parameterized by the conjugacy classes of holonomy representations, at least in a neighbourhood of the complete structure (see Sections 1.3, 1.4 for a precise definition of $\mathcal{T}(M)$):

Theorem. *Let M be a complete finite-volume hyperbolic 3-manifold with (possibly empty) geodesic boundary. Then the map which associates to any hyperbolic structure on M the conjugacy class of the corresponding holonomy representation induces a homeomorphism between an open neighbourhood of the complete structure on M and an open set of conjugacy classes of representations of the fundamental group of M into* $\mathrm{PSL}(2, \mathbb{C})$.

Chapter 2 is devoted to the description of the general theory of geometric triangulations of hyperbolic 3-manifolds with geodesic boundary. Our starting point is the idea of turning the construction of the hyperbolic structure on a manifold into an algebraic problem. In the setting of cusped manifolds one employs ideal tetrahedra, which are parameterized by complex numbers, and tries to solve hyperbolicity equations. We extend this scheme to the bounded case, with ideal tetrahedra replaced by truncated ones. The conditions under which a gluing of truncated tetrahedra defines a non-singular hyperbolic metric are encoded by the *consistency equations*, while *completeness equations* translate the conditions ensuring that such a metric is actually complete.

Properties of the deformation space The set of solutions of consistency equations naturally provides a nice deformation space for finite-volume hyperbolic structures with geodesic boundary on a fixed 3-manifold. Building on classical results in the cohomology theory of representations, we prove the following:

Theorem. *Let M be a complete hyperbolic 3-manifold with compact geodesic boundary and k cusps, let $\mathcal{T}$ be a geometric triangulation of M and denote by $\Omega(\mathcal{T})$ the set of solutions of the consistency equations for $\mathcal{T}$. Then $\Omega(\mathcal{T}) \subset \mathbb{R}^n$ is a smooth manifold of real dimension $2k$ near the point $x_0 \in \Omega(\mathcal{T})$ representing the complete structure on M. Moreover, local coordinates for $\Omega(\mathcal{T})$ near x_0 can be explicitly expressed in terms of the dihedral angles of the tetrahedra of $\mathcal{T}$.*

This theorem allows us to give a new proof of Thurston's hyperbolic Dehn filling Theorem which applies to all the hyperbolic manifolds with geodesic boundary which admit a genuine geometric triangulation. We are quite confident that our proof of Thurston's hyperbolic Dehn filling Theorem could be adapted along the strategy described in [PP00] in order to deal also with partially flat triangulations. This would give a complete and self-contained proof of Thurston's Theorem via deformation theory of geometric triangulations. However, all the examples constructed in this thesis admit genuine geometric triangulations, so we do not develop the study of partially flat triangulations here.

The Kojima decomposition We have already mentioned that just as for cusped 3-manifolds with empty boundary, which admit the canonical Epstein-Penner decomposition, it turns out that also for bounded manifolds a canonical decomposition exists which is due to Kojima. In Sections 2.6 and 2.7 we describe the main properties of the Kojima decomposition, and we explain how the moduli of a given geometric triangulation can be used to determine if such a triangulation is Kojima's canonical one.

In the **last chapter** we use the theory developed in Chapter 2 in order to describe certain classes $\mathcal{M}_{g,k}$ of hyperbolic 3-manifolds with geodesic boundary. Such classes were first introduced in [FMP03b], where the non-hyperbolic Dehn fillings of each M in $\mathcal{M}_{g,k}$ were completely classified in order to answer a question posed by Wu in [Wu98].

Similar surgeries In this thesis we deepen our analysis of the geometric properties of elements in $\mathcal{M}_{g,k}$ and of their Dehn fillings. Building on results from Chapter 2, we describe as explicitly as possible how small deformations of the complete hyperbolic structure defined on a manifold $M \in \mathcal{M}_{g,k}$ are induced by deformations of the shape of the geodesic partially truncated tetrahedra giving the Kojima decomposition of M. This approach allow us to compute (or to estimate) several invariants of the manifolds obtained by filling elements in $\mathcal{M}_{g,k}$ along "sufficiently complicated" curves. We deduce the following:

Theorem. *There exists an infinite family of complete non-compact hyperbolic* 3-*manifolds with geodesic boundary which admit non-homeomorphic hyperbolic Dehn fillings sharing the same volume, homology, cusp volume, cusp shape, Heegaard genus, complex length of the shortest geodesic, length of the shortest return path and Turaev-Viro invariants.*

Hyperbolic manifolds which share all the invariants mentioned in the statement of the last theorem are called *geometrically similar*, and were first studied in [HMW92] (the definition of geometric similarity introduced there is actually a bit different from ours, and regards cusped manifolds without geodesic boundary). In contrast with the examples provided in [HMW92], our geometrically similar Dehn fillings are typically non-commensurable with each other (however, examples are also provided of elements of $\mathcal{M}_{g,k}$ admitting non-homeomorphic geometrically similar commensurable Dehn fillings).

Isolation of cusps Neumann and Reid introduced in [NR93] the notion of *geometric isolation* for cusps in a hyperbolic 3-manifold: roughly speaking, if N is a complete finite-volume hyperbolic manifold with (possibly empty) geodesic boundary and cusps $C_1, \ldots, C_h, C_{h+1}, \ldots, C_k$, then $C_1, \ldots, C_h$ are geometrically isolated from $C_{h+1}, \ldots, C_k$ if any small deformation of the hyperbolic structure on N induced by Dehn filling $C_{h+1}, \ldots, C_k$ while keeping $C_1, \ldots, C_h$ complete does not affect the Euclidean structure at $C_1, \ldots, C_h$. Calegary described in [Cal01] several different ways of constructing manifolds with isolated cusps, also providing explanations for all the previously known examples of isolation phenomena. Here we prove the following:

Theorem. *Any set of cusps of any manifold $M \in \mathcal{M}_{g,k}$ is geometrically isolated from its complementary set of cusps.*

Apparently, isolation of cusps in our examples arises for different reasons from those described in [Cal01].

Non-isolation of the boundary The natural question if the geodesic boundary too of an element in $\mathcal{M}_{g,k}$ is isolated from the cusps is also answered. Examples of isolation of geodesic boundary from cusps in hyperbolic 3-manifolds were first provided in [NR93, Fuj93]. On the other hand, non-isolation phenomena were described in [Fuj92, FK97].

Theorem. *Any element in $\mathcal{M}_{g,k}$ admits an infinite family of hyperbolic Dehn fillings inducing non-trivial deformations of the hyperbolic structure on the geodesic boundary.*

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Chapter 1

Deforming hyperbolic 3-manifolds

In this chapter we give a quick review of the representation approach to the deformation theory of cusped hyperbolic 3-manifolds. We introduce in Section 1.1 the technology of (X, G)*-structures* and *holonomy representations*, also recalling some basic results about the local structure of the deformation space of (X, G)-structures on compact manifolds. In Sections 1.2 and 1.3 we concentrate our attention on the hyperbolic case, describing in detail the character variety associated to the space of $\mathrm{SL}(2, \mathbb{C})$-representations of the fundamental group of a hyperbolic 3-manifold. Most of the results presented in these sections date back to the beginning of the revolution in 3-dimensional geometry and topology inspired by Thurston in the late seventies. The main results of this chapter are Theorems 1.3.5 and 1.4.11, where we give a formal description of the relation between deformation of finite-volume hyperbolic structures on non-compact 3-manifolds and deformation of the corresponding holonomy representations.

The last section is devoted to an extension of Thurston's hyperbolic Dehn filling Theorem to the case of non-empty geodesic boundary. To this aim we introduce a suitably defined character variety which will be used also in the following chapters, and we describe how this variety parameterizes a small neighbourhood of the complete structure in the Teichmüller space of (incomplete) finite-volume hyperbolic structures with geodesic boundary supported by a fixed 3-manifold.

1.1 (X, G)-structures and holonomy

We begin by recalling the well-known definition of (X, G)-structure on a differentiable manifold.

(X, G)-atlases Let X be a real analytic manifold and G be a Lie group acting on X faithfully and analytically. Let N be a smooth manifold (possibly with boundary) having the same dimension as X.

Definition 1.1.1. An (X, G)*-atlas* for N is a collection of charts $\{\phi_\lambda : U_\lambda \to X\}_{\lambda\in\Lambda}$ satisfying the following conditions:

1. $\{U_\lambda\}_{\lambda\in\Lambda}$ is an open covering of N;
2. Each ϕ_λ is a smooth embedding in X (so ϕ_λ is a local diffeomorphism if $U_\lambda \cap \partial N = \emptyset$);
3. For each $x \in U_\lambda \cap U_\mu$, there are a neighbourhood W of x in $U_\lambda \cap U_\mu$ and an element $g \in G$, such that $\phi_\lambda|_W = g \circ \phi_\mu|_W$.

It is easily seen that an (X,G)-atlas defines a unique maximal (X,G)-atlas. An (X,G)-*structure* on N is a maximal (X,G)-atlas on N. A smooth map $f : M \to N$ between (X,G)-manifolds is an (X,G)-map if for any $x \in M$ the following holds:

- The maximal atlases $\{U_\nu,\phi_\nu\}_{\nu\in\Lambda}$, $\{U'_\mu,\phi'_\mu\}_{\mu\in\Lambda'}$, defining respectively the structures on M and N contain charts $\phi_\nu : U_\nu \to X$ and $\phi'_\mu : U'_\mu \to X$ with $x \in U_\nu \subset M$, $f(U_\nu) = U'_\mu \subset N$ and $\phi'_\mu \circ f|_{U_\nu} = g \circ \phi_\nu$ for some $g \in G$.

When X is a complete Riemannian manifold and G is a group of isometries of X, we say that N has a Riemannian (X,G)-structure. Under these circumstances, N has an induced Riemannian metric, and an (X,G)-map is a local isometry.

Developing map and holonomy Let N be a manifold with a fixed (X,G)-structure, let $\pi : \widetilde{N} \to N$ be a universal cover, and endow $\widetilde{N}$ with the unique (X,G)-structure which makes π an (X,G)-map. Let also $\widetilde{x}$ be a point in $\widetilde{N}$ and $\widetilde{\phi}$ be a chart at $\widetilde{x}$. Since $\widetilde{N}$ is simply connected, this chart analytically extends to a unique map $D : \widetilde{N} \to X$, which is called a *developing map* for the (X,G)-structure on N. Note that choosing a different germ of a chart at a possibly different point in $\widetilde{N}$ would only replace D with a new developing map $g \circ D$ for some $g \in G$.

Let now $\mathrm{Deck}(\widetilde{N})$ denote the group of covering transformations of $\pi : \widetilde{N} \to N$, and recall that there exists an isomorphism $\pi_1(N) \cong \mathrm{Deck}(\widetilde{N})$, which is canonical up to conjugacy. Let M be an (X,G)-structure on N and let $D : \widetilde{N} \to X$ be a fixed developing map for M. If γ is an element of $\mathrm{Deck}(\widetilde{N})$, then $D \circ \gamma$ is also a developing map for M, so there exists a unique element $H_D(\gamma) \in G$ such that $D \circ \gamma = H_D(\gamma) \circ D$. The map $H_D : \pi_1(N) \cong \mathrm{Deck}(\widetilde{N}) \to G$ just defined is a group homomorphism, called the *holonomy* of M (with respect to the fixed identification $\pi_1(N) \cong \mathrm{Deck}(\widetilde{N})$ and the developing map D). If D' is any other developing map for M, then there exists $g \in G$ with $D' = g \circ D$, so $H_{D'}(\gamma) = gH_D(\gamma)g^{-1}$ for any $\gamma \in \pi_1(N)$. This implies that a conjugacy class H_M of homomorphisms of $\pi_1(N)$ into G is defined which depends only on M. With a slight abuse, this class is also called the *holonomy* of M.

Definition 1.1.2. Let M be an (X,G)-structure on N. Then we say that M is *complete* (as an (X,G)-structure) if some (and then any) developing map $D : \widetilde{N} \to X$ for M is a covering projection.

A proof of the following results about (X,G)-structures can be found in [Rat94].

Theorem 1.1.3 (Hopf-Rinow). *If X is a complete Riemannian manifold, G is a group of isometries of X and $\partial N = \emptyset$, then an (X,G)-structure on N is complete if and only if it induces a complete metric on N.*

Proposition 1.1.4. *Suppose X is simply connected, let M be a complete (X,G)-structure on N, let $D : \widetilde{N} \to X$ be a developing map for M and $H_D : \pi_1(N) \to G$ be the induced holonomy representation. Then D gives an (X,G)-equivalence between the universal covering of M and X, the holonomy representation H_D is injective and $H_D(\pi_1(N))$ acts freely and properly discontinuously on X. This gives in turn an (X,G)-equivalence between M and $X/H_D(\pi_1(N))$.*

Definition 1.1.5. Let G' be a subgroup of G and M be an (X,G)-structure on N. We say that M *reduces* to an (X,G')-structure on N if it contains an (X,G')-atlas for N.

Proposition 1.1.6. *An (X,G)-structure M on N reduces to an (X,G')-structure if and only if there exists a holonomy representation $H : \pi_1(N) \to G$ for M with $H(\pi_1(N)) \subset G'$.*

A topology on developing maps and holonomies We now define a topology on the space $\Omega(N)$ of all the (X,G)-structures on N. To this aim, first observe that a map $D : \widetilde{N} \to X$ is a developing map for some (X,G)-structure on N if and only if D is a local diffeomorphism such that for any $\gamma \in \mathrm{Deck}(\widetilde{N})$ there exists an element $H(\gamma) \in G$ with $D \circ \gamma = H(\gamma) \circ D$. We endow the set of developing maps $\mathcal{D}(N)$ for N with the topology of C^∞-uniform convergence on compact subset of $\widetilde{N}$, and observe that composition on the left defines an action of G on $\mathcal{D}(N)$ whose quotient is canonically identified with $\Omega(N)$. We eventually give $\Omega(N)$ the quotient topology defined by this identification. We also put on $\mathcal{R}(\pi_1(N),G) := \mathrm{Hom}(\pi_1(N),G)$ the pointwise convergence topology.

Note that once an identification $\pi_1(N) \cong \mathrm{Deck}(\widetilde{N})$ is fixed, the holonomy defines a continuous map $H : \mathcal{D}(N) \to \mathcal{R}(\pi_1(N),G)$, where $H(D) = H_D$. We denote by H also the induced continuous map $H : \Omega(N) \to \mathcal{R}(\pi_1(N),G)/G$ from the space of (X,G)-structures on N to the space of conjugacy classes of homomorphisms of $\pi_1(N)$ into G.

Thickenings Let N be a smooth connected n-manifold possibly with boundary, and let M be an element of $\Omega(N)$.

Definition 1.1.7. A *thickening* of M is an (X,G)-structure on a smooth connected n-manifold without boundary, containing N as a submanifold, which induces the given (X,G)-structure on N.

The following proposition [CEG87] proves that if N is compact then a thickening exists and is unique in a suitable sense. Let M and N be as above.

Proposition 1.1.8. *Suppose N is compact. Then M has a thickening. Moreover, let $M \subset M_1$, $M \subset M_2$ be two thickenings. Then there exist U_1, U_2 with $M \subset U_i \subset M_i$, such that U_i is a thickening of M and there is a unique (X,G)-equivalence between U_1 and U_2 which extends the identity on N.*

The deformation space The following theorem clarifies the local structure of $\Omega(N)$ near a fixed (X,G)-structure M on a *compact* N. Observe however that in what follows we will be mainly interested in geometric structures on non-compact hyperbolic manifolds. Let $N_{\rm th}$ be the union of N with a collar $\partial N \times [0,1)$. Since $\pi_1(N) \cong \pi_1(N_{\rm th})$, the universal covering of N is a subset of the universal covering of $N_{\rm th}$, and developing maps for $N_{\rm th}$ restrict to developing maps for N.

Let now M be an (X,G)-structure on N and $M_{\rm th}$ be an extension of M to $N_{\rm th}$ (such an extension exists by Proposition 1.1.8). The following result is proved in [CEG87, Gol88].

Theorem 1.1.9. *Let $D_{\rm th} : \widetilde{N}_{\rm th} \to X$ be a fixed developing map for $M_{\rm th}$. Then a small neighbourhood W of $D := D_{\rm th}|_{\widetilde{N}}$ in the space of developing maps for N is homeomorphic to $\mathcal{A}\times\mathcal{B}$, where $\mathcal{A}$ is a small neighbourhood of the obvious embedding $N \hookrightarrow N_{\rm th}$ in the space of smooth embeddings, and $\mathcal{B}$ is a small neighbourhood of the holonomy $H_D : \pi_1(N) \to G$ in the space $\mathcal{R}(\pi_1(N),G)$. Moreover:*

1. *The projection of W onto $\mathcal{B}$ is given by the holonomy;*
2. *There exists a continuous section $\mathcal{B} \to \mathcal{D}(N_{\rm th})$;*
3. *The action of G on W corresponds to conjugation in $\mathcal{B}$;*
4. *Suppose $D_1, D_2 \in W$ determine the same holonomy representation, let M_j be the element of $\Omega(N)$ corresponding to D_j and $i_j : N \hookrightarrow N_{\rm th}$ be the projection of D_j into $\mathcal{A}$, $j = 1,2$. Then an (X,G)-structure $M'_{\rm th} \in \Omega(N_{\rm th})$ exists such that the embedding i_j is an (X,G)-map between M_j and $M'_{\rm th}$, $j = 1,2$.*

As an immediate consequence of Theorem 1.1.9 we have the following:

Corollary 1.1.10. *If N is compact, then the holonomy map $H : \Omega(N) \to \mathcal{R}(\pi_1(N),G)/G$ is open.*

Remark 1.1.11. Theorem 1.1.9 seems to suggest that holonomy should give a local homeomorphism between the set of (X,G)-structures on N up to homotopy and thickening and the space $\mathcal{R}(\pi_1(N),G)/G$. But this is false in general, since one has little control over the orbits of the action of G on $\mathcal{R}(\pi_1(N),G)$ (for example, these orbits could not be closed, see [JM87, Gol88] and [Apa00, Chapter 7] for more details).

1.2 Hyperbolic cone structures and Dehn filling singularities

In this section we closely study the geometric structures which arise when deforming complete finite-volume hyperbolic 3-manifolds without boundary. In order to describe such structures we begin by introducing the fundamental notion of *three-dimensional hyperbolic cone manifold*.

Cone manifolds For $\alpha \in (0, 2\pi)$ we construct the local model for a hyperbolic cone manifold with cone angle equal to α in the following way. Consider in $\mathbb{H}^3$ a solid angular sector S_α obtained by taking the union (if $\alpha \geqslant \pi$) or the intersection (if $\alpha \leqslant \pi$) of two half-spaces W_1 and W_2, and suppose the dihedral angle at the infinite edge $\ell = \partial W_1 \cap \partial W_2$ is equal to α. The cone manifold $\mathbb{H}^3_\alpha$ is the space obtained when we identify the faces of S_α by a rotation along ℓ, endowed with its path metric. The image of ℓ in $\mathbb{H}^3_\alpha$ gives the singular line $\Sigma \in \mathbb{H}^3_\alpha$. The metric on $\mathbb{H}^3_\alpha$ induces on $\mathbb{H}^3_\alpha \setminus \Sigma$ a smooth incomplete hyperbolic structure, whose completion is precisely $\mathbb{H}^3_\alpha$ itself. The restriction on α to belong to $(0, 2\pi)$ is inessential: a local model for cone singularities with angles bigger than 2π can be easily constructed just by gluing to each other N copies of $S_{\alpha/N}$. Alternatively, we can put on $\mathbb{H}^3 \setminus \ell$ cylindrical coordinates (r, θ, z) and describe $\mathbb{H}^3_\alpha$ as the metric completion of $\mathbb{H}^3 \setminus \ell$ endowed with the smooth hyperbolic metric

$$ds^2 = dr^2 + (\alpha/(2\pi))^2 \sinh^2(r) d\theta^2 + \cosh^2(r) dz^2.$$

Here r gives the distance from the singular locus, θ is the angular measure around Σ and z gives the distance along Σ.

Definition 1.2.1. Let C be a smooth 3-manifold endowed with a distance d. Then (C, d) is a *hyperbolic cone manifold of dimension three* if (C, d) is a path space and any point in C has an open neighbourhood which is isometric to an open subset of $\mathbb{H}^3_\alpha$, for some $\alpha \in \mathbb{R}^+$. The *singular locus* $\Sigma \subset C$ is the set of points having no neighbourhood isometric to a smooth Riemannian manifold. It is easily seen that Σ is a smooth link in C. Moreover, if K is a component of Σ there exists a unique $\overline{\alpha}$ such that any point of K has a neighbourhood isometric to an open subset of $\mathbb{H}^3_{\overline{\alpha}}$. Such an $\overline{\alpha}$ is called *the cone angle* of C along K.

We will see later that Thurston's hyperbolic Dehn filling Theorem provides many structures on a hyperbolic cusped 3-manifold whose completions are hyperbolic cone 3-manifolds.

Dehn filling singularities Let N be the interior of a compact orientable 3-manifold with boundary given by a finite number of tori, and suppose that N supports a complete finite-volume hyperbolic structure (which is essentially unique by Mostow-Prasad's rigidity Theorem). Let ℓ be a complete geodesic in $\mathbb{H}^3$, and C_ϵ be the set of points of $\mathbb{H}^3$ at distance at most ϵ from ℓ. Let J be the subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ leaving ℓ invariant and note that J acts on $C_\epsilon \setminus \ell$, so it lifts to a group $\widetilde{J}$ of isometries of the metric universal covering $\widetilde{C_\epsilon \setminus \ell}$ of $C_\epsilon \setminus \ell$. Let now M be a $(\mathbb{H}^3, \mathrm{Isom}(\mathbb{H}^3))$-structure on N, let $E \cong S^1 \times S^1 \times [0, \infty) \subset N$ be a closed end of N and $\widetilde{E} \subset \widetilde{N}$ be a universal covering of E. Following Hodgson [Hod86] we say that M induces a *Dehn filling singularity* on E if there exists a developing map $D : \widetilde{N} \to \mathbb{H}^3$ for M such that:

1. The image of $\widetilde{E}$ under D is equal to $C_\epsilon \setminus \ell$;
2. D lifts to a diffeomorphism $\widetilde{D} : \widetilde{E} \to \widetilde{C_\epsilon \setminus \ell}$ which is the developing map for an $(\widetilde{C_\epsilon \setminus \ell}, \widetilde{J})$-structure on E.

Let $\widetilde{\psi} : \pi_1(E) \to \widetilde{J}$ be the holonomy representation for the structure described in point (2) above. Then $\Gamma := \widetilde{\psi}(\pi_1(E))$ is a discrete subgroup of $\widetilde{J}$, and $\widetilde{C_\epsilon \setminus \ell} \to (\widetilde{C_\epsilon \setminus \ell})/\Gamma$ is a cover. Note also that $\widetilde{C_\epsilon \setminus \ell}$ is diffeomorphic to $\mathbb{R}^3$, and orientation-preserving isometries in $\widetilde{J}$ can be parameterized by elements in $\mathbb{R}^2$ in such a way that $(l, \theta) \in \mathbb{R}^2$ corresponds to the lift of an element of J acting as a translation of length l along ℓ followed by a rotation of angle θ around ℓ. Under this identification and the natural isomorphism $H_1(E;\mathbb{Z}) \cong \pi_1(E)$ the holonomy representation $\widetilde{\psi}$ provides a homomorphism $\widetilde{\psi} : H_1(E;\mathbb{Z}) \to \mathbb{R}^2$ with image the discrete group Γ. Such homomorphism extends to an isomorphism $\overline{\psi} : H_1(T;\mathbb{R}) \to \mathbb{R}^2$, and we define the *(generalized) Dehn filling coefficient* of M at E to be the unique class $s \in H_1(E;\mathbb{R})$ with $\overline{\psi}(s) = (0, 2\pi)$. This definition of s depends on the choice of a positive orientation on small loops around ℓ, *i.e.* on the choice of a preferred generator of $H_1(C_\epsilon \setminus \ell;\mathbb{Z}) \cong \mathbb{Z}$. The opposite orientation would give $-s$ instead of s, so we will sometimes think of s as of an element in $H_1(E;\mathbb{R})/\pm 1$.

Metric completion of Dehn filling singularities Recall now that N admits a natural compactification obtained by adding some tori $T_1, \ldots, T_k$, and that a *slope* on a torus is an isotopy class of simple closed unoriented essential curves. It is well-known that simple closed *oriented* essential curves on T bijectively correspond to indivisible elements in $H_1(T;\mathbb{Z})$, so slopes on T are canonically identified with the classes of such elements in $H_1(T;\mathbb{Z})/\pm 1$. Let $h \leqslant k$ and for any $i = 1, \ldots, h$ let s_i be a slope on T_i. We denote by $N(s_1, \ldots, s_h)$ the manifold obtained by *Dehn filling* T_i along s_i, *i.e.* by gluing a solid torus V_i to T_i so that a meridian curve of V_i is identified with a simple closed curve in s_i for $i = 1, \ldots, h$.

Fix now a hyperbolic structure M on N, let E_i be the end of N corresponding to T_i and suppose that M has a Dehn filling singularity at E_i with Dehn filling coefficient $s_i \in H_1(E_i;\mathbb{R}) \cong H_1(T_i;\mathbb{R})$ for $i = 1, \ldots, h \leqslant k$. Let also $N^* = N \cup T_1 \cup \ldots \cup T_h$ and observe that our assumption ensures that a developing map $D : \widetilde{N} \to \mathbb{H}^3$ for M smoothly extends to a map $D^* : \widetilde{N^*} \to \mathbb{H}^3$ which restricts to a submersion to a geodesic in $\mathbb{H}^3$ on each component of $\partial\widetilde{N^*}$. On each T_i, $i = 1, \ldots, h$ there is an induced foliation $\mathcal{F}_i$ which is defined as follows: take a component $\widetilde{T_i}$ of $\widetilde{N^*}$ covering T_i, and define the leaves of $\mathcal{F}_i$ to be the projections on T_i of the preimages of points under the map $D^*|_{\widetilde{T_i}} : \widetilde{T_i} \to \ell$, where ℓ is the geodesic mentioned above. Observe that the length function on ℓ induces a measure transverse to $\mathcal{F}_i$. Choosing an orientation on small loops around ℓ, we can also define a signed measure along each leaf of $\mathcal{F}_i$, given by the total angle of rotation around the axis ℓ. Then the generalized Dehn filling coefficient s_i is the homology class in $H_1(T_i;\mathbb{R})$ corresponding to a portion of a leaf of $\mathcal{F}_i$ having total angle equal to 2π (one should approximate multiples of this portion by simple closed curves to give a precise definition of this homology class). Note that as before choosing the opposite orientation around ℓ we would get the opposite element in $H_1(T_i;\mathbb{R})$.

We can now describe the metric completion of E_i as follows [Thu79, Hod86]. Extend the metric on E_i over T_i using the degenerate metric given by the transverse measure on $\mathcal{F}_i$, so that the distance between two points of T_i is the infimum of total measures of curves on T_i joining these points. Then the metric completion $\widehat{E}_i$ of E_i

is obtained from $E_i \cup T_i$ by identifying points at distance 0 apart. If $\mathcal{F}_i$ consists of simple closed curves then $\widehat{E}_i$ is obtained by adding a single loop L whose length is the measure of a shortest simple closed curve transverse to $\mathcal{F}_i$. Topologically, $\widehat{E}_i$ is obtained from E_i by Dehn filling: $\widehat{E}_i = E_i \cup V_i$ where V_i is a solid torus glued to T_i so that each leaf in $\mathcal{F}_i$ bounds a disc in V_i. Metrically, $\widehat{E}_i$ has cone type singularities along L, with cone angle equal to the total angular measure of each leaf in $\mathcal{F}_i$. So $\widehat{E}_i$ has a smooth hyperbolic structure exactly when s_i is an indivisible element of $H_1(T_i;\mathbb{Z}) \subset H_1(T_i;\mathbb{R})$. If $s_i = \lambda s_i'$, where s_i' is an indivisible element of $H_1(T_i;\mathbb{Z})$ and $\lambda \in \mathbb{R}^+$, then it is defined on $\widehat{E}_i$ a hyperbolic structure with cone-type singularity with cone angle $2\pi/\lambda$ along L. Otherwise, $\mathcal{F}_i$ gives a foliation of T_i by irrational slopes, and $\widehat{E}_i$ is the one-point compactification of E_i.

Isometries of Dehn filling singularity structures Let M be a Dehn filling singularity structure on $E \cong T \times [0,\infty)$, let $p : \widetilde{E} \to E$ be a universal covering and $D : \widetilde{E} \to C_\epsilon \setminus \ell$ be a developing map which lifts to a diffeomorphism $\widetilde{D} : \widetilde{E} \to \widetilde{C_\epsilon \setminus \ell}$. For any $0 < \epsilon' < \epsilon$ we set $\Sigma_{1/\epsilon'} = p(\widetilde{D}^{-1}(\partial C_{\epsilon'})) \subset E$, where $\partial C_{\epsilon'}$ is the set of points of $\mathbb{H}^3$ having distance from ℓ equal to ϵ'. Alternatively, $\Sigma_{1/\epsilon'}$ can be seen as the set of points of $\widehat{E}$ having distance from $\widehat{E} \setminus E$ equal to ϵ', where $\widehat{E}$ is the metric completion of E with respect to the metric defined by M. Note that $\Sigma_{1/\epsilon} = \partial E \subset E$ and that Σ_t is an embedded boundary parallel torus in E for any $t > 1/\epsilon$. For any $t, t' \geqslant 1/\epsilon$ we also have a natural identification $\varphi_{t',t} : \Sigma_t \to \Sigma_{t'}$ which sends any $p \in \Sigma_t$ into the point of $\Sigma_{t'}$ nearest to p. The diffeomorphism

$$\partial E \times [1/\epsilon, \infty) \to E, \quad (p,t) \mapsto \varphi_{t,1/\epsilon}(p)$$

induces on E a canonical product structure which depends solely on M.

Proposition 1.2.2. *Let M^i be a Dehn filling singularity structure on $E^i \cong T \times [0,\infty)$ for $i = 1,2$ and for sufficiently large $t > 0$ let $\Sigma_t^i \subset E^i$ be the set of points of $\widehat{E}^i$ at distance $1/t$ apart from $\widehat{E}^i \setminus E^i$, where $\widehat{E}^i$ is the metric completion of E^i with respect to the metric defined by M^i. Let U be an open regular neighbourhood of a level surface $\Sigma_{t_0}^1 \subset E^1$ and let $f : U \to E^2$ be a locally isometric embedding with respect to the metrics M^1 on U and M^2 on E^2. Then $f(\Sigma_{t_0}^1) = \Sigma_{t_0}^2$ and f extends to a global isometry $F : \partial E^1 \times [t_0,\infty) \to \partial E^2 \times [t_0, \infty)$, where $E^i \cong \partial E^i \times [1/\epsilon_i, \infty)$ is the canonical product structure induced on E^i by M^i, $i = 1,2$.*

Proof: Let ℓ^i be a complete geodesic in $\mathbb{H}^3$, $C_{\epsilon_i}^i$ the set of points in $\mathbb{H}^3$ at distance at most ϵ_i from ℓ_i and $D^i : \widetilde{E}^i \to C_{\epsilon_i}^i \setminus \ell^i$ be a developing map for M^i, $i = 1,2$. Since f is a local isometry, f lifts to a map $\widetilde{f} : \widetilde{E}^1 \supset \widetilde{U} \to \widetilde{E}^2$ such that $D^2 \circ \widetilde{f}$ is a developing map for the structure induced by M_1 on U. Thus we have $D^2 \circ \widetilde{f} = g \circ D^1|_U$ for some $g \in \mathrm{Isom}^+(\mathbb{H}^3)$. Since the holonomy representation relative to the developing map $D^1|_{\widetilde{U}}$ (resp. $D^2|_{\widetilde{f}(\widetilde{U})}$) completely determines ℓ^1 (resp. ℓ^2) we have $g(\ell^1) = \ell^2$ and $g(C_\epsilon^1) = g(C_\epsilon^2)$ for any $\epsilon > 0$. This easily gives $f(\Sigma_{t_0}^1) = \Sigma_{t_0}^2$. Moreover, since f is a local isometry it is easily seen that if $V = \partial V \times (t_0 - \delta, t_0 + \delta) \subset U$, then we have $f((p,t)) = (f_0(p), t)$, where $f_0 : \partial E^1 \to \partial E^2$ is the unique map such that $(f_0(p), t_0) = f((p, t_0))$ for any $p \in \partial E^1$. Since f is an embedding, f_0 is a diffeomorphism. We now show that the map

$$F : E^1 \supset \partial E^1 \times [t_0, \infty) \to \partial E^2 \times [t_0, \infty) \subset E^2, \quad F((p,t)) = (f_0(p), t)$$

extends f and is a local isometry between the structures induced by M^1, M^2 respectively on $\partial E^1 \times [t_0, \infty), \partial E^2 \times [t_0, \infty)$. Since F is bijective, this will prove the proposition.

Claim. Let $G : \partial E^1 \times [t_0, t_1) \to E_2$ be a local isometry coinciding with f on $\Sigma^1_{t_0} = \partial E^1 \times \{t_0\}$. Then G coincides with F on $\partial E^1 \times [t_0, t_1)$.
The same argument showing that $f(\Sigma^1_{t_0}) = \Sigma^2_{t_0}$ also proves that $G(\Sigma^1_t) = \Sigma^2_t$ for any $t \in [t_0, t_1)$, whence $G((p,t)) = (g_t(p), t)$ for any $(p,t) \in \partial E^1 \times [t_0, t_1)$, where $g_t : \partial E^1 \to \partial E^2$ is a diffeomorphism smoothly depending on t. But since G is a local isometry it is easily seen that $G(p,t)$ is the point of $\partial E^2 \times \{t\}$ nearest to $(f_0(p), t_0) \in \partial E^2 \times \{t_0\}$, whence $g_t = f_0$ for any $t \in [t_0, t_1)$ and the claim is proved.

Let now $\Lambda = \{t > t_0 :\ F \text{ is a local isometry on } \partial E^1 \times [t_0, t)\}$. Of course Λ is closed. To conclude the proof we are left to show that Λ is open. So let $\bar{t} \in \Lambda$. Then $F|_{\partial E^1 \times [t_0, \bar{t})}$ is a local isometry, whence F defines a local isometry also on the compact manifold with boundary $\partial E^1 \times [t_0, \bar{t}]$. Now we know from Proposition 1.1.8 that thickenings are unique, whence a local isometry $G : \partial E^1 \times [t_0, \bar{t} + \delta) \to E_2$ exists which extends $F|_{\partial E^1 \times [t_0, \bar{t}]}$. By the claim we have that $F = G$ on $E^1 \times [t_0, \bar{t} + \delta)$, whence Λ is open, and we are done. $\square$

Isometries of complete cusps For later purpose we now need to spell out the analogous result of Proposition 1.2.2 for complete structures on toric ends of hyperbolic manifolds. So let M be a complete finite-volume hyperbolic structure on $E \cong T \times [0, \infty)$, let $p : \widetilde{E} \to E$ be a universal covering and $D : \widetilde{E} \to \mathbb{H}^3$ a developing map which gives a diffeomorphism between $\widetilde{E}$ and a horoball $O = D(\widetilde{E}) \subset \mathbb{H}^3$ centred at $q \in \partial \mathbb{H}^3$. For any $t > 0$ we set $\Sigma_t = p(\widetilde{D}^{-1}(\partial O_t)) \subset E$, where ∂O_t is the set of points of $\mathbb{H}^3$ contained in O and having distance from ∂O equal to t. Note that $\Sigma_0 = \partial E \subset E$ and that Σ_t is an embedded boundary parallel torus in E for any $t > 0$. For any t, t' we also have a natural identification $\varphi_{t',t} : \Sigma_t \to \Sigma_{t'}$ which sends any $p \in \Sigma_t$ into the point of $\Sigma_{t'}$ nearest to p. The diffeomorphism

$$\partial E \times [0, \infty) \to E, \quad (p, t) \mapsto \varphi_{t,0}(p)$$

induces on E a canonical product structure which depends solely on M.

Proposition 1.2.3. *Let M^i be a complete finite-volume hyperbolic structure as above on $E^i \cong T \times [0, \infty)$ for $i = 1, 2$ and for any $t > 0$ let $\Sigma^i_t \subset E^i$ be the set of points at distance t apart from ∂E^i. Let U be an open regular neighbourhood of a level surface $\Sigma^1_{t_1} \subset E^1$ and let $f : U \to E^2$ be a locally isometric embedding with respect to the metrics M^1 on U and M^2 on E^2. Then $f(\Sigma^1_{t_1}) = \Sigma^2_{t_2}$ for some $t_2 > 0$ and f extends to an isometry $F : \partial E^1 \times [t_1, \infty) \to \partial E^2 \times [t_2, \infty)$, where $E^i \cong \partial E^i \times [0, \infty)$ is the canonical product structure induced on E^i by M^i, $i = 1, 2$.*

PROOF: Let O^i be a horoball in $\mathbb{H}^3$ centred in q^i and $D^i : \widetilde{E}^i \to O^i$ be a developing map for M^i, $i = 1, 2$. Since f is a local isometry, f lifts to a map $\widetilde{f} : \widetilde{E}^1 \supset \widetilde{U} \to \widetilde{E}^2$ such that $D^2 \circ \widetilde{f}$ is a developing map for the structure induced by M_1 on U. Thus we have $D^2 \circ \widetilde{f} = g \circ D^1|_U$ for some $g \in \text{Isom}^+(\mathbb{H}^3)$. Since the holonomy representation relative to the developing map $D^1|_{\widetilde{U}}$ (resp. $D^2|_{\widetilde{f}(\widetilde{U})}$) completely determines q^1 (resp. q^2) we have $g(q^1) = q^2$, so horoballs centred in q^1 are taken by g onto horoballs centred in q^2. This easily gives $f(\Sigma^1_{t_1}) = \Sigma^2_{t_2}$ for some $t_2 > 0$. Let now $f_0 : \partial E^1 \to \partial E^2$ be the unique map such that $(f_0(p), t_2) = f((p, t_1))$ for any $p \in \partial E^1$. An obvious modification of the argument in the proof of Proposition 1.2.2 now shows that the map

$$F : E^1 \supset \partial E^1 \times [t_1, \infty) \to \partial E^2 \times [t_2, \infty) \subset E^2, \quad F((p,t)) = (f_0(p), t + t_2 - t_1)$$

extends f and is an isometry between the structures induced by M^1, M^2 respectively on $\partial E^1 \times [t_1, \infty), \partial E^2 \times [t_2, \infty)$. □

1.3 Thurston's hyperbolic Dehn filling Theorem

This section will be devoted to describing small deformations of the complete finite-volume hyperbolic structure on a 3-manifold N. We will be interested in showing how such deformations induce Dehn filling singularities on the ends of N. Following [BP01], we will then outline a proof of Thurston's hyperbolic Dehn filling Theorem.

The representation variety We recall that a *complex linear group* G is a subgroup of $\text{GL}(n, \mathbb{C})$ which is also an affine subvariety with respect to the standard affine structure on $\text{GL}(n, \mathbb{C})$, and observe that such a group has a natural structure of complex Lie group.

If G is a complex linear group and Γ is a finitely generated group, then $\mathcal{R}(\Gamma, G)$ has a natural affine structure which is defined as follows: take a set $\{\gamma_1, \ldots, \gamma_n\}$ of generators for Γ and consider the map

$$\mathcal{R}(\Gamma, G) \ni \rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_n)) \in G^n.$$

It is straight-forward to recognize that this map provides a bijection between $\mathcal{R}(\Gamma, G)$ and an affine subset of G^n, thus inducing an affine structure on $\mathcal{R}(\Gamma, G)$. It is easily seen that such structure does not depend on the choice of generators, and that the Euclidean topology on $\mathcal{R}(\Gamma, G)$ coincides with the pointwise convergence topology.

Let now N be a (not necessarily compact) 3-manifold without boundary and M_0 a complete finite-volume $(\mathbb{H}^3, \text{Isom}^+(\mathbb{H}^3))$-structure on N (by Mostow-Prasad's rigidity Theorem this structure is essentially unique). Since $\text{Isom}^+(\mathbb{H}^3)$ can be identified with $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm 1\}$, the holonomy representation $H_{M_0} : \pi_1(N) \to \text{Isom}^+(\mathbb{H}^3)$ provides a representation $\overline{\rho}_0 : \pi_1(N) \to \text{PSL}(2, \mathbb{C})$ which is canonical up to conjugation by elements in $\text{PSL}(2, \mathbb{C})$. Moreover $\overline{\rho}_0$ is faithful, $\overline{\rho}_0(\pi_1(N))$ is discrete and torsion-free, and M is isometric to $\mathbb{H}^3/\overline{\rho}_0(\pi_1(N))$. Recall now that N compactifies to a manifold $\overline{N}$ whose boundary consists of tori. We say that a subgroup H of $\pi_1(N)$ is *peripheral* if it is conjugated to $\pi_1(T) \subset \pi_1(N)$ for some component T of $\partial\overline{N}$. If H is peripheral, then all the elements of $\overline{\rho}_0(H)$ are parabolic, and have traces (defined up to sign) equal to ± 2.

The following proposition is due to Thurston and Culler [CS83, Cul86].

Proposition 1.3.1. *Any holonomy representation $\overline{\rho} : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ lifts to a representation $\rho : \pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$.*

We denote by ρ_0 be a lift of $\overline{\rho}_0$ to a representation in $\mathrm{SL}(2,\mathbb{C})$, and recall that ρ_0 is a point in the affine set $\mathcal{R}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))$.

The space of characters Any representation $\rho \in \mathcal{R}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))$ defines a map from $\pi_1(N)$ to $\mathbb{C}$, which sends $\gamma \in \pi_1(N)$ to the trace of the matrix $\rho(\gamma)$. This map is the *character* of ρ. Moreover, for any $\gamma \in \pi_1(N)$ we have the regular function

$$\tau_\gamma : \mathcal{R}(\pi_1(N), \mathrm{SL}(2,\mathbb{C})) \to \mathbb{C}, \quad \tau_\gamma(\rho) = \mathrm{tr}\,\rho(\gamma).$$

If A is the ring generated by $\{\tau_\gamma : \ \gamma \in \pi_1(N)\}$, then A has finitely many generators $\tau_{\gamma_1}, \ldots, \tau_{\gamma_j}$ [CS83]. We define a map $t : \mathcal{R}(\pi_1(N), \mathrm{SL}(2,\mathbb{C})) \to \mathbb{C}^j$ by $t(\rho) := \chi_\rho := (\tau_{\gamma_1}(\rho), \ldots, \tau_{\gamma_j}(\rho))$ and set $X(\pi_1(N)) := t(\mathcal{R}(\pi_1(N), \mathrm{SL}(2,\mathbb{C})))$. Since $\tau_{\gamma_1}, \ldots, \tau_{\gamma_j}$ generate A, there is a bijection between points in $X(\pi_1(N))$ and characters of elements in $\mathcal{R}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))$. Moreover, $X(\pi_1(N))$ is a Zariski-closed subset of $\mathbb{C}^j$ [CS83], so characters of representations in $\mathcal{R}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))$ admit a natural structure of affine set (which does not depend on the choice of $\gamma_1, \ldots, \gamma_j$). The following theorem is proved in [CS83].

Theorem 1.3.2. *There exists a Zariski-open subset $Y \subset X(\pi_1(N))$ containing χ_{ρ_0} such that for any $\rho, \rho' \in t^{-1}(Y)$ we have $\chi_\rho = \chi_{\rho'}$ if and only if ρ is conjugated to ρ'.*

Let now $T_1, \ldots, T_k$ be the boundary tori of $\overline{N}$ and for each $i = 1, \ldots, k$ let μ_i be an element of $\pi_1(N) \cong \pi_1(\overline{N})$ representing (a conjugate of) a non-trivial loop on T_i. Given an element $\gamma \in \pi_1(N)$ we consider the regular function $I_\gamma : X(\pi_1(N)) \to \mathbb{C}$ given by $I_\gamma(\chi_\rho) = \tau_\gamma(\rho)$ for any χ_ρ in $X(\pi_1(N))$. The main result we need about the local structure of $X(\pi_1(N))$ near χ_{ρ_0} is the following:

Theorem 1.3.3. *The map*

$$I_\mu : X(\pi_1(N)) \to \mathbb{C}^k, \quad I_\mu(\chi_\rho)) = (I_{\mu_1}(\chi_\rho), \ldots, I_{\mu_k}(\chi_\rho))$$

is locally bianalytic at χ_{ρ_0}. Thus χ_{ρ_0} is a smooth point of $X(\pi_1(N))$ and the (complex) dimension of the irreducible component of $X(\pi_1(N))$ containing χ_{ρ_0} is equal to k.

PROOF: This theorem was first proved by Thurston using results from cohomology theory of representations [Wei64, Gar67, Rag72, Thu79, CS83, Hod86]. For a proof which relies on complex analysis rather than on cohomology theory see [Zho99, BP01]. □

From representations to structures We need now to promote representations in a small neighbourhood of $\overline{\rho}_0$ to holonomy representations for nice structures on N. Let M be an element in $\Omega(N)$. We say that M is *good* if for any end E of N one of the following conditions holds:

- Either M induces a complete metric on E (which is therefore isometric to the quotient of a horoball by a $\mathbb{Z}+\mathbb{Z}$ parabolic group),
- Or M induces a Dehn filling singularity on E.

We say that two good structures are equivalent if they are related by an $(\mathbb{H}^3, \mathrm{Isom}^+(\mathbb{H}^3))$-isomorphism which is homotopic to the identity of N and we denote by $\mathcal{T}(N)$ the space of equivalence classes of good structures in $\Omega(N)$, endowed with the quotient topology. Projection onto conjugacy classes of representations induces the map $\overline{\mathrm{Hol}} : \mathcal{T}(N) \to \mathcal{R}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$.

Recall now that M_0 is the complete finite-volume hyperbolic structure on N, let $[M_0]$ be the corresponding point in $\mathcal{T}(N)$ and set $[\overline{\rho}_0] = \overline{\mathrm{Hol}}(M_0)$. It is easily seen that a neighbourhood U of $[M_0]$ exists such that $\overline{\mathrm{Hol}}$ can be uniquely lifted to a continuous map $\mathrm{Hol} : \mathcal{T}(N) \to \mathcal{R}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C})$ with $\mathrm{Hol}([M_0]) = [\rho_0]$. We denote by $\mathrm{Hol}' : \mathcal{T}(N) \to X(\pi_1(N))$ the map induced by Hol and observe that $\chi_{\rho_0} = \mathrm{Hol}'([M_0])$.

Our next aim is to provide a parameterization of a neighbourhood of $[M_0]$ in $\mathcal{T}(N)$ in terms of holonomy representations. Let $\mathcal{D}^{\mathrm{good}}(N)$ be the set of developing maps for good structures on N, and fix a developing map $D_0 \in \mathcal{D}^{\mathrm{good}}(N)$ for the complete structure on N. Take a small neighbourhood V of D_0 in $\mathcal{D}^{\mathrm{good}}(N)$ and denote by $H : V \to \mathcal{R}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C}))$ the holonomy map. We set $\Phi = \{(D, D') \in V \times V :\ H(D) \text{ is conjugated to } H(D') \text{ in } \mathrm{PSL}(2,\mathbb{C})\}$.

Lemma 1.3.4. *For any $(D, D') \in \Phi$ a unique isometry $g(D, D') \in \mathrm{PSL}(2,\mathbb{C})$ exists such that $H(D)(\gamma) = g(D, D') \cdot H(D')(\gamma) \cdot g(D, D')^{-1}$ for any $\gamma \in \pi_1(N)$. The map*

$$\Phi \to \mathrm{PSL}(2,\mathbb{C}), \qquad (D, D') \mapsto g(D, D')$$

is continuous and takes the diagonal of $V \times V$ onto the identity of $\mathrm{PSL}(2,\mathbb{C})$.

PROOF: We concentrate our attention on the case in which N is non-compact, the case when $k = 0$ being easier and less interesting. Let (D, D') be a pair in Φ and $\{J_i\}_{i \in \mathbb{N}}$ the set of peripheral subgroups of $\pi_1(N)$, *i.e.* the set of subgroups conjugated to $\pi_1(T_1), \ldots, \pi_1(T_k) \subset \pi_1(N)$. Since $H(D), H(D')$ are conjugated to each other and close to $H(D_0)$, for $i \in \mathbb{N}$ we can choose an element $\gamma_i \in J_i$ such that $H(D)(\gamma_i)$, $H(D')(\gamma_i)$ are both loxodromic or both parabolic. Let $p_i, p_i' \in \partial\mathbb{H}^3$ be the attractive points of $H(D)(\gamma_i), H(D')(\gamma_i)$ respectively. Then if $g \in \mathrm{PSL}(2,\mathbb{C})$ conjugates $H(D)$ into $H(D')$ we have $g(p_i) = p_i'$. Since D is near to D_0 the p_i's are not contained in the boundary of any hyperbolic plane, thus g is uniquely determined by its behaviour on the p_i's and we can set $g(D, D') = g$. Since the p_i's and the p_i''s continuously depend on D, D' we also have that $g(D, D')$ continuously depend on $(D, D') \in \Phi$. $\square$

The following theorem provides the desired local description of $\mathcal{T}(N)$ in a neighbourhood of $[M_0]$.

Theorem 1.3.5. *Let $X(\pi_1(N))$ be endowed with the Euclidean topology. Then there exist an open neighbourhood U of $[M_0]$ in $\mathcal{T}(N)$ and an open neighbourhood Z of χ_{ρ_0} in $X(\pi_1(N))$ such that the following conditions hold:*

1. *For any $\rho, \rho' \in t^{-1}(Z)$ we have $\chi_\rho = \chi_{\rho'}$ if and only if ρ is conjugated to ρ';*
2. *The map $I_\mu : X(\pi_1(N)) \to \mathbb{C}^k$ defined in the statement of Theorem 1.3.3 restricts to a biholomorphism between Z and an open neighbourhood of 0 in $\mathbb{C}^k$;*
3. *The map $\mathrm{Hol}'|_U : U \to Z$ is a homeomorphism.*

PROOF: Points (1) and (2) readily follow from Theorems 1.3.2, 1.3.3. As a (not completely immediate) consequence of Theorem 1.1.9 we also have that $\mathrm{Hol}'|_U : U \to Z$ is open and surjective (see [PP00, BP01] for detailed proofs). Here we show that if $[M_1], [M_2]$ are two elements in $U \subset \mathcal{T}(N)$ with $\mathrm{Hol}'(M_1) = \mathrm{Hol}'(M_2)$ then we have $[M_1] = [M_2]$. In order to use Theorem 1.1.9 we have to deal with *compact* manifolds, so we set $N = N_c \cup C_1 \ldots \cup C_k$, where $N_c \cong \overline{N}$ is a compact core of N, $C_j \cong T_j \times [0, \infty)$, $N_c \cap C_j \cong T_j \times \{0\}$ and $C_j \cap C_l = \emptyset$ for $j, l = 1, \ldots, k$, $j \neq l$. Let V be a small neighbourhood of $D_0|_{N_c}$ in $\mathcal{D}(N_c)$ which contains developing maps $D_1, D_2 \in V$ for the structures induced on N_c by M_1, M_2 respectively. Recall we are assuming that holonomy representations corresponding to D_1, D_2 are conjugated by an element $g \in \mathrm{PSL}(2, \mathbb{C})$. Up to resizing U, by Lemma 1.3.4 we can suppose that g is near to the identity of $\mathrm{PSL}(2, \mathbb{C})$, thus $g \circ D_1$ and D_2 both lie in V and determine the same holonomy representation. Theorem 1.1.9 (4) now applies providing smooth embeddings $i_1, i_2 : N_c \hookrightarrow N$ with the following properties: i_j is close to the obvious inclusion $i_0 : N_c \hookrightarrow N$, and the map $i_2 \circ i_1^{-1} : i_1(N_c) \to i_2(N_c)$ is a local isometry from the structure induced on $i_1(N_c)$ by M_1 to the structure induced on $i_2(N_c)$ by M_2. Now Propositions 1.2.2, 1.2.3 easily imply that $i_2 \circ i_1^{-1}$ can be extended to a global diffeomorphism inducing an isometry between M_1 and M_2. Since i_1, i_2 are close to i_0 this diffeomorphism is homotopic to the identity of N, whence $[M_1] = [M_2]$, and we are done. □

The final step Let Z be as in the statement of Theorem 1.3.5 and for any $z \in Z$ let $\rho_z \in \mathcal{R}(\pi_1(N), \mathrm{SL}(2, \mathbb{C}))$ be a representation with $\chi_{\rho_z} = z$ (such a ρ_z is not canonically determined by z, we will take care of this fact later). We denote as usual by $\overline{\rho}_z : \pi_1(N) \to \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$ the representation induced by ρ_z.

For each $i = 1, \ldots, k$ let (μ_i, λ_i) be a fixed positively oriented basis of $H_1(T_i, \mathbb{Z}) \cong \pi_1(T_i, \mathbb{Z}) \subset \pi_1(N)$. Now $\pi_1(T_i)$ is Abelian and $\overline{\rho}_0|_{\pi_1(T_i)}$ is discrete faithful with image consisting of parabolic isometries. These facts easily imply that for any ρ_z, $z \in Z$ one of the following possibilities holds: either $\overline{\rho}_z(\pi_1(T_i))$ consists of elliptic and/or loxodromic isometries with exactly two common fixed points at infinity, or $\overline{\rho}_z|_{\pi_1(T_i)}$ is faithful and its image consists of a discrete group of parabolic elements all having the same fixed point at infinity. So let $p_i(z), q_i(z) \in \mathbb{C} \cup \{\infty\}$ be the fixed points for the action of $\overline{\rho}_z(\pi_1(T_i))$ on $\partial\mathbb{H}^3$, with $p_i(z) = q_i(z)$ if and only if $\overline{\rho}_z(\pi_1(T_i))$ is a parabolic group. (It can be seen we cannot require p_i, q_i to be continuous functions with respect to z in any neighbourhood of χ_{ρ_0}. Note also that in what follows the p_i's will play a different rôle with respect to the q_i's, so another choice is actually introduced here). We now identify $\mathbb{H}^3$ with the upper half-space model $\mathbb{C}\times(0,\infty)$ in such a way that $p_i(z)$ corresponds to ∞. Then $\overline{\rho}_z(\mu_i)$ and $\overline{\rho}_z(\lambda_i)$ induce affine transformations of $\mathbb{C}=\partial\mathbb{H}^3\setminus\{\infty\}$. Let $\overline{\rho}_z(\mu_i)(x)=a_i(\rho_z,p_i)x+b_i(\rho_z,p_i)$, $\overline{\rho}_z(\lambda_i)(x)=c_i(\rho_z,p_i)x+d_i(\rho_z,p_i)$ for any $x\in\mathbb{C}$. Note that the dilation factors $a_i(\rho_z,p_i)$, $b_i(\rho_z,p_i)$ do not depend on the identification $\mathbb{H}^3\cong\mathbb{C}\times(0,\infty)$, let ln be the branch of the complex logarithm defined on $\{z\in\mathbb{C}:\ \Re(z)>0\}$ such that $\ln 1=0$ and set

$$\begin{aligned} u_i(\rho_z,p_i)=\ln a_i(\rho_z,p_i),\quad v_i(\rho_z,p_i)=\ln c_i(\rho_z,p_i),\\ u(\rho_z,p_1,\ldots,p_k)=(u_1(\rho_z,p_1),\ldots,u_k(\rho_z,p_k)),\\ v(\rho_z,p_1,\ldots,p_k)=(v_1(\rho_z,p_1),\ldots,v_k(\rho_z,p_k)). \end{aligned} \tag{1.1}$$

Our heavy notation here is aimed at emphasizing that at this stage u and v could actually depend not only on z, but also on the choice of ρ_z and of the p_i's. More on this after the following:

Lemma 1.3.6. *Let E_i be the end of N corresponding to the torus T_i for $i=1,\ldots,k$. Then, up to resizing Z, for $z\in Z$ we have $u_i(\rho_z,p_i)=0$ if and only if $v_i(\rho_z,p_i)=0$.*

PROOF: Of course we have $u_i(\rho_z,p_i)=0$ (resp. $v_i(\rho_z,p_i)=0$) if and only if $\overline{\rho}_z(\mu_i)$ (resp. $\overline{\rho}_z(\mu_i)$) is parabolic. Moreover, both $\overline{\rho}_0(\mu_i)$, $\overline{\rho}_0(\lambda_i)$ are non-trivial parabolic and $\overline{\rho}_z(\mu_i)$ commutes with $\overline{\rho}_z(\lambda_i)$. This easily implies that there exists a neighbourhood of χ_{ρ_0} in Z such that for any z in this neighbourhood $\overline{\rho}_z(\mu_i)$, $\overline{\rho}_z(\lambda_i)$ are either both parabolic or both non-parabolic. □

As anticipated above we are now interested in understanding how u and v can depend on ρ_z and on the p_i's. So let ρ'_z be any representation with $\chi_{\rho'_z}=\chi_{\rho_z}=z$ and for each $i=1,\ldots,k$ let $p'_i(z)$ be a fixed point for $\overline{\rho}'_z(\pi_1(T_i))$ (we include the case $\rho'=\rho$, $p'_i=q'_i$ for some i). Then an easy computation shows that for $z\in Z$ and $i=1,\ldots,k$ we have $u_i(\rho_z,p_i)=\pm u_i(\rho'_z,p'_i)$, $v_i(\rho_z,p_i)=\pm v_i(\rho'_z,p'_i)$. Let $\sim$ be the equivalence relation on $\mathbb{C}^2$ defined by $(u,v)\sim(u',v')$ if and only if $(u,v)=\pm(u',v')$, set $W=\mathbb{C}^2/\sim$ and endow it with the quotient topology. Set also

$$\begin{array}{rclrcl} d_i & : & Z\to W, & d_i(z) & = & [(u_i(\rho_z,p_i),v_i(\rho_z,p_i))],\\ d & : & Z\to W^k, & d(z) & = & (d_1(z),\ldots,d_k(z)). \end{array} \tag{1.2}$$

It is easily seen that d is well-defined and continuous.

Neumann-Zagier parameters Our next task is to express v as a function of u, in a suitable sense. First of all, consider the map $\pi_\mu:\mathbb{C}^k\to\mathbb{C}^k$ given by $\pi_\mu(x_1,\ldots,x_k)=(\epsilon_1\cosh x_1,\ldots,\epsilon_k\cosh x_k)$, where $\epsilon_i=I_{\mu_i}(\chi_{\rho_0})=\pm2$ is the trace of $\rho_0(\mu_i)$. Let us now identify Z with a neighbourhood of $(\epsilon_1,\ldots,\epsilon_k)$ in $\mathbb{C}^k$ by the map $I_\mu:Z\to\mathbb{C}^k$ given by $\varphi_\mu(z)=(I_{\mu_1}(z),\ldots,I_{\mu_k}(z))$ (see Theorem 1.3.3). Of course π_μ defines a map $\overline{\pi}_\mu:W^k\to\mathbb{C}^k$ and an easy computation shows that for any $z\in Z$ we have $\overline{\pi}_\mu(\pm u_1(z),\ldots,\pm u_k(z))=z$, where $\pm u_i(z)$ is equal to $\pm u_i(\rho_z,p_i)$ for suitable $\rho_z\in\mathcal{R}(\pi_1(N),\mathrm{SL}(2,\mathbb{C}))$, $p_i\in\partial\mathbb{H}^3$. So fix a neighbourhood Y of $(0,\ldots,0)$ in $\mathbb{C}^k$ such that $\pi_\mu(Y)\subset Z$ and let $\overline{u}=(\overline{u}_1,\ldots,\overline{u}_k)$ be a point in Y. Take the representation $\rho_{\pi_\mu(\overline{u})}$ and let $\overline{p}_i$ be the the *only* fixed point of $\overline{\rho}_{\pi_\mu(\overline{u})}(\pi_1(T_i))$ with

$u_i(\rho_{\pi_\mu(\overline{u})}, \overline{p}_i) = \overline{u}_i$ (such a point is well-defined by Lemma 1.3.6). With a slight abuse we define a function $v : Y \to \mathbb{C}^k$ by setting

$$v_i(\overline{u}) = v_i(\rho_{\pi_\mu(\overline{u})}, \overline{p}_i), \qquad v(\overline{u}) = (v_1(\overline{u}), \ldots, v_k(\overline{u})). \tag{1.3}$$

Let now $\overline{u}$ be a point in Y with $\overline{u}_i \neq 0$ for all $i = 1, \ldots, k$. Continuity of v at $\overline{u}$ is easily checked. Moreover, consider the map $\pi_\lambda : \mathbb{C}^k \to \mathbb{C}^k$ given by $\pi_\lambda(x_1, \ldots, x_k) = (\epsilon'_1 \cosh x_1, \ldots, \epsilon'_k \cosh x_k)$, where $\epsilon'_i = I_{\lambda_i}(\chi_{\rho_0}) = \pm 2$ is the trace of $\rho_0(\lambda_i)$. As before, we identify Z with a neighbourhood of $(\epsilon'_1, \ldots, \epsilon'_k)$ in $\mathbb{C}^k$ by the map $I_\lambda : Z \to \mathbb{C}^k$ given by $I_\lambda(z) = (I_{\lambda_1}(z), \ldots, I_{\lambda_k}(z))$. By construction we have $\pi_\lambda \circ v = I_\lambda \circ I_\mu^{-1} \circ \pi_\mu$ on Y. Together with analyticity of $I_\lambda \circ I_\mu^{-1}$ (see Theorem 1.3.3), this implies the following:

Lemma 1.3.7. *The map $v : Y \to \mathbb{C}^k$ described above is complex analytic. Moreover, for $i = 1, \ldots, k$ the map $v_i : Y \to \mathbb{C}$ is odd with respect to u_i and even with respect to u_j for any $j \neq i$.*

For any u in Y let now $M(u)$ be a good structure in $\Omega(N)$ such that the character of a lift of a holonomy representation for $M(u)$ is equal to $\pi_\mu(u)$ (such a structure exists by Theorem 1.3.5). The following results were proved (in a slightly different context) in [NZ85].

Proposition 1.3.8. *Let j be a fixed integer in $\{1, \ldots, k\}$. We have:*

1. *$v_j(u) = 0$ if and only if $u_j = 0$ if and only if $M(u)$ induces a complete metric on E_j;*
2. *$\lim_{u_j \neq 0, u \to 0} v_j(u)/u_j$ exists and has positive imaginary part;*
3. *If $u_j \neq 0$ then $M(u)$ induces on E_j a Dehn filling singularity with coefficient $p_j(u) \cdot \mu_j + q_j(u) \cdot \lambda_j$, where $p_j(u), q_j(u)$ are the only real numbers such that $p_j(u)u_j + q_j(u)v_j(u) = 2\pi i$.*

Proof: Point (1) is an immediate consequence of Lemma 1.3.6.

For any $u \in Y$ let $\overline{\rho}_u : \pi_1(N) \to \mathrm{Isom}^+(\mathbb{C} \times (0, \infty))$ be a holonomy representation for $M(u)$ such that $\overline{\rho}_u(\pi_1(T_j))$ fixes ∞ and acts on $\mathbb{C} \cong \mathbb{C} \times \{0\}$ as follows:

$$\overline{\rho}_u(\mu_j)(x) = \exp(u_j) \cdot x + b_j(u), \quad \overline{\rho}_u(\lambda_j)(x) = \exp(v_j(u)) \cdot x + d_j(u).$$

Since $\overline{\rho}_u(\mu_j)$ commutes with $\overline{\rho}_u(\lambda_j)$ we get

$$\begin{aligned} &\exp(u_j) \cdot \exp(v_j(u)) \cdot x + \exp(u_j) \cdot d_j(u) + b_j(u) = \overline{\rho}_u(\mu_j) \circ \overline{\rho}_u(\lambda_j)(x) \\ &= \overline{\rho}_u(\lambda_j) \circ \overline{\rho}_u(\mu_j)(x) = \exp(v_j(u)) \cdot \exp(u_j) \cdot x + \exp(v_j(u)) \cdot b_j(u) + d_j(u), \end{aligned}$$

whence $d_j(u)(\exp(u_j) - 1) = b_j(u)(\exp(v_j(u)) - 1)$ and

$$\lim_{u_j \neq 0, u \to 0} v_j(u)/u_j = \lim_{u_j \neq 0, u \to 0} (\exp(v_j(u)) - 1)/(\exp(u_j) - 1) = d_j(0)/b_j(0).$$

Now $b_j(0)$ and $d_j(0)$ are the translation vectors of the parabolic isometries $\overline{\rho}_0(\mu_j)$ and $\overline{\rho}_0(\mu_j)$, so point (2) follows from the fact that (μ_j, λ_j) is a positively oriented basis of $H_1(T_j; \mathbb{Z})$.

Up to conjugating $\overline{\rho}_u$ by an isometry fixing ∞ we can suppose that $\overline{\rho}_u(\mu_j)(x) = \exp(u_j) \cdot x$, $\overline{\rho}_u(\lambda_j)(x) = \exp(v_j(u)) \cdot x$ for any $x \in \mathbb{C}$. Let now ℓ be the geodesic of $\mathbb{H}^3$ with endpoints $0, \infty$, let C_ϵ be the (closed) ϵ-neighbourhood of ℓ and J be the subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ leaving ℓ invariant. Recall that J lifts to a group $\widetilde{J}$ of isometries of $\widetilde{C_\epsilon \setminus \ell}$. It is easily seen that in our context the isomorphism $\overline{\psi}$: $H_1(T_j;\mathbb{R}) \to \widetilde{J} \cong \mathbb{R}^2$ defined in Section 1.2 takes the form $\overline{\psi}(p \cdot \mu_j + q \cdot \lambda_j) = pu_j + qv_j(u) = (\Re(pu_j + qv_j(u)), \Im(pu_j + qv_j(u)))$, whence the conclusion by the definition of (generalized) Dehn filling coefficient. □

Thurston's hyperbolic Dehn filling Theorem We are now ready to state Thurston's result about hyperbolic Dehn fillings of cusped hyperbolic 3-manifolds.

Proposition 1.3.9. *The map*

$$u \to ((p_1(u), q_1(u)), \ldots, (p_k(u), q_k(u)))$$

defined in Proposition 1.3.8 (3) gives a diffeomorphism between a neighbourhood of 0 *in* $\mathbb{C}^k$ *and a neighbourhood of* $(\infty, \ldots, \infty)$ *in* $S^2 \times \cdots \times S^2$.

PROOF: Point (2) of Proposition 1.3.8 ensures that this map has non-singular Jacobian matrix at 0. □

Corollary 1.3.10 (Thurston's hyperbolic Dehn filling Theorem). *For each* $i = 1, \ldots, k$ *there exists a finite set* $\mathcal{S}_i$ *of slopes on* T_i *with the following property: if* $h \leqslant k$ *and if* s_i *is a slope on* T_i *not belonging to* $\mathcal{S}_i$ *for* $i = 1, \ldots, h$, *then the filled manifold* $N(s_1, \ldots, s_h)$ *admits a complete finite-volume hyperbolic structure.*

1.4 The case of non-empty geodesic boundary

Let now $\overline{N}$ be a compact orientable 3-manifold with boundary $\partial\overline{N} = \Sigma_1 \sqcup \ldots \sqcup \Sigma_h \sqcup T_1 \sqcup \ldots \sqcup T_k$, where Σ_i is a surface of genus at least two and T_j is a torus for $i = 1, \ldots, h$, $j = 1, \ldots, k$. We set $N = \overline{N} \setminus (T_1 \sqcup \ldots \sqcup T_k)$ and suppose that N supports a complete finite-volume hyperbolic structure with geodesic boundary. Such a structure, which is essentially unique by Mostow-Prasad's rigidity Theorem, will be denoted by M_0. As before, we also denote by $\overline{\rho}_0 : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ a fixed holonomy representation for M_0, and by $\rho_0 : \pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$ a lift of $\overline{\rho}_0$. We denote by $\Omega^{\mathrm{geod}}(N)$ the set of $(\mathbb{H}^3, \mathrm{Isom}^+(\mathbb{H}^3))$-structures with totally geodesic boundary on N, and we endow $\Omega^{\mathrm{geod}}(N)$ with the topology it inherits from $\Omega(N)$. We say that a discrete torsion-free subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ is *Fuchsian* if its limit set is equal to a straight circle in $\partial\mathbb{H}^3$, and we set

$$\begin{aligned}
\mathcal{R}^{\mathrm{geod}}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C})) &:= \{\overline{\rho} : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C}) : \ \overline{\rho}(\pi_1(\Sigma_i)) \text{ is Fuchsian} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall i = 1, \ldots, h\}, \\
\mathcal{R}^{\mathrm{geod}}(\pi_1(N), \mathrm{SL}(2,\mathbb{C})) &:= \{\rho : \pi_1(N) \to \mathrm{SL}(2,\mathbb{C}) : \overline{\rho} \in \mathcal{R}^{\mathrm{geod}}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C}))\}.
\end{aligned}$$

We observe that $\mathcal{R}^{\mathrm{geod}}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))$ is invariant under conjugation by elements in $\mathrm{SL}(2,\mathbb{C})$ and we set $X^{\mathrm{geod}}(\pi_1(N)) := t(\mathcal{R}^{\mathrm{geod}}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))) \subset X(\pi_1(N))$, $\chi_{\rho_0} = t(\rho_0)$.

Remark 1.4.1. Let M, M' be (a priori non-necessarily homeomorphic) complete finite-volume hyperbolic 3-manifolds with compact geodesic boundary and let $\varphi : \pi_1(M) \to \pi_1(M')$ be a group isomorphism. Suppose $\partial M = \Sigma_1 \sqcup \ldots \sqcup \Sigma_h$, $\partial M' = \Sigma'_1 \sqcup \ldots \sqcup \Sigma'_{h'}$. Then the group $\varphi(\pi_1(\Sigma_i))$ is equal to (a conjugate of) $\pi_1(\Sigma'_{i'})$ for some $i' \in \{1, \ldots, h'\}$ (see [Joh79, Swa78, Fri04] for a proof). This means that $\mathcal{R}^{\text{geod}}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))$ and $X^{\text{geod}}(\pi_1(N))$ indeed depend only on $\pi_1(N)$ and not on N.

Lemma 1.4.2. *The holonomy map $\Omega(N) \to \mathcal{R}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$ takes $\Omega^{\text{geod}}(N)$ into $\mathcal{R}^{\text{geod}}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$.*

Proof: Let $D_M : \widetilde{N} \to \mathbb{H}^3$ be the developing map for a structure $M \in \Omega^{\text{geod}}(N)$, and $\overline{\rho}_M : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ be an associated holonomy representation. Let also $\widetilde{\Sigma}_i$ be the connected component of $\partial\widetilde{N}$ which is left invariant by $\pi_1(\Sigma_i) \subset \pi_1(N)$. Then the set $D_M(\widetilde{\Sigma}_i)$ is a hyperbolic plane, and $\overline{\rho}(\pi_1(\Sigma_i))$ acts freely and with compact quotient on $D_M(\widetilde{\Sigma}_i)$. This implies that the limit set of $\overline{\rho}_M(\pi_1(\Sigma_i))$ is equal to $\partial D_M(\widetilde{\Sigma}_i)$, whence the conclusion. □

The following lemma shows that $\mathcal{R}^{\text{geod}}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))$ and $X^{\text{geod}}(\pi_1(N))$ can be given a real algebraic structure, at least in a neighbourhood of ρ_0 and χ_{ρ_0}.

Lemma 1.4.3. *There exists a neighbourhood U of ρ_0 in $\mathcal{R}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))$ such that*

$$\begin{aligned} & U \cap \mathcal{R}^{\text{geod}}(\pi_1(N), \mathrm{SL}(2,\mathbb{C})) \\ = {} & U \cap \{\rho : \pi_1(N) \to \mathrm{SL}(2,\mathbb{C}) : \ \operatorname{tr}\rho(\gamma) \in \mathbb{R}\ \forall \gamma \in \pi_1(\Sigma_i),\ i = 1, \ldots, h\}. \end{aligned}$$

Proof: Fix $i \in \{1, \ldots, h\}$. It is well-known that a neighbourhood U of ρ_0 in $\mathcal{R}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))$ exists such that for any $\rho \in U$ the induced representation $\overline{\rho}|_{\pi_1(\Sigma_i)}$ is faithful and has discrete image. Let ρ belong to U and fix a non-trivial indivisible element $\gamma_0 \in \pi_1(\Sigma_i)$. We conjugate ρ by an element in $\mathrm{SL}(2,\mathbb{C})$ in such a way that $\overline{\rho}(\gamma_0)$ is a generator of the stabilizer in $\overline{\rho}(\pi_1(\Sigma_i))$ of the geodesic $\ell_0 \subset \mathbb{C} \times (0, \infty) \cong \mathbb{H}^3$ with endpoints 0 and ∞. Thus we have

$$\rho(\gamma_0) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

with $\lambda \in \mathbb{R} \setminus \{\pm 1\}$. For any element $\gamma \in \pi_1(\Sigma_i)$ we set

$$\rho(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}.$$

By imposing the traces of $\rho(\gamma)$ and $\rho(\gamma_0\gamma)$ to belong to $\mathbb{R}$ we obtain

$$a(\gamma) + d(\gamma) \in \mathbb{R}, \quad \lambda a(\gamma) + \lambda^{-1} d(\gamma) \in \mathbb{R},$$

whence $a(\gamma), d(\gamma) \in \mathbb{R}$. Moreover, since $a(\gamma)d(\gamma) - b(\gamma)c(\gamma) = 1 \in \mathbb{R}$ we can uniquely define functions $A, B, C, D : \pi_1(\Sigma_i) \to \mathbb{R}$ and $\theta : \pi_1(\Sigma_i) \to [0, \pi)$ such that

$$\rho(\gamma) = \begin{pmatrix} A(\gamma) & B(\gamma)e^{i\theta(\gamma)} \\ C(\gamma)e^{-i\theta(\gamma)} & D(\gamma) \end{pmatrix}$$

for any $\gamma \in \pi_1(\Sigma_i)$.

Recall now that the image of $\overline{\rho}|_{\pi_1(\Sigma_i)}$ in $\mathrm{PSL}(2,\mathbb{C})$ being discrete, any non-trivial isometry in $\overline{\rho}(\pi_1(\Sigma_i))$ either leaves ℓ_0 invariant or does not fix 0 nor ∞. This means that for any $\gamma \in \pi_1(\Sigma_i)$ either $B(\gamma) = C(\gamma) = 0$ or $B(\gamma)C(\gamma) \neq 0$. Moreover, it is easily seen that an element $\gamma_1 \in \pi_1(\Sigma_i)$ exists such that $B(\gamma_1)C(\gamma_1) \neq 0$. Now for any $\gamma \in \pi_1(\Sigma_i)$ we have $\mathbb{R} \ni A(\gamma_1\gamma) = A(\gamma_1)A(\gamma) + B(\gamma_1)C(\gamma)e^{i(\theta(\gamma_1)-\theta(\gamma))}$, whence either $B(\gamma) = C(\gamma) = 0$ or $\theta(\gamma) = \theta(\gamma_1)$. Hence the rotation of angle $-\theta(\gamma_1)$ along ℓ_0 conjugates $\overline{\rho}(\pi_1(\Sigma_i))$ in $\mathrm{PSL}(2,\mathbb{R})$. This in turn easily implies that $\overline{\rho}(\pi_1(\Sigma_i))$ is Fuchsian. □

As in the case of empty-boundary, characters determine conjugacy classes of representations in a neighbourhood of ρ_0 [CS83]:

Theorem 1.4.4. *There exists a Zariski-open subset $Y^{\mathrm{geod}} \subset X^{\mathrm{geod}}$ containing χ_{ρ_0} such that for any $\rho, \rho' \in t^{-1}(Y^{\mathrm{geod}})$ we have $\chi_\rho = \chi_{\rho'}$ if and only if ρ is conjugated to ρ'.*

Doubling structures Let DN (resp. $D\overline{N}$) be the double of N (resp. $\overline{N}$) along $\partial N = \Sigma_1 \sqcup \ldots \sqcup \Sigma_h$, *i.e.* the manifold obtained by mirroring N (resp. $\overline{N}$) along ∂N. Then N (resp. $\overline{N}$) is canonically embedded in DN (resp. $D\overline{N}$) and $D\overline{N}$ admits an involution σ which fixes $\partial N \subset N \subset D\overline{N}$ and interchanges N with its mirror copy. Set

$$\Omega^{\mathrm{sym}}(DN) = \{M \in \Omega(DN) \text{ s.t. } \sigma : DN \to DN \text{ is an isometry of } M\}.$$

We have an obvious bijection $\Omega^{\mathrm{geod}}(N) \to \Omega^{\mathrm{sym}}(DN)$ sending $M \in \Omega^{\mathrm{geod}}(N)$ to the hyperbolic structure $DN \in \Omega^{\mathrm{sym}}(DN)$ obtained by doubling M along ∂N. As in the empty boundary case, we say that M is *good* if for any toric end E of N either M induces a complete metric on E or M induces a Dehn filling singularity on E. We say that two good structures are equivalent if they are related by an $(\mathbb{H}^3, \mathrm{Isom}^+(\mathbb{H}^3))$-isomorphism which is homotopic to the identity of N and we denote by $\mathcal{T}^{\mathrm{geod}}(N)$ the space of equivalence classes of good structures in $\Omega^{\mathrm{geod}}(N)$, endowed with the quotient topology. Projection onto conjugacy classes of representations gives induced maps

$$\begin{array}{rlcl}
\overline{\mathrm{Hol}}: & \mathcal{T}^{\mathrm{geod}}(N) & \longrightarrow & \mathcal{R}^{\mathrm{geod}}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C}), \\
\mathrm{Hol}: & \mathcal{T}^{\mathrm{geod}}(N) & \longrightarrow & \mathcal{R}^{\mathrm{geod}}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C}), \\
\mathrm{Hol}': & \mathcal{T}^{\mathrm{geod}}(N) & \longrightarrow & X^{\mathrm{geod}}(\pi_1(N)).
\end{array}$$

Lemma 1.4.5. *The map $D : \Omega^{\mathrm{geod}}(N) \to \Omega^{\mathrm{sym}}(DN)$ induces a bijection*

$$D : \mathcal{T}^{\mathrm{geod}}(N) \to \mathcal{T}^{\mathrm{sym}}(DN),$$

where $\mathcal{T}^{\mathrm{sym}}(DN)$ is the space of equivalence classes of good structures in $\Omega^{\mathrm{sym}}(DN)$.

PROOF: Let us first prove that $D : \mathcal{T}^{\mathrm{geod}}(N) \to \mathcal{T}^{\mathrm{sym}}(DN)$ is well-defined. So let $M_1, M_2 \in \Omega^{\mathrm{geod}}(N)$ be equivalent good structures on N. Then there exists a diffeomorphism $\varphi : N \to N$ which is homotopic to the identity of N and induces an isometry from M_1 to M_2. It is easily seen that φ can be modified to a smooth map which extends to a diffeomorphism $\overline{\varphi} : \overline{N} \to \overline{N}$ via a homotopy with support far from ∂N. Now a result of Waldhausen [Wal68] applies ensuring that $\overline{\varphi}$ is isotopic to the identity of $\overline{N}$, whence φ can be homotoped to the identity of N via a homotopy which preserves the boundary of N. This easily implies that the diffeomorphism $D\varphi : DN \to DN$ obtained by doubling φ is homotopic to the identity of DN. Moreover $D\varphi$ induces an isometry between the structures DM_1 and DM_2 on DN, whence $[DM_1] = [DM_2]$ in $\mathcal{T}^{\mathrm{sym}}(DN)$ and $D : \mathcal{T}^{\mathrm{geod}}(N) \to \mathcal{T}^{\mathrm{sym}}(DN)$ is well-defined.

Surjectivity of $D : \mathcal{T}^{\mathrm{geod}}(N) \to \mathcal{T}^{\mathrm{sym}}(DN)$ is obvious, so we turn our attention to injectivity. Let $M_1, M_2 \in \Omega^{\mathrm{geod}}(N)$ be good structures on N and suppose that $[DM_1] = [DM_2]$ in $\mathcal{T}^{\mathrm{sym}}(DN)$. Then a diffeomorphism $\psi : DN \to DN$ exists which is homotopic to the identity of DN and induces an isometry between DM_1 and DM_2. Since ∂N is totally geodesic both in DM_1 and in DM_2 it is readily seen that $\psi(\partial N) = \partial N$. Since ψ is homotopic to the identity, this easily implies that $\psi(N) = N$, so ψ restricts to a diffeomorphism $\psi|_N : N \to N$ inducing an isometry between M_1 and M_2. Now let $r : DN \to N$ be the retraction of DN to N which is the identity on $N \subset DN$ and coincides with the reflection σ on the mirror copy of N in DN. Composing the homotopy between ψ and the identity of DN with r and restricting to N we obtain a homotopy between $\psi|_N$ and the identity of N, whence $[M_1] = [M_2]$, and we are done. $\square$

For each $i = 1, \ldots, k$ let s_i be a simple closed oriented loop on $T_i \subset \partial\overline{N} \subset \partial D\overline{N}$ and μ_i be an element of $\pi_1(DN) \cong \pi_1(D\overline{N})$ representing s_i (such an element is well-defined only up to conjugacy). Recall that we have $\partial D\overline{N} = T_1 \sqcup \ldots \sqcup T_k \sqcup \hat{T}_1 \sqcup \ldots \sqcup \hat{T}_k$, where $\hat{T}_i = \sigma(T_i)$ for $i = 1, \ldots, k$. For $i = 1, \ldots, k$ the loop $s_i \subset T_i$ determines a mirror loop $\hat{s}_i = \sigma(s_i) \subset \hat{T}_i$, and we denote by $\hat{\mu}_i = \sigma_*(\mu_i)$ an element of $\pi_1(DN)$ representing $\hat{s}_i$.

We now set

$$\begin{aligned}\mathcal{R}^{\mathrm{sym}}(\pi_1(DN), \mathrm{SL}(2,\mathbb{C})) = \ & \{\rho : \pi_1(DN) \to \mathrm{SL}(2,\mathbb{C}) : \ \mathrm{tr}\,\rho(\mu_i) = \mathrm{tr}\,\rho(\hat{\mu}_i) \\ & \qquad\qquad\qquad\qquad\qquad \text{for all } i = 1, \ldots, k\},\end{aligned}$$

and observe that $\mathcal{R}^{\mathrm{sym}}(\pi_1(DN), \mathrm{SL}(2,\mathbb{C}))$ is invariant under conjugation by elements in $\mathrm{SL}(2,\mathbb{C})$. We also set $X^{\mathrm{sym}}(\pi_1(DN)) := t(\mathcal{R}^{\mathrm{sym}}(\pi_1(DN), \mathrm{SL}(2,\mathbb{C})) \subset X(\pi_1(DN)))$, where $t : \mathcal{R}(\pi_1(DN), \mathrm{SL}(2,\mathbb{C})) \to X(\pi_1(DN))$ is the natural projection.

Proposition 1.4.6. *The natural projection* $\mathrm{Hol}' : \mathcal{T}(DN) \to X(\pi_1(DN))$ *induces a map (still denoted by* Hol'*) from* $\mathcal{T}^{\mathrm{sym}}(DN)$ *to* $X^{\mathrm{sym}}(DN)$.

PROOF: Let DM be a structure in $\mathcal{T}^{\mathrm{sym}}(DN)$ and $\overline{\rho}$ be a holonomy representation for DM. Let $\widetilde{\sigma} : \widetilde{DN} \to \widetilde{DN}$ be a lift of σ with $\hat{\mu}_i = \widetilde{\sigma} \circ \mu_i \circ \widetilde{\sigma}^{-1}$ in $\mathrm{Aut}(\widetilde{DN})$. Since σ is an isometry of DM it is easily seen that an isometry $\overline{\sigma} \in \mathrm{Isom}^+(\mathbb{H}^3)$ exists such that $\overline{\rho}(\widetilde{\sigma} \circ \gamma \circ \widetilde{\sigma}^{-1}) = \overline{\sigma} \circ \overline{\rho}(\gamma) \circ \overline{\sigma}^{-1}$. This readily implies that $\mu_i, \hat{\mu}_i$ are conjugated in $\mathrm{SL}(2,\mathbb{C})$, whence $\mathrm{tr}\,\rho(\hat{\mu}_i) = \mathrm{tr}\,\rho(\mu_i)$ and $\chi_\rho \in X^{\mathrm{sym}}(\pi_1(DN))$. $\square$

Remark 1.4.7. Let M, M' be complete finite-volume hyperbolic 3-manifolds with geodesic boundary, denote by DM, DM' the doubles of M, M' along their geodesic boundary and by $\sigma_M, \sigma_{M'}$ the symmetries of DM, DM' with respect to $\partial M \subset$

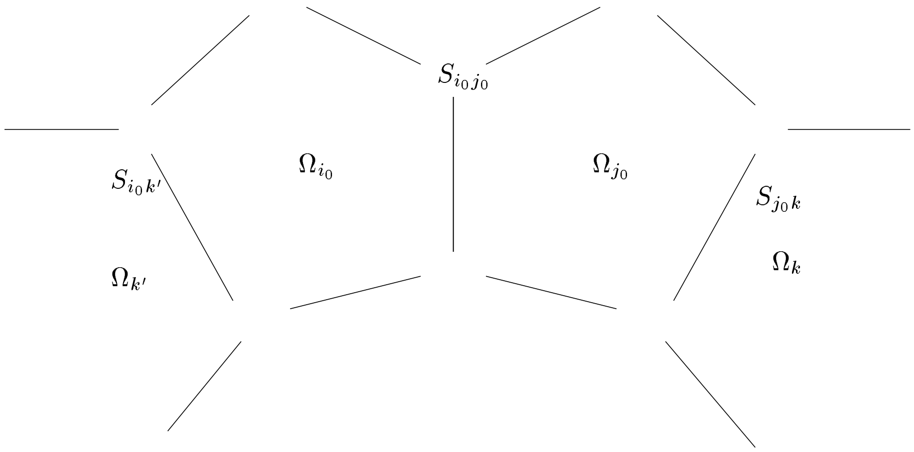

Figure 1.1: The configuration of regions described in the proof of Proposition 1.4.8.

$DM, \partial M' \subset DM'$. Suppose DM is isometric to DM'. Then we *cannot* conclude that σ_M is conjugated to $\sigma_{M'}$ by a homeomorphism between DM and DM'. Namely, we cannot even conclude that M is homeomorphic to M'. This shows that the definition of $\mathcal{R}^{\mathrm{sym}}(\pi_1(DN), \mathrm{SL}(2,\mathbb{C}))$ and $X^{\mathrm{sym}}(\pi_1(DN))$ actually depend not only on $\pi_1(DN)$, but also on the embedding of ∂N in DN.

Proposition 1.4.8. *There exists a continuous injective map*

$$\overline{D} : \mathcal{R}^{\mathrm{geod}}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C})) \to \mathcal{R}^{\mathrm{sym}}(\pi_1(DN), \mathrm{PSL}(2,\mathbb{C}))$$

such that if ρ is a holonomy representation for an element $M \in \Omega^{\mathrm{geod}}(N)$, then $\overline{D}\rho$ is a holonomy representation for $DM \in \Omega^{\mathrm{sym}}(DN)$.

Proof: Consider the universal covering $p : \widetilde{DN} \to DN$ and let $\{\Omega_i\}_{i\in\mathbb{N}}$ be the connected components of $p^{-1}(DN \setminus \partial N)$. Set also $\Phi = \{(i,j) \in \mathbb{N}\times\mathbb{N} : \overline{\Omega}_i \cap \overline{\Omega}_j \neq \emptyset\}$ and $A(i) = \{j \in \mathbb{N} : (i,j) \in \Phi\}$ for any $i \in \mathbb{N}$ (so Φ is the set of pairs of indices of adjacent regions, while $A(i)$ is the set of indices of regions adjacent to Ω_i). Let us denote by S_{ij} the connected component of $p^{-1}(\partial N)$ given by $S_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$, whenever $(i,j) \in \Phi$. We now fix a representation $\overline{\rho} \in \mathcal{R}^{\mathrm{geod}}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C}))$. Our first aim is to define a hyperbolic plane $P_{ij} \subset \mathbb{H}^3$ for any $(i,j) \in \Phi$: such a plane should be thought as the image of S_{ij} under a developing map with holonomy $\overline{D}\overline{\rho}$.

So let Ω_0 be the component of $p^{-1}(DN \setminus \partial N)$ which is left invariant by $\pi_1(N) \subset \pi_1(DN)$. For any $j \in A(0)$ let Γ_{0j} be the stabilizer of S_{0j} in $\pi_1(N)$. Our hypothesis says that $\overline{\rho}(\Gamma_{0j})$ is Fuchsian, so we can define P_{0j} as the unique hyperbolic plane whose boundary at infinity is equal to the limit set of $\overline{\rho}(\Gamma_{0j})$.

Now let $i_0 \in \mathbb{N}$ and suppose that P_{i_0j} has already been defined for all $j \in A(i_0)$. Fix $j_0 \in A(i_0)$ and let k belong to $A(j_0)$ (see Fig. 1.1). Consider the symmetry $\sigma : DN \to DN$ and let $\widetilde{\sigma}_{i_0j_0}$ be the lift of σ which fixes $S_{i_0j_0}$. We have $\widetilde{\sigma}_{i_0j_0}(S_{j_0k}) = S_{i_0k'}$ for some $k' \in A(i_0)$, and we define $P_{j_0k'}$ to be the image of P_{i_0k} under the isometric reflection of $\mathbb{H}^3$ with respect to the hyperbolic plane $P_{i_0j_0}$. Since the dual graph of the decomposition of $\widetilde{DN}$ into the Ω_i's is a tree, this procedure provides a well-defined hyperbolic plane P_{ij} for any $(i,j) \in \Phi$.

Let now γ be an element in $\pi_1(DN) \cong \mathrm{Aut}(\widetilde{DN})$ and let $r_* : \pi_1(DN) \to \pi_1(N)$ be the map induced by the obvious retraction $DN \to N$. It is easily seen that a finite sequence of indices $\{0 = l_0, l_1 \ldots, l_m\} \subset \mathbb{N}$ exists such that $\gamma = \widetilde{\sigma}_{l_m l_{m-1}} \circ \widetilde{\sigma}_{l_{m-1} l_{m-2}} \circ \cdots \circ \widetilde{\sigma}_{l_1 0} \circ r_*(\gamma)$, where $(l_j, l_{j+1}) \in \Phi$ for any $j = 0, \ldots, m-1$. We then set

$$\overline{D}\overline{\rho}(\gamma) = \overline{\sigma}_{l_m l_{m-1}} \circ \overline{\sigma}_{l_{m-1} l_{m-2}} \circ \cdots \circ \overline{\sigma}_{l_1 0} \circ \overline{\rho}(r_*(\gamma)),$$

where $\overline{\sigma}_{ij}$ is the isometric reflection of $\mathbb{H}^3$ with respect to the hyperbolic plane P_{ij}, for any $(i,j) \in \Phi$. We want to show that $\overline{D\rho}$ is a homomorphism. Note first that by the very definition of the $\overline{\sigma}_{ij}$'s we have

$$\widetilde{\sigma}_{ij}(S_{hk}) = S_{h'k'} \Longrightarrow \overline{\sigma}_{ij}(P_{hk}) = P_{h'k'} \quad \text{if } \{i,j\} \cap \{h,k\} \neq \emptyset. \tag{1.4}$$

Claim 1. Let m be an even integer number and let $(l_j, l_{j+1}) \in \Phi$ for any $j = 0, \ldots, m-1$, so that the map $\widetilde{\sigma}_{l_m l_{m-1}} \circ \widetilde{\sigma}_{l_{m-1} l_{m-2}} \circ \cdots \circ \widetilde{\sigma}_{l_1 0}$ gives an element of $\mathrm{Aut}(\widetilde{DN})$. Let also $\gamma \in \mathrm{stab}\, \Omega_k \subset \mathrm{Aut}(\widetilde{DN}) \cong \pi_1(DN)$ for some $k \in \mathbb{N}$ and suppose that $\widetilde{\sigma}_{l_m l_{m-1}} \circ \widetilde{\sigma}_{l_{m-1} l_{m-2}} \circ \cdots \circ \widetilde{\sigma}_{l_1 0} \circ \gamma = 1$ in $\mathrm{Aut}(\widetilde{DN}) \cong \pi_1(DN)$. Then $\widetilde{\sigma}_{l_m l_{m-1}} \circ \widetilde{\sigma}_{l_{m-1} l_{m-2}} \circ \cdots \circ \widetilde{\sigma}_{l_1 0} = \gamma = 1$ in $\mathrm{Aut}(\widetilde{DN})$.
It is easily seen that $r_*(\widetilde{\sigma}_{l_m l_{m-1}} \circ \widetilde{\sigma}_{l_{m-1} l_{m-2}} \circ \cdots \circ \widetilde{\sigma}_{l_1 0}) = 1$, whence

$$1 = r_*(\widetilde{\sigma}_{l_m l_{m-1}} \circ \widetilde{\sigma}_{l_{m-1} l_{m-2}} \circ \cdots \circ \widetilde{\sigma}_{l_1 0} \circ \gamma) = r_*(\gamma).$$

Now $\mathrm{stab}\, \Omega_k$ is conjugated to $\mathrm{stab}\, \Omega_0$ and $r_*|_{\mathrm{stab}\, \Omega_0}$ is injective, so $\gamma = 1$, and the claim is proved.

Claim 2. Let $(i,j), (h,k) \in \Phi$. If $\widetilde{\sigma}_{ij}(S_{hk}) = S_{h'k'}$, then $\overline{\sigma}_{ij}(P_{hk}) = P_{h'k'}$.
We can choose $\gamma \in \mathrm{stab}\, \Omega_k$ and $k = l_0, l_1, \ldots, l_m \in \mathbb{N}$ such that $\widetilde{\sigma}_{ij} = \widetilde{\sigma}_{l_m l_{m-1}} \circ \widetilde{\sigma}_{l_{m-1} l_{m-2}} \circ \cdots \circ \widetilde{\sigma}_{l_1 0} \circ \gamma$, where $(l_j, l_{j+1}) \in \Phi$ for any $j = 0, \ldots, m-1$. By Claim 1 we get $\widetilde{\sigma}_{ij} = \widetilde{\sigma}_{l_m l_{m-1}} \circ \widetilde{\sigma}_{l_{m-1} l_{m-2}} \circ \cdots \circ \widetilde{\sigma}_{l_1 0}$. By the very definition of the $\overline{\sigma}_{ij}$'s we now have $\overline{\sigma}_{ij} = \overline{\sigma}_{l_m l_{m-1}} \circ \overline{\sigma}_{l_{m-1} l_{m-2}} \circ \cdots \circ \overline{\sigma}_{l_1 0}$, whence $\overline{\sigma}_{ij}(P_{hk}) = P_{h'k'}$ by repeatedly using equation (1.4).

Claim 3. Let $(i_1, j_1), \ldots, (i_l, j_l) \in \Phi$ and suppose that $\widetilde{\sigma}_{i_1 j_1} \circ \widetilde{\sigma}_{i_2 j_2} \circ \cdots \circ \widetilde{\sigma}_{i_l j_l} = 1$. Then $\overline{\sigma}_{i_1 j_1} \circ \overline{\sigma}_{i_2 j_2} \circ \circ \cdots \circ \overline{\sigma}_{i_l j_l} = 1$.
Take a connected component S of $p^{-1}(\partial N)$ and let P be the corresponding hyperbolic plane in $\mathbb{H}^3$. By Claim 2, since $\widetilde{\sigma}_{i_1 j_1} \circ \cdots \circ \widetilde{\sigma}_{i_l j_l}(S) = S$ we have $\overline{\sigma}_{i_1 j_1} \circ \circ \cdots \circ \overline{\sigma}_{i_l j_l}(P) = P$. Moreover, since any $\widetilde{\sigma}_{ij}$ is orientation-reversing l must be even, whence $\overline{\sigma}_{i_1 j_1} \circ \circ \cdots \circ \overline{\sigma}_{i_l j_l}$ is orientation-preserving and is therefore equal to the identity of $\mathbb{H}^3$.

Let us now fix some notation: for any $(i,j) \in \Phi$ and $\gamma \in \pi_1(DN)$ we set $\gamma(ij) = rs$, where $(r,s) \in \Phi$ is such that $\gamma(S_{ij}) = S_{rs}$. Since $\gamma \circ \widetilde{\sigma}_{ij}, \widetilde{\sigma}_{\gamma(ij)} \circ \gamma : \widetilde{DN} \to \widetilde{DN}$ both lift $\sigma : DN \to DN$ and coincide on S_{ij}, we have

$$\gamma \circ \widetilde{\sigma}_{ij} = \widetilde{\sigma}_{\gamma(ij)} \circ \gamma, \quad \gamma \in \pi_1(DN), \ (i,j) \in \Phi. \tag{1.5}$$

In the same spirit, by Claim 2 the orientation-reversing isometries $\overline{\rho}(\gamma)\circ\overline{\sigma}_{ij},\overline{\sigma}_{\gamma(ij)}\circ\overline{\rho}(\gamma):\mathbb{H}^3\to\mathbb{H}^3$ coincide on P_{ij}, so

$$\overline{\rho}(\gamma)\circ\overline{\sigma}_{ij}=\overline{\sigma}_{\gamma(ij)}\circ\overline{\rho}(\gamma),\quad \gamma\in\pi_1(DN),\ (i,j)\in\Phi. \tag{1.6}$$

So let

$$\begin{array}{rcl}\pi_1(DN)\ni\gamma &=& \widetilde{\sigma}_{l_m l_{m-1}}\circ\widetilde{\sigma}_{l_{m-1}l_{m-2}}\circ\cdots\circ\widetilde{\sigma}_{l_1 0}\circ r_*(\gamma),\\ \pi_1(DN)\ni\gamma' &=& \widetilde{\sigma}_{l'_{m'}l'_{m'-1}}\circ\widetilde{\sigma}_{l'_{m'-1}l'_{m'-2}}\circ\cdots\circ\widetilde{\sigma}_{l'_1 0}\circ r_*(\gamma').\end{array}$$

By definition of $\overline{D\rho}$ we have

$$\begin{array}{rcl}\overline{D}\overline{\rho}(\gamma) &=& \overline{\sigma}_{l_m l_{m-1}}\circ\overline{\sigma}_{l_{m-1}l_{m-2}}\circ\cdots\circ\overline{\sigma}_{l_1 0}\circ\overline{\rho}(r_*(\gamma)),\\ \overline{D}\overline{\rho}(\gamma') &=& \overline{\sigma}_{l'_{m'}l'_{m'-1}}\circ\overline{\sigma}_{l'_{m'-1}l'_{m'-2}}\circ\cdots\circ\overline{\sigma}_{l'_1 0}\circ\overline{\rho}(r_*(\gamma')),\end{array}$$

whence

$$\begin{array}{rcl}\overline{D}\overline{\rho}(\gamma)\overline{D}\overline{\rho}(\gamma') &=& \overline{\sigma}_{l_m l_{m-1}}\circ\cdots\circ\overline{\sigma}_{l_1 0}\circ r_*(\gamma)\circ\overline{\sigma}_{l'_{m'}l'_{m'-1}}\circ\cdots\circ\overline{\sigma}_{l'_1 0}\circ\overline{\rho}(r_*(\gamma))\\ &=& \overline{\sigma}_{l_m l_{m-1}}\circ\cdots\circ\overline{\sigma}_{l_1 0}\circ\overline{\sigma}_{\gamma(l'_{m'}l'_{m'-1})}\circ\cdots\circ\overline{\sigma}_{\gamma(l'_1 0)}\circ\overline{\rho}(r_*(\gamma\gamma'))\end{array} \tag{1.7}$$

by an application of formula (1.6).

Moreover, by equation (1.5) we have

$$\begin{array}{rcl}\gamma\gamma' &=& \widetilde{\sigma}_{l_m l_{m-1}}\circ\cdots\circ\widetilde{\sigma}_{l_1 0}\circ r_*(\gamma)\circ\widetilde{\sigma}_{l'_{m'}l'_{m'-1}}\circ\cdots\circ\widetilde{\sigma}_{l'_1 0}\circ r_*(\gamma)\\ &=& \widetilde{\sigma}_{l_m l_{m-1}}\circ\cdots\circ\widetilde{\sigma}_{l_1 0}\circ\widetilde{\sigma}_{\gamma(l'_{m'}l'_{m'-1})}\circ\cdots\circ\widetilde{\sigma}_{\gamma(l'_1 0)}\circ r_*(\gamma\gamma').\end{array} \tag{1.8}$$

Let now

$$\gamma\gamma'=\widetilde{\sigma}_{l''_{m''}l''_{m''-1}}\circ\cdots\circ\widetilde{\sigma}_{l''_1 0}\circ r_*(\gamma\gamma') \tag{1.9}$$

be the decomposition of $\gamma\gamma'$ which gives

$$\overline{D}\overline{\rho}(\gamma\gamma')=\overline{\sigma}_{l''_{m''}l''_{m''-1}}\circ\cdots\circ\overline{\sigma}_{l''_1 0}\circ\overline{\rho}(r_*(\gamma\gamma')),$$

where $(l_j,l_{j+1})\in\Phi$ for any $j=0,\ldots,m-1$. By equations (1.8), (1.9) we have

$$\widetilde{\sigma}_{l_m l_{m-1}}\circ\cdots\circ\widetilde{\sigma}_{l_1 0}\circ\widetilde{\sigma}_{\gamma(l'_{m'}l'_{m'-1})}\circ\cdots\circ\widetilde{\sigma}_{\gamma(l'_1 0)}=\widetilde{\sigma}_{l''_{m''}l''_{m''-1}}\circ\cdots\circ\widetilde{\sigma}_{l''_1 0},$$

whence

$$\overline{\sigma}_{l_m l_{m-1}}\circ\cdots\circ\overline{\sigma}_{l_1 0}\circ\overline{\sigma}_{\gamma(l'_{m'}l'_{m'-1})}\circ\cdots\circ\overline{\sigma}_{\gamma(l'_1 0)}=\overline{\sigma}_{l''_{m''}l''_{m''-1}}\circ\cdots\circ\overline{\sigma}_{l''_1 0}$$

by Claim 3. Thus $\overline{D}\overline{\rho}(\gamma\gamma')=\overline{D}\overline{\rho}(\gamma)\overline{D}\overline{\rho}(\gamma')$ and $\overline{D}\overline{\rho}$ is a homomorphism.

By the very construction of $\overline{D}\overline{\rho}$ it follows that if $\overline{\rho}$ is a holonomy representation for an element $M\in\Omega^{\mathrm{geod}}(N)$, then $\overline{D}\overline{\rho}$ is a holonomy representation for $DM\in\Omega^{\mathrm{sym}}(DN)$. Injectivity of $\overline{D}$ is obvious. □

Corollary 1.4.9. *There exists a neighbourhood U of ρ_0 in $\mathcal{R}^{\mathrm{geod}}(\pi_1(N),\mathrm{SL}(2,\mathbb{C}))$ with the following properties: U projects onto $Z\subset X^{\mathrm{geod}}(\pi_1(N))$ and the map $\overline{D}$ introduced in the previous proposition induces continuous injective maps*

$$\begin{array}{rl}D: & U\to\mathcal{R}^{\mathrm{sym}}(\pi_1(DN),\mathrm{SL}(2,\mathbb{C})),\\ D': & Z\to X^{\mathrm{sym}}(\pi_1(DN)).\end{array}$$

PROOF: Since $\overline{D}\overline{\rho}_0$ lifts to a holonomy representation for the hyperbolic structure DM_0 on DN, the existence of a continuous $D : U \to \mathcal{R}(\pi_1(DN), \mathrm{SL}(2,\mathbb{C}))$ lifting $\overline{D}$ follows from general facts about continuous lifts of families of representations (see *e.g.* [Cul86]). Moreover, it is easily seen that for $i = 1, \ldots, k$ and $\overline{\rho} \in \mathcal{R}^{\mathrm{geod}}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C}))$ the isometries $\overline{D}\overline{\rho}(\mu_i)$ and $\overline{D}\overline{\rho}(\hat{\mu}_i)$ are conjugated by a reflection with respect to some hyperbolic plane, so $D\rho$ belongs indeed to $\mathcal{R}^{\mathrm{sym}}(\pi_1(DN), \mathrm{SL}(2,\mathbb{C}))$. Finally if $\rho, \rho' \in U$ are conjugated representations, then $D\rho, D\rho'$ are also conjugated, whence existence and continuity of D'. Injectivity of D' is an immediate consequence of Proposition 1.4.8 and Theorem 1.4.4. □

The following proposition is inspired by [FK97, Lemma 1]. We keep the notation from the previous corollary.

Proposition 1.4.10. *Up to resizing Z, the map $D' : Z \to X^{\mathrm{sym}}(\pi_1(DN))$ induces a homeomorphism between Z and an open neighbourhood of $\chi_{D\rho_0} = D'\chi_{\rho_0}$ in $X^{\mathrm{sym}}(\pi_1(DN))$.*

PROOF: Inclusion of N into DN induces a map $i : \mathcal{R}(\pi_1(DN), \mathrm{SL}(2,\mathbb{C})) \to \mathcal{R}(\pi_1(N), \mathrm{SL}(2,\mathbb{C}))$ which gives in turn a map $i' : X(\pi_1(DN)) \to X(\pi_1(N))$. We claim that i' sends a neighbourhood of $D'\chi_{\rho_0}$ in $X^{\mathrm{sym}}(\pi_1(DN))$ into $X^{\mathrm{geod}}(\pi_1(N))$. So choose a neighbourhood $W \subset \mathcal{R}^{\mathrm{sym}}(\pi_1(DN), \mathrm{SL}(2,\mathbb{C}))$ of $D\rho_0$ such that any element of W is the lift of a holonomy representation for a good structure in $\Omega(DN)$ (such a W exists by Theorem 1.3.5). Let now $\rho_M \in W$ be a representation corresponding to a good structure $M \in \Omega(DN)$ with *integral indivisible* Dehn filling coefficients. Then the metric completion of M is isometric to a complete finite-volume hyperbolic manifold. Let $\sigma_* : \pi_1(DN) \to \pi_1(DN)$ be the map induced by the topological symmetry $\sigma : DN \to DN$. Up to resizing Z, since ρ_M belongs to $\mathcal{R}^{\mathrm{sym}}(\pi_1(DN), \mathrm{SL}(2,\mathbb{C}))$ Theorem 1.3.5 (3) ensures that the representations ρ_M and $\rho_M \circ \sigma_*$ are conjugated to each other. By Mostow-Prasad's rigidity Theorem (applied to the metric completion of M) this gives an isometric involution $\hat{\sigma} : M \to M$ which is homotopic to $\sigma : DN \to DN$. Now a result of Tollefson [Tol81] applies ensuring that $\hat{\sigma}$ fixes an embedded closed surface $\Sigma \subset DN$ which is isotopic to $\partial N \subset DN$. Such a Σ has to be totally geodesic, thus cutting M along Σ we obtain a hyperbolic structure with geodesic boundary M' on N whose character $\chi_{M'} = i'(\chi_M)$ lies in $X^{\mathrm{geod}}(\pi_1(N))$. Note also that $D'(\chi_{M'}) = \chi_M$. Since characters of good structures in $\Omega^{\mathrm{sym}}(DN)$ corresponding to integral indivisible Dehn filling coefficients are Zariski-dense in a neighbourhood of $D\chi_{\rho_0}$ in $X^{\mathrm{sym}}(\pi_1(DN))$, this proves that $i' : X(\pi_1(DN)) \to X(\pi_1(N))$ sends a neighbourhood of $D\chi_{\rho_0}$ into $X^{\mathrm{geod}}(\pi_1(N))$. Now it is easily seen that i' and D' are one the inverse to the other in sufficiently small neighbourhoods of $\chi_{D\rho_0}$ and χ_{ρ_0}, whence the conclusion. □

Coordinates near the complete structure The following result gives a useful description of the local structure of $\mathcal{T}(N)$ and $X^{\mathrm{geod}}(\pi_1(N))$ in a neighbourhood of $[M_0]$ and χ_{ρ_0} respectively.

Theorem 1.4.11. *Let $X^{\mathrm{geod}}(\pi_1(N))$ be endowed with the Euclidean topology. Then there exist an open neighbourhood V of $[M_0]$ in $\mathcal{T}(N)$ and an open neighbourhood Z of χ_{ρ_0} in $X^{\mathrm{geod}}(\pi_1(N))$ such that the following conditions hold:*

- *For any* $\rho, \rho' \in t^{-1}(Z)$ *we have* $\chi_\rho = \chi_{\rho'}$ *if and only if* ρ *is conjugated to* ρ'*;*

- *The map* $I_\mu : X^{\mathrm{geod}}(\pi_1(N)) \to \mathbb{C}^k$ *given by* $I_\mu(\chi_\rho) = (I_{\mu_1}(\chi_\rho), \ldots, I_{\mu_k}(\chi_\rho))$ *restricts to a bianalytic map between* Z *and an open neighbourhood of* 0 *in* $\mathbb{C}^k$*;*

- *The map* $\mathrm{Hol}'|_V : V \to Z$ *is a homeomorphism.*

PROOF: By Lemma 1.4.5 and Propositions 1.4.6, 1.4.10 we can choose open sets $V \subset \mathcal{T}^{\mathrm{geod}}(N)$, $Z \subset X^{\mathrm{geod}}(\pi_1(N))$ containing respectively $[M_0]$, $D'\chi_{\rho_0}$ such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{T}^{\mathrm{geod}}(N) \supset & V & \xrightarrow{D|_V} & D(V) & \subset \mathcal{T}^{\mathrm{sym}}(DN) \\
& \Big\downarrow{\scriptstyle \mathrm{Hol}'|_V} & & \Big\downarrow{\scriptstyle \mathrm{Hol}'|_{D(V)}} & \\
X^{\mathrm{geod}}(\pi_1(N)) \supset & Z & \xrightarrow{D'|_Z} & D'(Z) & \subset X^{\mathrm{sym}}(\pi_1(N)).
\end{array}$$

By Lemma 1.4.5 and Propositions 1.4.6, 1.4.10 we can assume the horizontal lines to be homeomorphisms, while Theorem 1.3.5 implies that we can also suppose that $\mathrm{Hol}'|_{D(V)}$ is a homeomorphism. This implies that $\mathrm{Hol}'|_V$ is a homeomorphism too, whence the conclusion. □

Thurston's hyperbolic Dehn filling Theorem Let now $\pi_\mu : \mathbb{C}^k \to \mathbb{C}^k$ be the map given by $\pi_\mu(x_1, \ldots, x_k) = (\epsilon_1 \cosh x_1, \ldots, \epsilon_k \cosh x_k)$, where $\epsilon_j = I_{\mu_j}(\chi_{\rho_0}) = \pm 2$ is the trace of $\rho_0(\mu_j)$. If $Z \subset X^{\mathrm{geod}}(\pi_1(N))$ is as in Theorem 1.4.11, then we can identify Z with a neighbourhood of $(\epsilon_1, \ldots, \epsilon_k)$ in $\mathbb{C}^k$ by the map $I_\mu : Z \to \mathbb{C}^k$ given by $I_\mu(z) = (I_{\mu_1}(z), \ldots, I_{\mu_k}(z))$. Let also $\pi_\lambda : \mathbb{C}^k \to \mathbb{C}^k$ be the map given by $\pi_\lambda(x_1, \ldots, x_k) = (\epsilon'_1 \cosh x_1, \ldots, \epsilon'_k \cosh x_k)$, where $\epsilon'_j = I_{\lambda_j}(\chi_{\rho_0}) = \pm 2$ is the trace of $\rho_0(\lambda_j)$.

Arguing in the very same way as in the preceding section we can now prove the following:

Theorem 1.4.12. *There exist a neighbourhood* U *of* 0 *in* $\mathbb{C}$ *and analytic maps* $v : U \to \mathbb{C}^k$, $\tau_1, \ldots, \tau_k : U \to \mathbb{C}$ *such that:*

- $\tau_j(0)$ *has positive imaginary part for* $i = 1, \ldots, k$*;*

- τ_j *is an even function of* u_l *for* $i, l = 1, \ldots, k$*;*

- $v_j(u) = \tau_j(u) \cdot u_j$ *for* $u \in U$, $i = 1, \ldots, k$*;*

- *For* $u \in U$ *a good structure* $M(u) \in \Omega^{\mathrm{geod}}(N)$ *exists such that the character of the lift of a holonomy representation* ρ_u *for* $M(u)$ *is equal to* $I_\mu^{-1}(\pi_\mu(u))$*;*

- *For* $u \in U$ *we have* $I_\lambda(\chi_{\rho_u}) = \pi_\lambda(v(u))$*;*

- *For any* $u \in U$, $i = 1, \ldots, k$ *we have* $v_j(u) = 0$ *if and only if* $u_j = 0$ *if and only if* $M(u)$ *induces a complete structure on* E_j*;*

- *Let* $d_j : U \to S^2$ *be the map defined as follows:* $d_j(u) = \infty$ *if* $u_j = 0$, *and* $d_j(u) = (p_j(u), q_j(u)) \in \mathbb{R}^2 \subset S^2$, *where* $p_j(u), q_j(u)$ *are the only real numbers with* $p_j(u) \cdot u_j + q_j(u) \cdot v_j(u) = 2\pi i$. *Then* d_j *is well-defined and the map* $d := (d_1, \ldots, d_k) : U \to S^2 \times \ldots \times S^2$ *is a diffeomorphism of* U *onto an open neighbourhood of* $(\infty, \ldots, \infty) \in S^2 \times \ldots \times S^2$*;*
- *If* $u \in U$ *and* $u_j \neq 0$ *then* $M(u)$ *induces a Dehn filling singularity on the end* E_j *of* N *with Dehn filling coefficient equal to* $p_j(u) \cdot \mu_j + q_j(u) \cdot \lambda_j$.

We conclude the section stating Thurston's hyperbolic Dehn filling Theorem for hyperbolic 3-manifolds with compact geodesic boundary.

Corollary 1.4.13 (Thurston's hyperbolic Dehn filling Theorem). *For each* $i = 1, \ldots, k$ *there exists a finite set* $\mathcal{S}_j$ *of slopes on* T_j *with the following property: if* $h \leqslant k$ *and* s_j *is a slope on* T_j *not belonging to* $\mathcal{S}_j$ *for* $i = 1, \ldots, k$, *then the filled manifold* $N(s_1, \ldots, s_h)$ *admits a complete finite-volume hyperbolic structure.*

Chapter 2

Geometric triangulations of hyperbolic 3-manifolds with geodesic boundary

The idea of constructing hyperbolic structures on manifolds by suitably gluing to each other geodesic polyhedra dates back to Thurston [Thu79]. In the case of cusped manifolds, this strategy has been systematically exploited by the software SnapPea [Sna]. In Sections 2.1, 2.2 and 2.3 we explain how this approach can be adapted to treat the case of non-empty geodesic boundary. In the setting of cusped manifolds one employs ideal tetrahedra, which are parameterized by complex numbers, and tries to solve consistency and completeness equations. In the bounded case one has to consider truncated tetrahedra, and moduli get more complicated, but basically the whole scheme extends. Most of the results described here already appeared in [FP04], and were used in [FMP04] to provide a census of all the hyperbolic manifolds with geodesic boundary which admit a topological ideal triangulation with at most four tetrahedra.

The set of solutions of the consistency equations naturally provides a nice deformation space for finite-volume hyperbolic structures with geodesic boundary on a fixed 3-manifold. Building on classical results in cohomology theory of representations, in Section 2.4 we prove that the complete structure is a smooth point of this deformation space and we explicitly describe local coordinates around it. This allows us to give a proof of Thurston's hyperbolic Dehn filling Theorem which applies to all the hyperbolic manifolds with geodesic boundary which admit a genuine geometric triangulation. It is actually conjectured that any complete finite-volume hyperbolic 3-manifold with geodesic boundary admits a genuine triangulation; moreover, any such manifold admits a *partially flat* triangulation, and following [PP00] we could probably adapt our proof of Thurston's hyperbolic Dehn filling Theorem to deal also with this kind of triangulations. This would give a complete and self-contained proof of the filling Theorem via deformation theory of geometric triangulations. However, to spell out in detail this approach we should go into several technicalities: namely, small deformations of a flat tetrahedron naturally give rise to flat, positively oriented and negatively oriented tetrahedra, so we should consider the issue of understanding when a triangulation with some negative tetrahedron defines a hyperbolic structure; moreover, we should look for a convenient parameterization of the isometry classes of non-positively oriented tetrahedra. Since all the hyperbolic manifolds explicitly described in this thesis admit genuine geometric triangulations, we will not develop the study of partially flat triangulations here.

It is maybe worth mentioning that the deformation variety defined by the consistency equations for a cusped 3-manifold without boundary has already been studied by several authors. In particular, in this context the smoothness at the complete structure was first claimed by Neumann and Zagier in [NZ85]. Choi has recently provided in [Cho04] a complete proof of the fact that in the cusped empty-boundary case the deformation variety is a smooth complex manifold at any point representing a non-degenerate (*i.e.* neither partially flat nor partially negatively oriented) geodesic ideal triangulation.

After *constructing* hyperbolic manifolds with boundary via moduli and equations, one is naturally faced with the issue of *recognizing* these manifolds, *i.e.* of understanding when different triangulations actually define different manifolds. It turns out that for both cusped and bounded manifolds a certain *canonical* decomposition exists, due to Epstein and Penner [EP88] in the former case and to Kojima [Koj90, Koj92] in the latter. One natural strategy to recognize a manifold decomposed into geometric pieces is then to modify the decomposition until the canonical one is reached. A detailed description of how this method works in the non-empty geodesic boundary case can be found in [FP04]. Here we describe in Section 2.6 the main properties of the Kojima decomposition, and we explain in Section 2.7 how the moduli of a given geometric triangulation can be used to determine if such triangulation is Kojima's canonical one.

2.1 Topological and geometric triangulations

In this section we prove some preliminary facts about the topology and geometry of a finite-volume orientable hyperbolic 3-manifold with geodesic boundary, explaining also what do we mean by a triangulation of such a manifold. Some interesting points emerge when the boundary of a finite-volume hyperbolic manifold is itself non-compact. However, since the main results of this thesis only concern hyperbolic manifolds with compact boundary, we do not deal here with the general case, which is treated in [FP04].

Topological restrictions Let N be a complete finite-volume orientable hyperbolic 3-manifold with *compact* geodesic boundary (from now on we will usually summarize all this information saying just that N *is hyperbolic*). We denote by DN the double of N, *i.e.* the manifold obtained by mirroring N in its boundary, and observe that within DN we have the surface ∂N which cuts DN into two isometric copies of N. Now DN is an orientable finite-volume hyperbolic 3-manifold without boundary, so it consists of a compact portion containing ∂N together with several cusps of the form $T \times [0, \infty)$, where T is the torus —see *e.g.* [Thu79, BP92]. This easily implies that N admits a natural compactification $\overline{N}$ obtained by adding some boundary tori.

Since the components of ∂N are hyperbolic surfaces we have the following:

Proposition 2.1.1. *The components of ∂N have negative Euler characteristic.*

The following restrictions are easily deduced from the well-known topological properties of hyperbolic manifolds without boundary (see [FP04] for a proof).

Proposition 2.1.2. *The compact manifold $\overline{N}$ is irreducible and geometrically atoroidal. Moreover $\overline{N}$ is boundary-incompressible and anannular.*

Thurston's Hyperbolization Theorem for Haken manifolds implies that the conditions in Propositions 2.1.1 and 2.1.2 are actually sufficient to ensure hyperbolicity of N, when N has non empty boundary (see *e.g.* [McM90, FP04]):

Proposition 2.1.3. *Let $\overline{M}$ be a compact orientable 3-manifold and let M be obtained from $\overline{M}$ by removing the toric components of $\partial\overline{M}$. Suppose that every component of $\partial M \neq \emptyset$ has negative Euler characteristic, and that $\overline{M}$ is irreducible, boundary-irreducible, geometrically atoroidal and anannular. Then M is hyperbolic.*

It is easily seen that Mostow-Prasad's rigidity Theorem applies also to hyperbolic manifolds with geodesic boundary:

Theorem 2.1.4. *On any given 3-manifold there exists, up to isometry, at most one complete finite-volume hyperbolic structure with geodesic boundary.*

PROOF: See [FP04, Fri04]. □

Partially truncated tetrahedra Recall that, when N is finite-volume non-compact hyperbolic and $\partial N = \emptyset$, it is typically possible to decompose N into pieces isometric to geodesic ideal tetrahedra in $\mathbb{H}^3$, and in practice the hyperbolic structure of N is constructed by first taking a topological ideal triangulation and then choosing the geometric shape of the tetrahedra so that their structures match under the gluings and give a complete structure on N. Our wish in the rest of this section is to extend the notion of ideal triangulation to the case of hyperbolic manifolds with geodesic boundary. We begin by describing the pieces into which manifolds will be decomposed, first topologically and then geometrically.

Definition 2.1.5. We call *partially truncated tetrahedron* a pair $(\Delta, \mathcal{I})$ where Δ is a tetrahedron and $\mathcal{I}$ is a set of vertices of Δ. The elements of $\mathcal{I}$ will be called *ideal vertices* for a reason to be explained soon. In the sequel we will always refer to Δ itself as a partially truncated tetrahedron, tacitly implying that a certain $\mathcal{I}$ is also fixed.

Definition 2.1.6. Given a partially truncated tetrahedron Δ we define its *topological realization* as the space Δ^* obtained by removing from Δ the ideal vertices and small open stars of the non-ideal vertices. We will call *lateral hexagon* and *truncation triangle* the intersection of Δ^* respectively with a face of Δ and with the link in Δ of a non-ideal vertex. The edges of the truncation triangles, which also belong to the lateral hexagons, will be called *boundary edges*. The other edges of the lateral hexagons will be called *internal edges*.

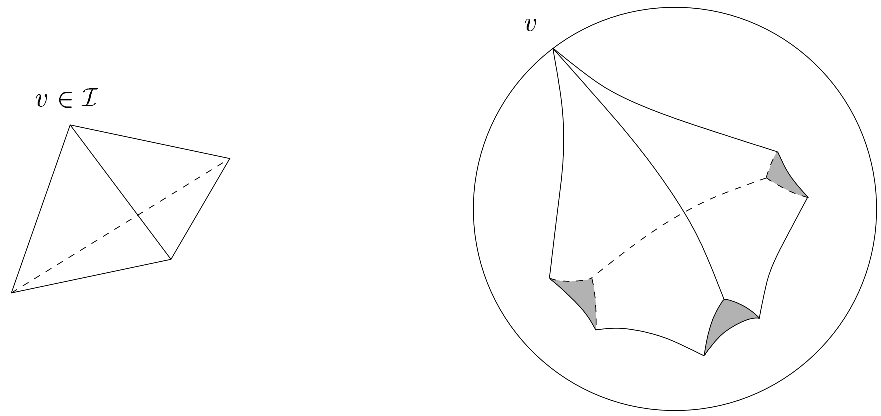

Figure 2.1: A geometric tetrahedron with one ideal vertex.

Note that, if Δ has ideal vertices, a lateral hexagon of Δ^* may not quite be a hexagon, because some of its (closed) edges may be missing.

Definition 2.1.7. Given a partially truncated tetrahedron Δ we call *geometric realization* of Δ an embedding of Δ^* in $\mathbb{H}^3$ such that:

1. The truncation triangles are geodesic triangles;
2. The lateral hexagons are geodesic polygons, with ideal vertices corresponding to missing edges;
3. Truncation triangles and lateral hexagons lie at right angles to each other.

An example of geometric realization is shown in Fig. 2.1, where truncation triangles are shadowed.

Triangulations In the language introduced above, the classical notion of *ideal triangulation* of a compact 3-manifold with boundary is a realization of the interior of the manifold as a gluing of some Δ^*'s, where the corresponding Δ's have all ideal vertices and the gluing is induced by a simplicial pairing of the faces of the Δ's. We can now easily extend this notion to the situation we are interested in.

Definition 2.1.8. Let $\overline{N}$ be a compact orientable manifold and let N be obtained from $\overline{N}$ by removing the toric components of $\partial\overline{N}$. We define a *partially truncated triangulation* of N to be a realization of N as a gluing of some Δ^*'s along a pairing of the lateral hexagons induced by a simplicial pairing of the faces of the Δ's.

Remark 2.1.9. 1. Under the pairing of the faces of the Δ's, ideal vertices are matched to each other.

2. The truncation triangles of the Δ^*'s give a triangulation of ∂N.
3. The links of the ideal vertices of the Δ's give a triangulation of the toric components of $\partial\overline{N}$.

Definition 2.1.10. Let N as above be endowed with a hyperbolic structure. A partially truncated triangulation of N is called *geometric* if, for each tetrahedron Δ of the triangulation, the pull-back to Δ^* of the Riemannian metric of N defines a geometric realization of Δ. Equivalently, the hyperbolic structure of N should be obtained by gluing geometric realizations of the Δ's along isometries of their lateral hexagons.

In Section 2.6 we will briefly describe Kojima's result [Koj90] according to which every hyperbolic N as above has a *canonical* decomposition into partially truncated *polyhedra*, rather than tetrahedra. Just as it happens with the Epstein-Penner decomposition [EP88] of non-compact manifolds with empty boundary, in the vast majority of cases the Kojima decomposition actually consists of tetrahedra, or at least can be subdivided into a geometric partially truncated triangulation. For instance, it is proved in [FMP04] that there exist exactly 5192 hyperbolic manifolds with non-empty geodesic boundary which can be (topologically) triangulated by at most four partially truncated tetrahedra: their Kojima decomposition can always be subdivided into a triangulation, and is itself a triangulation in 5108 cases.

The following definition will prove useful later:

Definition 2.1.11. A partially truncated triangulation $\mathcal{T}$ of an orientable 3-manifold N is *good* if any tetrahedron in $\mathcal{T}$ has at most one ideal vertex.

As we shall see later, any polyhedron in the Kojima decomposition of a hyperbolic N has at most one ideal vertex, so Kojima decompositions which can be subdivided into triangulations are examples of good triangulations.

2.2 Moduli and equations for partially truncated triangulations

In this section we introduce moduli for the geometric realizations of a partially truncated tetrahedron, and we describe the equations which ensure that a gluing of geometric tetrahedra gives rise to a (not necessarily complete) hyperbolic structure with geodesic boundary.

Moduli for partially truncated tetrahedra The following result implies that the dihedral angles can be used as moduli for geometric tetrahedra.

Theorem 2.2.1. *Let Δ be a partially truncated tetrahedron and let $\Delta^{(1)}$ be the set of edges of Δ. The geometric realizations of Δ are parameterized up to isometry by the functions $\theta : \Delta^{(1)} \to (0, \pi)$ such that for each vertex v of Δ, if e_1, e_2, e_3 are the edges that emanate from v, then $\theta(e_1) + \theta(e_2) + \theta(e_3)$ is equal to π for ideal v and less than π for non-ideal v.*

The map θ corresponding to a geometric realization Δ^* associates to each edge e the dihedral angle $\theta(e)$ of Δ^* along e.

Proof: See [Fuj90, FP04]. □

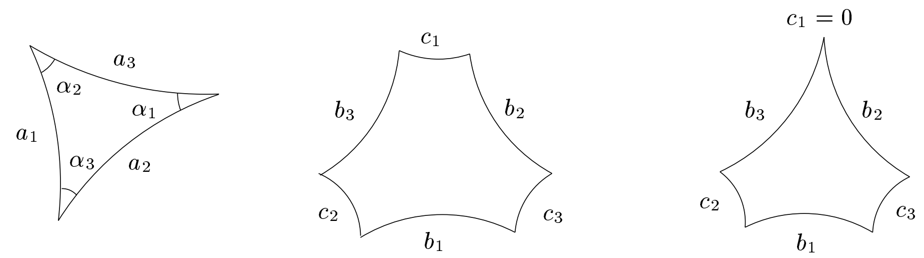

Figure 2.2: A triangle, a hexagon and a degenerate hexagon.

Having introduced moduli for geometric tetrahedra, our next task is to determine, given a triangulated orientable manifold, which values of moduli define a global hyperbolic structure on the manifold. To this aim we first compute the lengths of internal and boundary edges in terms of the dihedral angles.

From angles to lengths We begin by recalling two well-known results in hyperbolic trigonometry.

Lemma 2.2.2. *Let $\alpha_1, \alpha_2, \alpha_3$ and a_1, a_2, a_3 be respectively the angles and the lengths of edges of the hyperbolic triangle in Fig. 2.2-left. Then we have*

$$\cosh a_1 = \frac{\cos\alpha_2 \cdot \cos\alpha_3 + \cos\alpha_1}{\sin\alpha_2 \cdot \sin\alpha_3}.$$

PROOF: See [Bea95, The Cosine Rule II, pag. 148]. □

Lemma 2.2.3. *Let $b_1, c_1, b_2, c_2, b_3, c_3$ be the lengths of edges of the hyperbolic right-angled hexagon in Fig. 2.2-centre. Then we have*

$$\cosh b_1 = \frac{\cosh c_2 \cdot \cosh c_3 + \cosh c_1}{\sin c_2 \cdot \sinh c_3}.$$

This relation is still true when $c_1 = 0$, i.e. *when the hexagon degenerates into a pentagon with four right angles and one ideal vertex (see Fig. 2.2-right).*

PROOF: See [Bea95, Theorem 7.19.2, pag. 161]. □

Let now Δ be a partially truncated tetrahedron with edges $e_1, \ldots, e_6$ as in Fig. 2.3. We fix a geometric realization θ of Δ determined by the dihedral angles $\theta_i = \theta(e_i)$ for $i = 1, \ldots, 6$ and we denote by L^θ the length with respect to this realization. The boundary edges of the lateral hexagons of Δ correspond to the pairs of distinct non-opposite edges $\{e_i, e_j\}$, and will be denoted by e_{ij}. Note that e_{ij} disappears towards infinity, so it has length 0, when the common vertex of e_i and e_j is ideal. Lemma 2.2.2 readily implies

$$\cosh L^\theta(e_{12}) = \frac{\cos\theta_1 \cdot \cos\theta_2 + \cos\theta_3}{\sin\theta_1 \cdot \sin\theta_2}. \tag{2.1}$$

Note that this result is correct (and obvious) also when the common end of e_1 and e_2 is ideal.

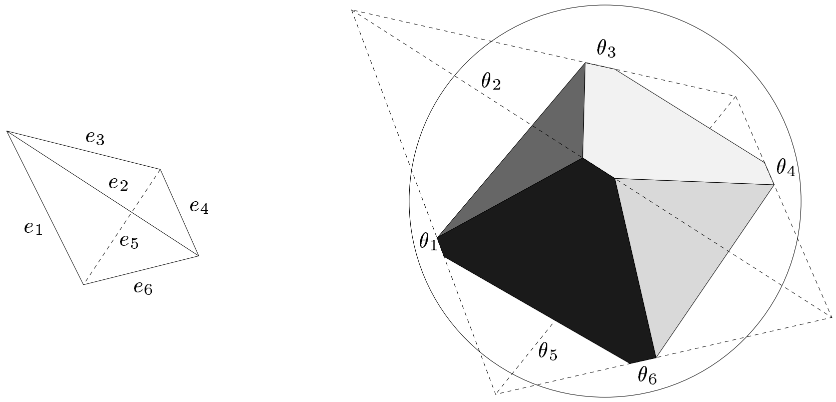

Figure 2.3: Notation for edges and dihedral angles of a truncated tetrahedron. The geometric figure on the right uses the projective model of $\mathbb{H}^3$, as explained below in Section 2.6.

Turning to the length of an internal edge, we note that the edge is an infinite half-line or an infinite line when one or both its ends are ideal, respectively. Otherwise the length is computed using Lemma 2.2.2 and Lemma 2.2.3. To state the result of the computation we need to introduce certain functions that will be used again below. With notation as in Fig. 2.3, and defining v_{ijk} as the vertex from which the edges e_i, e_j, e_k emanate, we set:

$$\begin{aligned} c^\theta(e_1) &= \cos\theta_1 \cdot (\cos\theta_3 \cdot \cos\theta_6 + \cos\theta_2 \cdot \cos\theta_5) \\ &\quad + \cos\theta_2 \cdot \cos\theta_6 + \cos\theta_3 \cdot \cos\theta_5 + \cos\theta_4 \cdot \sin^2\theta_1; \end{aligned} \tag{2.2}$$

$$d^\theta(v_{123}) = 2\cos\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3 + \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 - 1. \tag{2.3}$$

Using the combinatorial symmetries of the tetrahedron we can easily use these formulae to compute the value of c^θ and d^θ respectively on any internal edge and any vertex of Δ.

Remark 2.2.4. Using Lemma 2.2.2 it is easily seen that $d^\theta(v_{123}) = \sin^2\theta_1 \cdot \sin^2\theta_2 \cdot \sinh^2 L^\theta(e_{12})$.

Lemma 2.2.5. *$d^\theta(v_{123}) = 0$ if and only if the vertex v_{123} is ideal.*

Proposition 2.2.6. *If v_{123} and v_{156} are both non-ideal then*

$$\cosh L^\theta(e_1) = c^\theta(e_1) \Big/ \sqrt{d^\theta(v_{123}) \cdot d^\theta(v_{156})}. \tag{2.4}$$

Conditions for geometric gluing Let N be obtained from an orientable compact $\overline{N}$ by removing all the tori in $\partial\overline{N}$ and let $\mathcal{T}$ be a partially truncated triangulation of N. Let also θ be a geometric realization of the tetrahedra in $\mathcal{T}$ and denote by L^θ the length with respect to this realization. We now describe the conditions under which the realization θ defines a hyperbolic structure on the whole of N. For our purposes it will be sufficient to deal with *good* triangulations only, so we assume from now on that $\mathcal{T}$ is good. The general case is treated in [FP04].

In order to define a global hyperbolic structure on N, the tetrahedra of $\mathcal{T}$ must satisfy two obvious necessary conditions, which in fact are also sufficient. Namely, we should be able to glue the lateral hexagons by isometries, and we should have a total dihedral angle of 2π around each edge of the manifold. The first condition ensures that the hyperbolic structure defined by θ on the complement of the 2-skeleton of $\mathcal{T}$ extends to the complement of the 1-skeleton. Since $\mathcal{T}$ is good, the second one ensures that such structure glues up without singularities also along the edges. The second condition is directly expressed in terms of moduli, and we will explain in a moment how to translate the first one into an equation on dihedral angles.

Remark 2.2.7. If $\mathcal{T}$ were not good, requiring the dihedral angles around each edge to sum up to 2π would not be sufficient to obtain a non-singular hyperbolic metric on the 1-skeleton of $\mathcal{T}$. The point is that when some geometric tetrahedra are arranged one after the other around an edge e with two ideal endpoints, the first face of the first tetrahedron and the second face of the last tetrahedron may overlap without coinciding. Namely, the isometry which pairs these two faces may be a translation along e instead of being the identity. Of course the isometry has to be the identity if at least one endpoint of e is not ideal.

Consistency equations Since $\mathcal{T}$ is good, Lemma 2.2.3 readily implies that the isometry class of a lateral hexagon of a tetrahedron in $\mathcal{T}$ is completely determined by the lengths of its boundary edges, which are computed in equation (2.1) in terms of the dihedral angles of the tetrahedra. This means we can write down a set of explicit algebraic conditions on the moduli which are satisfied if and only if the parameterization θ defines a hyperbolic structure on the whole of N. Let us define for an internal edge e

$$\alpha^{\theta}(e) = \sum\{\theta(e') : \ e' \text{ is glued to } e\}. \tag{2.5}$$

The above discussion is summarized by the following:

Theorem 2.2.8. *The geometric realization θ defines on N a hyperbolic structure with geodesic boundary if and only if the following conditions hold:*

1. *$\alpha^{\theta}(e) = 2\pi$ for all internal edges e;*
2. *$L^{\theta}(e) = L^{\theta}(e')$ for all pairs (e, e') of matching boundary edges.*

An easy computation shows that the number of equations arising from the conditions stated in Theorem 2.2.8 is much bigger than the number of parameters describing the geometric realization of the tetrahedra of $\mathcal{T}$. Thus it seems quite reasonable to look for a smaller set of equations which still would ensure the geometric structure on the tetrahedra to glue up giving a global hyperbolic metric on N. First of all, we recall that by default gluings match ideal vertices to each other, because these notions are part of the initial topological information about a triangulation. When a pairing glues two *compact* hexagons, *i.e.* when there are no ideal vertices involved, to be sure that the gluing is an isometric one we may equivalently require the lengths of the internal edges or those of the boundary edges to match under the

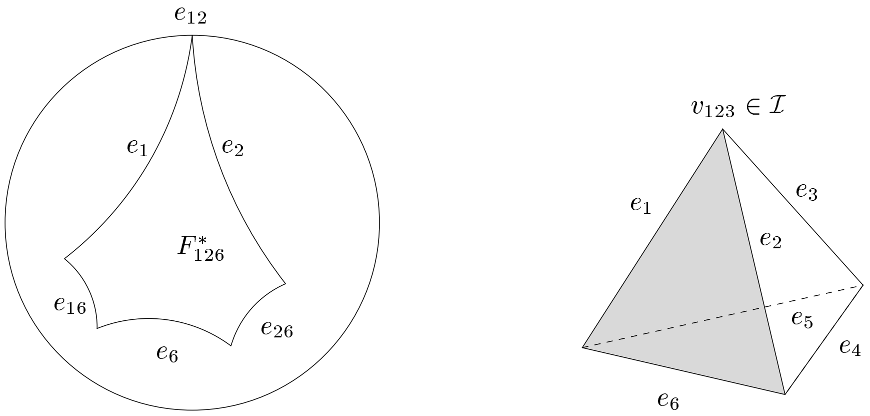

Figure 2.4: An exceptional hexagon.

gluing. On the other hand, since $\mathcal{T}$ is a good triangulation, a non-compact lateral hexagon F is actually a pentagon with four right angles and an ideal vertex: we shall say in this case that F is an *exceptional* lateral hexagon. As we have seen before, (the isometry class of) an exceptional lateral hexagon is determined by the lengths of its boundary edges. In the following paragraph we describe an alternative parameterization of exceptional hexagons which will prove useful later.

Exceptional hexagons To parameterize exceptional hexagons we need now to be slightly more careful about orientation than we have been so far. Namely, we choose on the tetrahedra an orientation compatible with a global orientation of the manifold. As a result also the lateral hexagons and the truncation triangles have a fixed orientation, and the gluing maps reverse the orientation of the hexagons.

So, let us consider an exceptional hexagon F^*_{126} as in Fig. 2.4, and recall that the hexagon is oriented and embedded in $\mathbb{H}^3$ by θ. We consider now the horospheres O_1 and O_2 centred at e_{12} and passing through the non-ideal ends of e_1 and e_2 respectively. We define $\sigma^\theta(F_{126})$ to be $\pm\mathrm{dist}(O_1, O_2)$, the sign being positive if e_2, e_{12}, e_1 are arranged positively on ∂F^*_{126} and O_1 is contained in the horoball bounded by O_2, or if e_2, e_{12}, e_1 are arranged negatively on ∂F^*_{126} and O_2 is contained in the horoball bounded by O_1, and negative otherwise.

Lemma 2.2.9. *We have* $\sigma^\theta(F_{126}) = \ln\sinh L^\theta(e_{16}) - \ln\sinh L^\theta(e_{26})$.

PROOF: Let α_1, α_2 and R, R_1, R_2 be the angles and lengths shown in Fig. 2.5. An easy computation in the upper half-plane model of $\mathbb{H}^2$ shows that $L^\theta(e_{i6}) = \ln\cot(\alpha_i/2)$ for $i = 1, 2$. Moreover we have $R = R_1 \cdot \tan\alpha_1 = R_2 \cdot \tan\alpha_2$, so $\exp(\sigma^\theta(F_{126})) = R_1/R_2 = \tan\alpha_2/\tan\alpha_1$. Now for $i = 1, 2$ we have $\tan\alpha_i = (2\cot(\alpha_i/2))/(\cot^2(\alpha_i/2) - 1) = (2\exp L^\theta(e_{i6}))/(\exp(2L^\theta(e_{i6})) - 1)$. Combining these equations we finally get $\exp(\sigma^\theta(F_{126})) = \sinh L^\theta(e_{16})/\sinh L^\theta(e_{26})$, and we are done. $\square$

Using equation (2.1) and Lemma 2.2.9 we can easily compute σ in terms of the dihedral angles:

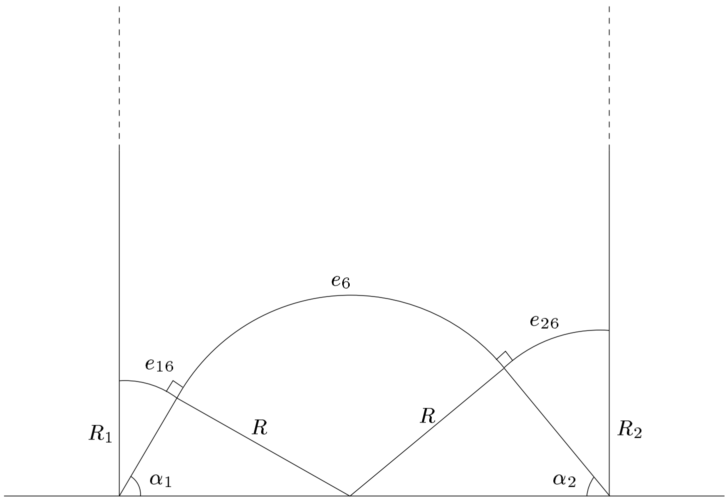

Figure 2.5: Notation for the proof of Lemma 2.2.9.

Proposition 2.2.10. *We have*

$$\sigma^\theta(F_{126}) = (1/2)\ln(d^\theta(v_{156})/d^\theta(v_{246})) + \ln(\sin\theta_2/\sin\theta_1), \tag{2.6}$$

where d^θ is the function defined in equation (2.3).

We now define $\ell^\theta(F_{126})$ to be the length of e_6. The next proposition shows that the functions σ and ℓ provide a parameterization of isometry classes of exceptional hexagons.

Proposition 2.2.11. *Let F and F' be paired exceptional lateral hexagons. Their pairing can be realized by an isometry if and only if $\sigma^\theta(F)+\sigma^\theta(F') = 0$ and $\ell^\theta(F) = \ell^\theta(F')$.*

PROOF: We concentrate on the "if" part of the statement, the "only if" part being obvious.

Let f_1, f_2 and f'_1, f'_2 be the boundary edges of of F and F' respectively, and assume that the orientation-reversing pairing between F and F' glues f_1 to f'_2 and f'_1 to f_2. Since $\ell^\theta(F) = \ell^\theta(F')$, Lemma 2.2.3 easily implies that $L(f_1) \leqslant L(f'_2)$ if and only if $L(f_2) \geqslant L(f'_1)$. Moreover the assumption that $\sigma^\theta(F) = -\sigma^\theta(F')$ and Lemma 2.2.9 give $L(f_1) \leqslant L(f'_2)$ if and only if $L(f_2) \leqslant L(f'_1)$. This forces $L(f_1) = L(f'_2)$ and $L(f_2) = L(f'_1)$. □

Consistency equations revisited Recall now that we are considering a candidate hyperbolic 3-manifold N endowed with a good triangulation $\mathcal{T}$, and that we have fixed a geometric realization θ of the tetrahedra in $\mathcal{T}$.

Theorem 2.2.12. *The parameterization θ defines on N a hyperbolic structure with geodesic boundary if and only if the following conditions hold:*

1. $\alpha^\theta(e) = 2\pi$ *for all internal edges* e*;*

2. $L^\theta(e) = L^\theta(e')$ *for all pairs* (e, e') *of matching internal edges;*

3. $\sigma^\theta(F) + \sigma^\theta(F') = 0$ *for all pairs* (F, F') *of matching exceptional hexagons.*

PROOF: Since the isometry class of a compact hexagon is determined by the lengths of its internal edges, condition (2) implies that matching compact hexagons are isometric. Moreover Proposition 2.2.11 and condition (3) ensure that also matching exceptional hexagons can be glued by isometries. □

Computing the number of equations Let N be obtained from an orientable compact $\overline{N}$ by removing all the tori in $\partial\overline{N}$ and let $\mathcal{T}$ be a good triangulation of N. From now on we denote by $\mathcal{C}(\mathcal{T})$ the set of consistency equations described in Theorem 2.2.12. Our next task is to compare the number of equations in $\mathcal{C}(\mathcal{T})$ with the dimension of the moduli space of geometric realizations of the tetrahedra of $\mathcal{T}$.

Let t^C (resp. t^I) be the number of compact (resp. non-compact) tetrahedra of $\mathcal{T}$. Theorem 2.2.1 readily implies that the number of parameters for the geometric realizations of $\mathcal{T}$ is equal to $6t^C + 5t^I$. Let now $e_1, \ldots, e_k$ be the edges of $\mathcal{T}$ without ideal endpoints, and for any $i = 1, \ldots, k$ let x_i be the valence of e_i, *i.e.* the number of tetrahedra of $\mathcal{T}$ incident to e_i, with multiplicity. Of course we have $\sum_{i=1}^k x_i = 6t^C + 3t^I$. If we denote by j the number of edges of $\mathcal{T}$ with exactly one ideal endpoint, the union of the boundary tori of $\overline{N}$ admits a triangulation with j vertices and t^I triangles, so $j = t^I/2$. Let us now turn to the number of equations. Theorem 2.2.12 determines $k+j$ equations arising from conditions (1), $\sum_{i=1}^k (x_i - 1)$ equations arising from conditions (2) and $3t^I/2$ equations arising from conditions (3). Since

$$\begin{array}{rlcl} & k + j + \left(\sum_{i=1}^k (x_i - 1)\right) + 3t^I/2 & = & j + \left(\sum_{i=1}^k x_i\right) + 3t^I/2 \\ = & t^I/2 + 6t^C + 3t^I + 3t^I/2 & = & 6t^C + 5t^I, \end{array}$$

we get the following:

Proposition 2.2.13. *Given a manifold* N *with a good triangulation* $\mathcal{T}$*, the number of equations in* $\mathcal{C}(\mathcal{T})$ *is equal to the dimension of the moduli space of geometric realizations of the tetrahedra of* $\mathcal{T}$*.*

Reducing the number of equations When N has cusps, some equations in $\mathcal{C}(\mathcal{T})$ turn out to be actually redundant. Let T be a fixed toric component of $\partial\overline{N}$ and let t be the number of tetrahedra asymptotic to T. Since any such tetrahedron contributes to the triangulation of T with a Euclidean triangle, the sum of all the dihedral angles along all the edges of $\mathcal{T}$ incident to T is equal to $t\pi$. So if we require condition (1) of Theorem 2.2.12 to hold for *all but one* edge incident to T, then the same condition is automatically satisfied also along the remaining edge. This allows us to discard from $\mathcal{C}(\mathcal{T})$ one equation for each cusp of N.

Moreover, let Δ be a partially truncated tetrahedron with an ideal vertex v incident to T, and let F_1, F_2, F_3 be the faces of Δ incident to v. By the very definition of σ it follows that $\sigma(F_1) + \sigma(F_2) + \sigma(F_3) = 0$. This implies that if $\mathcal{F}$ is the set of all the faces incident to T of tetrahedra of $\mathcal{T}$, then we have $\sum_{F\in\mathcal{F}}\sigma(F) = 0$. So if we require condition (3) of Theorem 2.2.12 to hold for *all but one* pair of matching exceptional hexagons incident to T, then the same condition also holds for the remaining pair. This means that another equation of $\mathcal{C}(\mathcal{T})$ for each cusp of N can be discarded.

The above discussion is summarized by the following:

Proposition 2.2.14. *Let N compactify to an orientable $\overline{N}$ with k boundary tori and let $\mathcal{T}$ be a good triangulation of N. Then we can discard $2k$ equations from $\mathcal{C}(\mathcal{T})$ thus obtaining an equivalent set $\mathcal{C}^*(\mathcal{T})$ of $n-2k$ equations, where n is the dimension of the moduli space of geometric realizations of the tetrahedra of $\mathcal{T}$.*

2.3 Completeness

Let t be the total number of tetrahedra of $\mathcal{T}$. Fixing an ordering on the set of edges of these tetrahedra, we can identify the moduli space of their geometric realizations with a subset of $\mathbb{R}^{6t}$. Let $\Omega(\mathcal{T}) \subset \mathbb{R}^{6t}$ be the set of solutions of the consistency equations $\mathcal{C}^*(\mathcal{T})$. We now determine the equations on $x \in \Omega(\mathcal{T})$ representing the condition that the hyperbolic structure associated to x is complete. Our discussion here follows [Thu79, BP92, Rat94].

The geometric structure on the boundary tori We begin by recalling that the group of orientation-preserving similarities of the real plane coincides, under the canonical identification $\mathbb{R}^2 = \mathbb{C}$, with the group $\mathrm{Aff}(\mathbb{C})$ of affine automorphisms of the complex line $\mathbb{C}$. We call *similarity structure* on a torus an $(\mathbb{C}, \mathrm{Aff}(\mathbb{C}))$-structure, according to the general definition given in Section 1.1. A similarity structure is *Euclidean* if it contains an $(\mathbb{R}^2, \mathrm{Isom}^+(\mathbb{R}^2))$-structure.

Fix now a point $x \in \Omega(\mathcal{T})$ and for any tetrahedron Δ in $\mathcal{T}$ let Δ^*_x be the geometric realization of Δ corresponding to x. Let v be an ideal vertex of Δ^*_x and choose a horosphere O_v based at v that meets only the faces of Δ^*_x incident to v. The *link* of v in Δ^*_x is the set $L_x(v) = O_v \cap \Delta^*_x$. Note that $L_x(v)$ is a compact Euclidean triangle in the horosphere O_v, and that the similarity type of $L_x(v)$ does not depend on the choice of O_v. Suppose now that N is obtained from a compact manifold $\overline{N}$ by removing all the tori $T_1, \ldots, T_k$ in $\partial\overline{N}$, and for any $i = 1, \ldots, k$ consider the family $\mathcal{L}^i_x = \{L_x(v) : \ v \text{ is an ideal vertex of } \mathcal{T} \text{ asymptotic to } T_i\}$. Since $x \in \Omega(\mathcal{T})$ we have that triangles in $\mathcal{L}^i_x$ can be nicely glued together along the edges, thus defining a similarity structure on the torus T_i. The following result is proved in [Thu79, BP92, Rat94].

Proposition 2.3.1. *Let $E_i \cong T_i \times [0, \infty)$ be the end of N corresponding to T_i. Then the hyperbolic structure associated with $x \in \Omega(\mathcal{T})$ induces on E_i a complete metric if and only if the similarity structure induced on T_i is Euclidean.*

Since N is the union of $E_1 \cup \ldots \cup E_k$ and a compact set, we readily have the following:

Corollary 2.3.2. *The hyperbolic structure associated to $x \in \Omega(\mathcal{T})$ is complete if and only if all the similarity structures induced by x on the T_i's are Euclidean.*

We are now going to translate the equivalent conditions just described into equations on moduli. Let us recall that the similarity structure on T_i associated to $x \in \Omega(\mathcal{T})$ induces a holonomy homomorphism $\rho_x^i : \pi_1(T_i) \to \mathrm{Aff}(\mathbb{C})$, well-defined up to conjugation.

Lemma 2.3.3. *The similarity structure induced by $x \in \Omega(\mathcal{T})$ on T_i is Euclidean if and only if $\rho_x^i(\pi_1(T_i))$ consists of translation.*

PROOF: By Theorem 1.1.3 and Proposition 1.1.4, if T_i is Euclidean then $\rho_x^i(\pi_1(T_i))$ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{R}^2)$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. This easily implies that $\rho_x^i(\pi_1(T_i))$ consists of translations. On the other hand, if $\rho_x^i(\pi_1(T_i))$ consists of translations then the similarity structure on T_i reduces to a Euclidean one by Proposition 1.1.6. $\square$

As stated above, the group of orientation-preserving similarities of $\mathbb{R}^2$ is canonically identified with $\mathrm{Aff}(\mathbb{C})$. If φ is the element in $\mathrm{Aff}(\mathbb{C})$ such that $\varphi(z) = az + b$ for any $z \in \mathbb{C}$, we say that a is the *dilation coefficient* of φ. Note that the dilation coefficient is constant on conjugacy classes of elements in $\mathrm{Aff}(\mathbb{C})$, so the map

$$\delta_x^i : \pi_1(T_i) \to \mathbb{C}^*, \quad \delta_x^i(\gamma) = \text{dilation coeff. of } \rho_x^i(\gamma) \tag{2.7}$$

gives a well-defined homomorphism (not only a conjugacy class of homomorphisms).

For any $i = 1, \ldots, k$ we now choose (μ_i, λ_i) to be a positively oriented basis of $\pi_1(T_i) \cong H_1(T_i)$ and we choose representatives of μ_i, λ_i which are simple simplicial oriented loops with respect to the triangulation induced by $\mathcal{T}$ on T_i. With a slight abuse, we denote such representatives by μ_i and λ_i too. The following proposition is just a re-statement of Corollary 2.3.2 and Lemma 2.3.3.

Proposition 2.3.4. *An element $x \in \Omega(\mathcal{T})$ represents a complete structure on N if and only if $\delta_x^i(\mu_i) = \delta_x^i(\lambda_i) = 1$ for $i = 1, \ldots, k$.*

Our next task is to translate conditions arising from Proposition 2.3.4 into explicit equations in x.

Horospherical cross-sections Let Δ be a tetrahedron in $\mathcal{T}$, let v be an ideal vertex of Δ and $L_x(v)$ be the (similarity class of the) Euclidean triangle obtained by intersecting the geometric realization of Δ parameterized by x with a small horosphere centred at v. The tetrahedron being oriented, this triangle is also oriented, so, once a vertex p of $L_x(v)$ is fixed, we can associate to the similarity structure of $L_x(v)$ the unique complex number $z_x(L(v), p)$ such that $L_x(v)$ is carried to the Euclidean triangle with vertices $0, 1, z_x(L(v), p)$ by an orientation-preserving similarity sending p to 0. Note that $z_x(L(v), p)$ has positive imaginary part, *i.e.* it lies in the upper half-plane $\mathbb{H} \subset \mathbb{C}$.

Proposition 2.3.5. *For an oriented tetrahedron as in Fig. 2.3, assume v_{123} is ideal and e_1, e_2, e_3 are positively arranged around v_{123}. Let also $L(v_{123})$ be the intersection of the tetrahedron with a small horosphere centred at v_{123}, and set $p = L(v_{123}) \cap e_1$. Then:*

$$z_x(L(v_{123}), p) = \frac{\sin\theta_2}{\sin\theta_3} \cdot e^{i\theta_1}. \tag{2.8}$$

Completeness equations According to Proposition 2.3.4, we need now to compute the value of δ^i_x on μ_i and λ_i in terms of x. Let us fix some notation: if γ is an oriented simplicial loop on T_i and p is a vertex of γ, then the set of all triangles touching γ in p and lying on the right of γ is well-defined and will be denoted by $R(\gamma,p)$. Moreover, we shall denote by $V(\gamma)$ the set of vertices of γ and by γ itself the element of $\pi_1(T_i)$ represented by the loop γ. The following result is proved in [Thu79, BP92].

Proposition 2.3.6. *We have*

$$\delta^i_x(\gamma) = (-1)^{\#V(\gamma)} \cdot \prod_{p\in V(\gamma)} \prod_{T\in R(\gamma,p)} z_x(T,p). \tag{2.9}$$

Putting together Propositions 2.3.6 and 2.3.4 we finally get the following:

Theorem 2.3.7. *A point $x \in \Omega(\mathcal{T})$ represents a complete structure on N if and only if we have*

$$(-1)^{\#V(\mu_i)} \cdot \prod_{p\in V(\mu_i)} \prod_{T\in R(\mu_i,p)} z_x(T,p) = (-1)^{\#V(\lambda_i)} \cdot \prod_{p\in V(\lambda_i)} \prod_{T\in R(\lambda_i,p)} z_x(T,p) = 1$$

for $i = 1,\ldots,k$. Moreover, by Proposition 2.3.5 we can explicitly compute $z_x(T,p)$ in terms of x for any pair (T,p) where T is a triangle in the triangulation of a toric component of $\partial\overline{N}$ and p is a vertex of T.

Uniqueness As an application of Theorem 2.1.4 we get that a solution of consistency and completeness equations, if any, is unique:

Theorem 2.3.8. *Given a triangulated N as in Theorems 2.2.8 and 2.2.12, there exists at most one choice of θ that turns N into a complete manifold with geodesic boundary.*

Proof: See [FP04]. □

2.4 Smoothness at the complete structure

Let now suppose that x_0 is the *unique* point in $\Omega(\mathcal{T})$ which satisfies consistency and completeness equations. For $i = 1,\ldots,k$ and $x \in \Omega(\mathcal{T})$ let us define

$$u_i(x) = \ln \delta^i_x(\mu_i), \qquad v_i(x) = \ln \delta^i_x(\lambda_i), \tag{2.10}$$

where (μ_i,λ_i) is a positively oriented basis for $\pi_1(T_i)$, the map $\delta^i_x : \pi_1(T_i) \to \mathbb{C}^*$ is defined in equation (2.7), and ln is the branch of the complex logarithm defined on $\{z \in \mathbb{C} : \Re(z) > 0\}$ such that $\ln 1 = 0$.

Lemma 2.4.1. *Let $U = \{x \in \Omega(\mathcal{T}) : \Re(\delta^i_x(\mu_i)) > 0,\ \Re(\delta^i_x(\lambda_i)) > 0,\ \rho^i_x(\mu_i) \neq \mathrm{Id}_\mathbb{C},\ \rho^i_x(\lambda_i) \neq \mathrm{Id}_\mathbb{C}\}$. Then for any $x \in U$ the following conditions are equivalent:*

1. $u_i(x) = 0$;

2. $v_i(x) = 0$;

3. *The hyperbolic structure corresponding to x induces a complete metric on the end of N corresponding to T_i.*

PROOF: (1) $\iff$ (2): If $u_i(x) = 0$, then $\delta^i_x(\mu_i) = 1$ and $\rho^i_x(\mu_i)$ is a translation. Since an element of $\mathrm{Aff}(\mathbb{C})$ commuting with a non-trivial translation is itself a translation we have $\delta^i_x(\lambda_i) = 1$, so $v_i(x) = 0$. After switching the rôles of μ_i and λ_i, the very same argument also gives that $v_i(x) = 0$ implies $u_i(x) = 0$.

$((1) \wedge (2)) \implies (3)$: If $u_i(x) = v_i(x) = 0$ then $\rho^i_x(\pi_1(T_i))$ consists of translations, so by Lemma 2.3.3 x induces a Euclidean structure on T_i, whence a complete hyperbolic structure on the end corresponding to T_i (see Proposition 2.3.1).

$(3) \implies ((1) \wedge (2))$: The hypothesis implies that $\rho^i_x(\pi_1(T_i))$ consists of translations, so $u_i(x) = v_i(x) = 0$. □

Recall now that the moduli space of geometric realizations of a partially truncated tetrahedron Δ is given by an open convex subset of $\mathbb{R}^6$ if Δ has no ideal vertices, or by an open convex subset of an affine hyperplane of $\mathbb{R}^6$ if Δ has one ideal vertex. So the moduli space of geometric realizations of tetrahedra in $\mathcal{T}$ is identified with a subset $\mathcal{W}$ of $\mathbb{R}^{6t}$, where t is the number of tetrahedra of $\mathcal{T}$. More precisely, $\mathcal{W}$ is an open convex subset of an affine subspace of dimension equal to $6t-c$, where c is the number of ideal vertices of tetrahedra in $\mathcal{T}$. Reduced consistency equations $\mathcal{C}^*(\mathcal{T})$ can be divided into two groups, the first one forcing the matching faces to be isometric to each other and the second one forcing the total dihedral angles along all internal edges to be equal to 2π. If e is the number of compact edges of $\mathcal{T}$, then equations of the first type are in number of $6t - e - (3/2)c - k$, while equations of the second type are in number of $e + c/2 - k$ (see Proposition 2.2.14). As described in Proposition 2.2.12, the solutions of these equations can be represented as follows: the angles satisfying the conditions of the first type are given by the zero locus of a smooth function $F : \mathcal{W} \to \mathbb{R}^{6t-e-(3/2)c-k}$, while the angles satisfying the conditions of the second type are given by the zero locus of a smooth function $A : \mathcal{W} \to \mathbb{R}^{e+c/2-k}$. Any point $x \in F^{-1}(0)$ canonically corresponds to a cone structure on N having cone singularities along the edges of $\mathcal{T}$, so we define $\Omega_{\mathrm{cone}}(\mathcal{T}) = F^{-1}(0)$ to be the space parameterizing these cone structures. Of course we have $\Omega_{\mathrm{cone}}(\mathcal{T}) \supset \Omega(\mathcal{T}) = F^{-1}(0) \cap A^{-1}(0)$.

We now set

$$\mathrm{Compl} : \mathcal{W} \to \mathbb{R}^{6t-c-2k} \times \mathbb{C}^k, \quad \mathrm{Compl}(x) = (F(x), A(x), u_1(x), \ldots, u_k(x)).$$

Recall that $\dim_{\mathbb{R}} \mathcal{W} = 6t-c$, and that by Theorem 2.3.8 we have $\mathrm{Compl}^{-1}(0) = \{x_0\}$.

The rest of this section will be devoted to the proof of the following:

Theorem 2.4.2. *We have* $\mathrm{Ker}\ d\mathrm{Compl}_{x_0} = \{0\}$, *so* Compl *induces a diffeomorphism of an open neighbourhood of x_0 in $\mathcal{W}$ onto an open neighbourhood of* 0 *in* $\mathbb{R}^{6t-c-2k} \times \mathbb{C}^k$.

Before going into the proof of Theorem 2.4.2 we list some of its immediate consequences:

Corollary 2.4.3. *There exists an open neighbourhood V of x_0 in $\mathcal{W}$ such that the following holds: $\Omega(\mathcal{T}) \cap V$ is a smooth submanifold of real dimension $2k$ in V and the map $u : \Omega(\mathcal{T}) \to \mathbb{C}^k$ given by*

$$u(x) = (u_1(x), \ldots, u_k(x))$$

is a diffeomorphism of $\Omega(\mathcal{T}) \cap V$ onto an open neighbourhood of 0 in $\mathbb{C}^k$.

Corollary 2.4.4. *Let $X^{\mathrm{geod}}(\pi_1(N))$ be the space of characters defined in Section 1.4. Then there exists an open neighbourhood U of x_0 in $\Omega(\mathcal{T})$ such that the natural projection $\pi : U \to X^{\mathrm{geod}}(\pi_1(N))$ has the following property: $\pi(U)$ is an open neighbourhood of $\pi(x_0)$ and $\pi|_U : U \to \pi(U)$ is a branched covering of degree 2^k (*i.e.* a map locally modeled on the holomorphic map $(z_1, \ldots, z_k) \mapsto (z_1^2, \ldots, z_k^2)$).*

Different proofs of the following result can be found in [HK98, Luo].

Corollary 2.4.5. *Let $L \subset N$ be the union of the edges of $\mathcal{T}$ and denote by $\Omega_{\mathrm{cone}}(N, L)$ the space of hyperbolic cone-structures on N with singular locus contained in L. If N is compact, then $\Omega_{\mathrm{cone}}(N, L)$ is parameterized by cone angles along the components of L in a neighbourhood of the non-singular structure.*

The space $\Omega_{\mathrm{cone}}(\mathcal{T})$ As a first step towards the proof of Theorem 2.4.2 we have the following:

Proposition 2.4.6. *The tangent map $dF_{x_0} : T_{x_0}(\mathcal{W}) \to \mathbb{R}^{6t-e-(3/2)c-k}$ has maximal rank, so $\Omega_{\mathrm{cone}}(\mathcal{T})$ is a manifold of real dimension $e + c/2 + k$ near x_0.*

In order to prove this proposition we begin by describing a new parameterization for partially truncated tetrahedra.

Let Δ be a partially truncated tetrahedron and let $e_1, \ldots, e_6$ be the edges of Δ as in Fig. 2.3. Suppose for the moment that Δ has no ideal vertices. Let L^θ be the length function for edges in the geometric realization of Δ determined by the dihedral angles $\theta = (\theta_1, \ldots, \theta_6)$. The following lemma shows that lengths can be used as moduli for the geometric realizations of Δ. We set

$$\begin{aligned} D = \{x \in (0,\pi)^6 \quad : \quad & x_1 + x_2 + x_3 < \pi,\ x_1 + x_5 + x_6 < \pi, \\ & x_2 + x_4 + x_6 < \pi,\ x_3 + x_4 + x_5 < \pi\}. \end{aligned}$$

Lemma 2.4.7. *The map*

$$D \ni \theta \mapsto L^\theta \in \mathbb{R}^6$$

defines a diffeomorphism of D onto an open subset of $\mathbb{R}^6$.

PROOF: The map associating lengths to angles was shown to be smooth in Section 2.2. Similar calculations in hyperbolic trigonometry also show that lengths of edges of Δ determine its dihedral angles, and that angles depend smoothly on lengths. □

Let now v_{ijk} be the vertex of Δ from which e_i, e_j, e_k emanate and suppose that v_{123} is the only ideal vertex of Δ. Let F_1, F_2, F_3 be the exceptional hexagons of Δ

and for any angle assignment θ for Δ let $\sigma^\theta(F_i)$ be the function defined in Section 2.2. Let also D', D'' be the open portions of affine hyperplanes of $\mathbb{R}^6$ defined as follows:

$$\begin{aligned} D' &= \{x \in (0,\pi)^6 \ : \ x_1+x_2+x_3=\pi,\ x_1+x_5+x_6<\pi, \\ &\qquad\qquad x_2+x_4+x_6<\pi,\ x_3+x_4+x_5<\pi\}, \\ D'' &= \{y \in \mathbb{R}^6 \ : \ y_4+y_5+y_6=0\}. \end{aligned}$$

Another simple calculation implies the following:

Lemma 2.4.8. *The map*

$$D' \ni \theta \mapsto (L^\theta(e_4), L^\theta(e_5), L^\theta(e_6), \sigma^\theta(F_1), \sigma^\theta(F_2), \sigma^\theta(F_3)) \in D''$$

induces a diffeomorphism of D' onto an open subset of D''.

We can now easily prove Proposition 2.4.6. Recall we are parameterizing the geometric realizations of tetrahedra in $\mathcal{T}$ by dihedral angles, thus obtaining a bijection between the space of such realizations and an open convex subset $\mathcal{W}$ of a $(6t-c)$-dimensional affine space. Let us now describe the tetrahedra in $\mathcal{T}$ using lengths of finite edges and the parameter σ for exceptional hexagons. By Lemmas 2.4.7, 2.4.8 this corresponds to introducing new coordinates $(y_1,\ldots,y_{6t-c})$ on $\mathcal{W}$ which smoothly depend on the old ones. Moreover, with respect to these new coordinates the function $F=(F_1,\ldots,F_{6t-e-(3/2)c-k}) : \mathcal{W} \to \mathbb{R}^{6t-e-(3/2)c-k}$ takes the form $F_j(y) = y_{i_j} \pm y_{h_j}$ for certain $i_j, h_j \in \{1,\ldots,6t-e-(3/2)c-k\}$. Now it is straight-forward to show that dF_{x_0} has maximal rank.

Deforming cone structures Let now w be a tangent vector to $\Omega_{\mathrm{cone}}(\mathcal{T})$ at x_0. By Proposition 2.4.6 a smooth arc $\alpha : (-\varepsilon,\varepsilon) \to \Omega_{\mathrm{cone}}(\mathcal{T})$ exists such that $\dot\alpha(0)=w$. We will study the deformation associated to α using tools from the cohomology theory of representations: notation is as in Appendix A, where we also give some basic definitions and results.

From now on we denote by N' the non-compact manifold obtained by drilling from N all the edges of $\mathcal{T}$. For any $t\in(-\varepsilon,\varepsilon)$ the point $\alpha(t)$ determines a smooth hyperbolic structure M_t on N', whose completion gives a hyperbolic cone structure on N. First of all we describe how to deduce the shape of the geometric tetrahedra corresponding to the point $\alpha(t)$ just from the geometric structure M_t on N'. To this end we fix for $t\in(-\varepsilon,\varepsilon)$ a developing map $D_t : \widetilde{N}' \to \mathbb{H}^3$ with associated holonomy representation $\rho_t : \pi_1(N') \to \mathrm{PSL}(2,\mathbb{C})$ (note that we can choose D_t and ρ_t to vary smoothly with t). Recall that N compactifies to a manifold $\overline{N}$ with k boundary tori, and denote by $\overline{N}'$ the non-compact manifold obtained by drilling from $\overline{N}$ the *closed* properly embedded arcs corresponding to the edges of $\mathcal{T}$. For any $i=1,\ldots,k$ let T_i' be the punctured torus $T_i' = T_i \cap \overline{N}'$ and denote by $\mathcal{S}_p$ the family of all the boundary components of the universal covering of $\overline{N}'$ projecting to some T_i'. We say that $P \subset \pi_1(N') \cong \pi_1(\overline{N}')$ is a *parabolic peripheral subgroup* of $\pi_1(N')$ if P is the stabilizer of some boundary component $S \in \mathcal{S}_p$ of the universal covering of $\overline{N}'$. We also denote by $\mathcal{P}_p$ the class of all parabolic peripheral subgroups of $\pi_1(N')$, and observe that if $P\in\mathcal{P}_p$ then $\rho_0(P)$ is a $\mathbb{Z}+\mathbb{Z}$ parabolic subgroup of

$\mathrm{PSL}(2,\mathbb{C})$. Let now $\partial_{\mathrm{geod}}\overline{N}'$ be the components of $\partial\overline{N}'$ corresponding to the geodesic boundary of N, *i.e.* let $\partial_{\mathrm{geod}}\overline{N}' = \partial\overline{N}' \cap \partial N$, and denote by $\mathcal{S}_F$ the family of all the boundary components of the universal covering of $\overline{N}'$ projecting to some component of $\partial_{\mathrm{geod}}\overline{N}'$. We say that $P \subset \pi_1(N')$ is a *Fuchsian peripheral subgroup* of $\pi_1(N')$ if P is the stabilizer of some component $S \in \mathcal{S}_F$, we denote by $\mathcal{P}_F$ the class of all Fuchsian peripheral subgroups of $\pi_1(N')$, and observe that if $P \in \mathcal{P}_F$ then $\rho_0(P)$ is a Fuchsian subgroup of $\mathrm{PSL}(2,\mathbb{C})$.

Dual vectors to planes and horospheres If $S \in \mathcal{S}_F$, by construction the image of S under the developing map D_t is contained in a totally geodesic immersed surface in $\mathbb{H}^3$. It is easily seen that such a surface must in turn be contained in a geodesic plane $\hat{S}(t)$ of $\mathbb{H}^3$. We now need to associate to each $\hat{S}(t)$ a suitable *ultra-ideal* point $b_S(t)$, which will be called the *dual* point of $\hat{S}(t)$. Such point naturally lies in 4-dimensional Minkowsky space, so we fix some notation about this space.

We denote by $\mathbb{M}^{3+1}$ the space $\mathbb{R}^4$ with coordinates x_0, x_1, x_2, x_3 endowed with the Lorentzian inner product $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$. We set

$$\begin{aligned} \mathcal{H}^3_- &= \{x \in \mathbb{M}^{3+1} : \langle x, x\rangle = -1,\ x_0 > 0\}, \\ \mathcal{H}^3_+ &= \{x \in \mathbb{M}^{3+1} : \langle x, x\rangle = 1\}, \\ L^3_+ &= \{x \in \mathbb{M}^{3+1} : \langle x, x\rangle = 0,\ x_0 > 0\}. \end{aligned}$$

We recall that $\mathcal{H}^3_-$ is the upper sheet of the two-sheeted hyperboloid, and that $\langle\,\cdot\,,\,\cdot\,\rangle$ restricts to a Riemannian metric on $\mathcal{H}^3_-$. With this metric, $\mathcal{H}^3_-$ is the so-called *hyperboloid model* $\mathbb{H}^3_{\mathrm{hyp}}$ of hyperbolic space. The one-sheeted hyperboloid $\mathcal{H}^3_+$ turns out to have a bijective correspondence with the set of hyperbolic half-spaces in $\mathbb{H}^3_{\mathrm{hyp}}$. Given $b \in \mathcal{H}^3_+$, the corresponding half-space, called the *dual* of b, is given by

$$\{v \in \mathcal{H}^3_- : \langle v, b\rangle \leqslant 0\}.$$

Similarly, the cone L^3_+ of future-oriented light-like vectors of $\mathbb{M}^{3+1}$ corresponds to the set of horospheres in $\mathbb{H}^3_{\mathrm{hyp}}$. The horosphere dual to $u \in L^3_+$ is given by

$$\{v \in \mathcal{H}^3_- : \langle v, u\rangle = -1\}.$$

Note now that for any $S \in \mathcal{S}_F$ the set $D_t(\widetilde{N})$ locally lies on a definite side of $\hat{S}(t) \supseteq D_t(S)$, so we can define $b_S(t)$ to be the dual vector to the half-space that locally contains $D_t(\widetilde{N})$ and is bounded by $\hat{S}(t)$.

When S belongs to $\mathcal{S}_p$ a vector $b_S(t) \in L^3_+$ can also be defined as follows: take an oriented edge $\widetilde{f}$ of $\widetilde{\mathcal{T}}$ ending in S and set $b_S(t)$ to be the unique point in L^3_+ with $x_0(b_S(t)) = 1$ which projects to the endpoint of $D_t(\widetilde{f})$ in $\partial\mathbb{H}^3$. Such an endpoint exists because $D_t(\widetilde{f})$ is a geodesic, and is clearly independent of the choice of $\widetilde{f}$, so $b_S(t)$ is indeed well-defined. Since developing maps vary smoothly with respect to t we have the following:

Lemma 2.4.9. *For any $S \in \mathcal{S}_F \cup \mathcal{S}_p$ the map $t \mapsto b_S(t)$ is smooth with respect to t.*

Lifting geometric tetrahedra Let now Δ be a tetrahedron in $\mathcal{T}$, and denote by $\widetilde{\Delta}$ a lift of Δ to $\widetilde{N}'$. Let S_i be the boundary component of the universal covering of $\overline{N}'$ that corresponds to v_i, where $v_1, \ldots, v_4$ are the vertices of $\widetilde{\Delta}$. If $\overline{\Delta}(t)$ is the convex hull of $b_{S_1}(t), \ldots, b_{S_4}(t)$ in $\mathbb{M}^{3+1}$, then projecting $\overline{\Delta}(t)$ onto $\mathbb{H}^3$ and truncating its infinite-volume ends with the corresponding $\hat{S}_i(t)$'s gives back a truncated tetrahedron $\Delta^*(t)$ isometric to the geometric realization of Δ parameterized by $\alpha(t)$. It is easily seen that the dihedral angles of $\Delta^*(t)$ smoothly depend on $b_{S_1}(t), \ldots, b_{S_4}(t)$, so Theorem 2.4.2 is now reduced to the following:

Proposition 2.4.10. *Suppose* $\dot{\alpha}(0)$ *belongs to* Ker $d\mathrm{Compl}_{x_0}$. *Then we can choose the* D_t*'s in such a way that* $\dot{b}_S(0) = 0$ *for any* $S \in \mathcal{S}_F \cup \mathcal{S}_p$.

The rest of this section is devoted to the proof of this proposition.

The tangent vector to ρ_t Let us consider the double DN' of N' obtained by mirroring N' along its boundary (which is now given by some punctured surfaces of negative Euler characteristic). Since $\partial N'$ is totally geodesic with respect to the hyperbolic structure M_t, this structure can be doubled to a smooth hyperbolic metric DM_t on DN'.

Let $D\rho_t : \pi_1(DN') \to \mathrm{PSL}(2,\mathbb{C})$ be a holonomy representation corresponding to a developing map for DM_t. It is easily seen that we can assume $D\rho_t$ to vary smoothly with respect to t. We set

$$\dot{D\rho} : \pi_1(DN') \to \mathfrak{sl}(2,\mathbb{C}), \quad \dot{D\rho}(\gamma) = \frac{d}{dt}\Big|_{t=0} \left(D\rho_t(\gamma) D\rho_0(\gamma)^{-1}\right).$$

As explained in Appendix A, we have $\dot{D\rho} \in Z^1(\pi_1(DN'); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ D\rho_0)$. Our next aim is to prove the following:

Proposition 2.4.11. *Suppose* $\dot{\alpha}(0)$ *belongs to* Ker $d\mathrm{Compl}_{x_0}$. *Then* $\dot{D\rho}$ *belongs to* $B^1(\pi_1(DN'); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ D\rho_0)$.

From now on we suppose $\dot{\alpha}(0) \in$ Ker $d\mathrm{Compl}_{x_0}$. Let $f_1, \ldots, f_m \in DN'$ be the doubles of the edges of $\mathcal{T}$ and for all $j = 1, \ldots, m$ let ℓ_j be a small loop in DN' encircling f_j. We denote by γ_j an element in $\pi_1(DN')$ representing ℓ_j (such a γ_j is well-defined only up to conjugation). Now our hypothesis implies that if $\theta_j(t)$ is the cone angle of (the completion of) DM_t along f_j then $\dot{\theta}_j(0) = 0$. Also observe that we have

$$D\rho_t(\gamma_j) = g_t \cdot \left[\begin{pmatrix} \exp(i\theta_j(t)/2) & 0 \\ 0 & \exp(-i\theta_j(t)/2) \end{pmatrix}\right] \cdot g_t^{-1}$$

for some smooth path $g : (-\varepsilon, \varepsilon) \to \mathrm{PSL}(2,\mathbb{C})$. Differentiating this relation we easily get $\dot{D\rho}(\gamma_j) = 0$. Let now K be the the normal subgroup of $\pi_1(DN')$ generated by $\gamma_1, \ldots, \gamma_m$. Since the natural projection $\pi_1(DN') \to \pi_1(DN)$ induces an isomorphism $\pi_1(DN) \cong \pi_1(DN')/K$, by Lemmas A.2, A.3 we obtain the following:

Proposition 2.4.12. *For any* $\gamma \in K$ *we have* $D\rho_0(\gamma) = 1$ *and* $\dot{D\rho}(\gamma) = 0$, *so* $\dot{D\rho}$ *naturally defines a cocycle* $z_\rho \in Z^1(\pi_1(DN); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ D\overline{\rho}_0)$, *where* $D\overline{\rho}_0 : \pi_1(DN) \to \mathrm{PSL}(2,\mathbb{C})$ *is the natural representation associated to* $D\rho_0$. *Moreover,* $\dot{D\rho}$ *belongs to* $B^1(\pi_1(DN'); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ D\rho_0)$ *if and only if* z_ρ *belongs to* $B^1(\pi_1(DN); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ D\overline{\rho}_0)$.

Now if N is compact, *i.e.* if no cusps are involved, Theorem A.4 directly applies concluding the proof of Proposition 2.4.11. When there are cusps, Proposition 2.4.11 can be deduced from Theorem A.6 and the following:

Proposition 2.4.13. *When N is non-compact, we have*

$$[z_\rho] \in H^1_{\mathrm{par}}(\pi_1(DN); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ D\overline{\rho}_0) \subset H^1(\pi_1(DN); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ D\overline{\rho}_0).$$

Proof: Let $\overline{\gamma} \in \pi_1(DN)$ be such that $D\overline{\rho}_0(\overline{\gamma})$ is non-trivial parabolic, and let $\langle\overline{\gamma}\rangle$ be the infinite cyclic group generated by $\overline{\gamma}$. We have to check that z_ρ restricts to a coboundary in $B^1(\langle\overline{\gamma}\rangle; \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ D\overline{\rho}_0 \circ i)$, where $i : \langle\overline{\gamma}\rangle \to \pi_1(DN)$ is the natural inclusion.

Since for any $t \in (-\varepsilon, \varepsilon)$ an isometry $\psi_t : DM_t \to DM_t$ exists which depends smoothly on t and interchanges the two copies of M_t in DM_t, it is easily seen that we can suppose $\overline{\gamma} \in \pi_1(T_i) \subset \pi_1(N) \subset \pi_1(DN)$ for some $i = 1, \ldots, k$. Recall now that we have chosen a preferred element $\mu_i \in \pi_1(T_i)$ such that $u_i(x) = \ln \delta^i_x(\mu_i)$. Set $u_i(t) = u_i(\alpha(t))$ and observe that condition $\dot{\alpha}(0) \in \mathrm{Ker}\, d\mathrm{Compl}_{x_0}$ implies that $\dot{u}_i(0) = 0$. Let $\hat{\mu}_i, \gamma$ be elements in $\pi_1(T_i')$ projecting respectively to $\mu_i, \overline{\gamma}$. By Lemma 2.4.9 a smooth path $g : (-\varepsilon, \varepsilon) \to \mathrm{PSL}(2,\mathbb{C})$ exists such that $g_t^{-1} \cdot \rho_t(\gamma') \cdot g_t$ fixes $\infty \in \partial\mathbb{H}^3$ for any $\gamma' \in \pi_1(T_i')$. In particular we have

$$\begin{aligned} D\rho_t(\hat{\mu}_i) &= g_t \cdot \left[\begin{pmatrix} \exp(u_i(t)/2) & \tau_i(t) \\ 0 & \exp(-u_i(t)/2) \end{pmatrix}\right] \cdot g_t^{-1}, \\ D\rho_t(\gamma) &= g_t \cdot \left[\begin{pmatrix} a(t) & b(t) \\ 0 & a(t)^{-1} \end{pmatrix}\right] \cdot g_t^{-1}, \end{aligned}$$

where $\tau_i, a, b : (-\varepsilon, \varepsilon) \to \mathbb{C}$ are smooth arcs with $a(0) = 1, b(0) \neq 0, \tau_i(0) \neq 0$.

Since $\rho_0(\pi_1(T_i)) \cong \mathbb{Z} + \mathbb{Z}$ is Abelian, an element $k \in K$ exists such that $\hat{\mu}_i \cdot \gamma = k \cdot \gamma \cdot \hat{\mu}_i$ in $\pi_1(DN')$ (recall that K is the kernel of the natural projection $\pi_1(DN')\ to \pi_1(DN)$). By Proposition 2.4.12, this readily implies

$$\left.\frac{d}{dt}\right|_{t=0} (D\rho_t(\hat{\mu}_i) D\rho_t(\gamma)) = \left.\frac{d}{dt}\right|_{t=0} (D\rho_t(\gamma) D\rho_t(\hat{\mu}_i)),$$

which after some computations gives $\dot{a}(0) = 0$ (here we use $\dot{u}_i(0) = 0$). Let us consider the deformation $\varphi_t : \langle\gamma\rangle \to \mathrm{PSL}(2,\mathbb{C})$ defined by $\varphi_t(\gamma^n) = D\rho_t(\gamma^n)$ for any $n \in \mathbb{N}$.

We claim that $\dot{\varphi} = 0$ in $H^1(\langle\gamma\rangle; \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ \varphi_0)$. This will easily give that z_ρ restricts to a coboundary in $B^1(\langle\overline{\gamma}\rangle; \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ D\overline{\rho}_0 \circ i)$, whence the conclusion. Since derivatives of conjugated deformations differ by a coboundary, we can suppose

$$\varphi_t(\gamma) = \left[\begin{pmatrix} a(t) & b(t) \\ 0 & a(t)^{-1} \end{pmatrix}\right].$$

Setting

$$v = \begin{pmatrix} \frac{\dot{b}(0)}{2b(0)} & 0 \\ 0 & -\frac{\dot{b}(0)}{2b(0)} \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C}),$$

an easy computation shows that $\dot{\varphi}(\gamma) = v - \mathrm{Ad}(\varphi_0(\gamma))(v)$. This readily implies that $\dot{\varphi}$ is a coboundary, and we are done. $\square$

For any $\gamma \in \pi_1(DN')$ let now $\mathrm{tr}_\gamma : (-\varepsilon, \varepsilon) \to \mathbb{C}$ be the map defined as follows: $\mathrm{tr}_\gamma(t) = \mathrm{trace}(D\hat{\rho}_t(\gamma))$, where $t \mapsto D\hat{\rho}_t(\gamma) \in \mathrm{SL}(2,\mathbb{C})$ is a smooth lift of $t \mapsto D\rho_t(\gamma) \in \mathrm{PSL}(2,\mathbb{C})$ (so tr_γ is well-defined only up to the sign). Also recall that D_t and ρ_t are respectively a developing map and a holonomy representation for the hyperbolic structure M_t on N', and observe that an analogous definition of $\mathrm{tr}_\gamma(t)$ (with $D\rho_t$ replaced by ρ_t) also works when γ belongs to $\pi_1(N')$ (we have of course $\mathrm{tr}_\gamma(t) = \mathrm{tr}_{j(\gamma)}(t)$ for any $\gamma \in \pi_1(N')$, where $j : \pi_1(N') \to \pi_1(DN')$ is the natural inclusion). The following result can be easily deduced from the proof of the previous proposition.

Lemma 2.4.14. *Let $\gamma \in \pi_1(DN')$ be such that $D\rho_0(\gamma)$ is non-trivial parabolic. Then $\ddot{\mathrm{tr}}_\gamma(0) = 0$.*

As a consequence of Proposition 2.4.11 and Lemma 2.4.14 we obtain the following:

Proposition 2.4.15. *D_t and ρ_t can be chosen in such a way that*

$$\left.\frac{d}{dt}\right|_{t=0} \rho_t(\gamma) = 0$$

for any $\gamma \in \pi_1(N')$. Moreover, if γ is a non-trivial element of a peripheral parabolic subgroup of $\pi_1(N')$ then $\ddot{\mathrm{tr}}_\gamma(0) = 0$.

The final step We begin with two technical lemmas. Let $\gamma \in \pi_1(N')$ be such that $\rho_0(\gamma) \neq 1$ and suppose $r_\gamma : (-\varepsilon, \varepsilon) \to \partial\mathbb{H}^3$ is a smooth path such that $r_\gamma(t)$ is a fixed point for $\rho_t(\gamma)$ for any $t \in (-\varepsilon, \varepsilon)$. We want to study how the derivative of $r(t)$ is related to the derivative of $\rho_t(\gamma)$. We identify $\partial\mathbb{H}^3$ with $\mathbb{C} \cup \{\infty\}$ and we set

$$\rho_t(\gamma) = \left[\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\right].$$

Without loss of generality we can suppose that $\rho_0(\gamma)$ does not fix ∞ and that $r_\gamma(0) = 0 \in \mathbb{C} \subset \partial\mathbb{H}^3$, so $b(0) = 0$, $c(0) \neq 0$. An easy computation now shows that

$$r_\gamma(t) = \frac{a(t) - d(t) \pm \sqrt{(\mathrm{tr}_\gamma(t) + 2)(\mathrm{tr}_\gamma(t) - 2)}}{2c(t)}.$$

From this formula we can readily deduce the following lemmas.

Lemma 2.4.16. *Suppose $\rho_0(\gamma)$ is non-trivial loxodromic and $\dot{\rho}(\gamma) = 0$. Then $\dot{r}_\gamma(0) = 0$.*

Lemma 2.4.17. *Let $\gamma \in \pi_1(N')$ be such that $\rho_0(\gamma)$ is non-trivial parabolic. Also assume that $\dot{\rho}(\gamma) = 0$ and that $\ddot{\mathrm{tr}}_\gamma(0) = 0$. Then $\dot{r}_\gamma(0) = 0$.*

We can now conclude the proof of Proposition 2.4.10. Choose D_t and ρ_t as in the statement of Proposition 2.4.15. Let S be a boundary component of the universal covering of $\overline{N}'$ which belongs to $\mathcal{S}_p$, denote by P_S the stabilizer of S in $\pi_1(N')$ and choose an element $\gamma \in P_S$ with $\rho_0(\gamma) \neq 1$. By construction the projection of $b_S(t)$ to $\partial\mathbb{H}^3$ is fixed by $\rho_t(\gamma)$, so Lemma 2.4.17 applies ensuring $\dot{b}_S(0) = 0$.

When S belongs to $\mathcal{S}_F$ a slightly subtler argument is needed. Let P_S be the stabilizer of S in $\pi_1(N')$ and for any $\gamma \in P_S$ with $\rho_0(\gamma) \neq 1$ let $p_\gamma(t), q_\gamma(t)$ be the fixed points of $\rho_t(\gamma)$ on $\partial\mathbb{H}^3$. Note that since $\rho_0(\gamma)$ is loxodromic we can choose p_γ, q_γ to vary smoothly with respect to t, at least in a small neighbourhood of 0. This gives $\dot{p}_\gamma = \dot{q}_\gamma = 0$ by Lemma 2.4.16. Let now $\hat{p}_\gamma(t)$ (resp. $\hat{q}_\gamma(t)$) be the unique point in $L^3_+ \subset \mathbb{M}^{3+1}$ with x_0-coordinate equal to 1 which projects onto $p_\gamma(t)$ (resp. onto $q_\gamma(t)$). Of course we have $\dot{\hat{p}}_\gamma(0) = \dot{\hat{q}}_\gamma(0) = 0$, so differentiating the relation $\langle b_S(t), \hat{p}_\gamma(t)\rangle = \langle b_S(t), \hat{q}_\gamma(t)\rangle = 0$ and evaluating at 0 we obtain

$$\langle \dot{b}_S(0), \hat{p}_\gamma(0)\rangle = \langle \dot{b}_S(0), \hat{q}_\gamma(0)\rangle = 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the Lorentzian product on $\mathbb{M}^{3+1}$. Now a standard result in Kleinian group theory ensures that the set

$$\{p_\gamma(0), q_\gamma(0):\ \gamma \in P_S,\ \rho_0(\gamma) \neq 1\}$$

is dense in the closure at infinity of $D_0(S) \subset \mathbb{H}^3$, which is a straight circle. This easily implies that $\dot{b}_S(0)$ is a multiple of $b_S(0)$, whence $\dot{b}_S(0) = 0$ since $||b_S(t)|| = 1$ for any $t \in (-\varepsilon, \varepsilon)$.

An interesting consequence We know from Mostow's rigidity Theorem that compact hyperbolic 3-manifolds do not admit deformations. The following results seem to suggest that in the non-compact case deformations take place mostly near the cusps: even if it has to be affected by any non-trivial deformation, the compact core offers resistance to changing its shape. More precisely, we now show that deformations of $\mathcal{T}$ near the complete structure affect compact tetrahedra only at the second order.

Let f be any *compact* edge of $\mathcal{T}$. For any $x \in \Omega(\mathcal{T})$ we denote by $\ell^f(x)$ the length of f with respect to the metric structure defined by x. By Lemmas 2.4.7, 2.4.8 the map $\ell^f : \Omega(\mathcal{T}) \to \mathbb{R}$ is smooth.

Proposition 2.4.18. *We have* $d\ell^f_{x_0} = 0$.

PROOF: Let V be a small neighbourhood of x_0 in $\Omega(\mathcal{T})$ and for any $x \in V$ let us fix a developing map $D_x : \widetilde{N} \to \mathbb{H}^3$ and a holonomy representation $\rho_x : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ for the hyperbolic structure defined by x. Take a lift $\widetilde{f}$ of f in $\widetilde{N}$, let S_1, S_2 be the components of $\partial\widetilde{N}$ containing the endpoints of $\widetilde{f}$ and set $\hat{S}_i(x) = D_x(S_i)$, $i = 1,2$. Let also P_i be the stabilizer of S_i in $\pi_1(N)$, and recall that $\hat{S}_i(x)$ is the hyperbolic convex hull of the limit set of $\rho_x(P_i)$. Moreover, since $D_x(\widetilde{f})$ intersects both $\hat{S}_1(x)$ and $\hat{S}_2(x)$ orthogonally we have $\ell^f(x) = d_{\mathbb{H}^3}(\hat{S}_1(x), \hat{S}_2(x))$.

Let now $u = (u_1, \ldots, u_k) : V \to \mathbb{C}^k$ be the local coordinates defined in Corollary 2.4.3. Up to resizing V, for any $x \in V$ we can uniquely define a point $-x \in V$ by setting $-x := u^{-1}(-u(x))$. Since the differential of $x \mapsto -x$ acts as minus the identity on $T_{x_0}\Omega(\mathcal{T})$, to conclude the proof it is sufficient to show that $\ell^f(x) = \ell^f(-x)$ for any $x \in V$.

Now by Corollary 2.4.4 we have that x and $-x$ project onto the same character in $X^{\mathrm{geod}}(\pi_1(N))$, so ρ_x is conjugated to ρ_{-x} and in particular $\rho_{-x}(P_1) = g \cdot \rho_x(P_1) \cdot g^{-1}$, $\rho_{-x}(P_2) = g \cdot \rho_x(P_2) \cdot g^{-1}$ for some $g \in \mathrm{PSL}(2, \mathbb{C})$. This gives in turn $\hat{S}_i(-x) = g(\hat{S}_i(x))$, whence

$$\ell^f(-x) = d_{\mathbb{H}^3}(g(\hat{S}_1(x)), g(\hat{S}_2(x))) = d_{\mathbb{H}^3}(\hat{S}_1(x), \hat{S}_2(x)) = \ell^f(x).$$

$\square$

Since the length of any boundary edge of any compact lateral hexagon smoothly depends on the lengths of its internal edges, we have the following:

Corollary 2.4.19. *Fix a boundary edge e of a* compact *lateral hexagon of a tetrahedron of $\mathcal{T}$ and let $\ell^e : \Omega(\mathcal{T}) \to \mathbb{R}$ be the function that associates to any $x \in \Omega$ the length of e in the geometric realization of $\mathcal{T}$ parameterized by x. Then ℓ^e is smooth and $d\ell^e_{x_0} = 0$.*

The dihedral angles of a compact truncated tetrahedron smoothly depend on the lengths of the internal edges. Thus Proposition 2.4.18 also implies the following:

Corollary 2.4.20. *Fix an internal edge f in a* compact *tetrahedron of $\mathcal{T}$ and let $a^f : \Omega(\mathcal{T}) \to \mathbb{R}$ be the function that associates to any $x \in \Omega(\mathcal{T})$ the dihedral angle along f in the geometric realization of $\mathcal{T}$ parameterized by x. Then a^f is smooth and $da^f_{x_0} = 0$.*

2.5 Dehn filling equations

In this section we show that hyperbolic structures parameterized by points in a neighbourhood of x_0 in $\Omega(\mathcal{T})$ induce on any end of N either a complete metric or a Dehn filling type singularity. This allows us to describe the manifolds obtained by Dehn filling N along sufficiently complicated curves as the metric completions of small deformations of the geometric triangulation of N defined by x_0.

Neumann-Zagier parameters revisited Let us consider the functions $u, v : \Omega(\mathcal{T}) \to \mathbb{C}^k$ defined in the preceding section and fix a neighbourhood V of x_0 in $\Omega(\mathcal{T})$ such that $u : V \to u(V)$ is a diffeomorphism onto an open neighbourhood of 0 in $\mathbb{C}^k$. Then we can think of v as of a function of u, just by setting $\overline{v} : \mathbb{C}^k \supset u(V) \to \mathbb{C}^k$, $\overline{v}(\overline{u}) = v(u^{-1}(\overline{u}))$ for any $\overline{u} \in u(V)$. From now on we will use only the symbol v to denote both the function $v : \Omega(\mathcal{T}) \to \mathbb{C}$ and the function $\overline{v} : u(V) \to \mathbb{C}$, the context making clear which map we are actually referring to. By the very definitions it easily follows that the function $v : u(V) \to \mathbb{C}^k$ just defined coincides with the one defined in Section 1.3, equation (1.3).

The structure at the infinity For any $x \in V$ let us denote by $M(x)$ the hyperbolic structure defined on N by x. Recall also that $E_j \cong T_j \times [0,\infty)$ is the end of N corresponding to the boundary torus $T_j \subset \partial\overline{N}$. The proof of the following proposition is essentially taken from [Rat94].

Proposition 2.5.1. *Up to resizing V, for any $x \in V$ with $u_j(x) \neq 0$ the structure $M(x)$ induces on E_j a Dehn filling singularity with coefficient $p_j(x)\mu_j + q_j(x)\lambda_j$, where $p_j(x), q_j(x)$ are the only real numbers such that $p_j(x)u_j(x) + q_j(x)v_j(x) = 2\pi i$.*

Proof: By Proposition 1.3.8, we only have to prove that $M(x)$ induces a Dehn filling singularity on E_j. To lighten notation a bit we drop the index j, thus setting $E := E_j$, $T := T_j$ and $\mu := \mu_j$.

Recall that for each ideal vertex v of $\mathcal{T}$ a Euclidean triangle $L_x(v)$ is well-defined up to similarity, and that a similarity structure on T is defined just by considering T as obtained by gluing the triangles in the family $\mathcal{L}_x = \{L_x(v) :$ v is an ideal vertex of $\mathcal{T}$ asymptotic to $T\}$. Also recall that this structure naturally defines a developing map $D_x^T : \widetilde{T} \to \mathbb{C}$, well-defined up to post-composition with elements of $\mathrm{Aff}(\mathbb{C})$, and a holonomy homomorphism $\rho_x^T : \pi_1(T) \to \mathrm{Aff}(\mathbb{C})$, well-defined up to conjugation.

By Lemma 2.4.1 the end E is non-complete, so by Proposition 2.3.1 and Theorem 1.1.6 the image of ρ_x^T contains an element which is not an isometry. By conjugating ρ_x^T we may suppose that this element fixes $0 \in \mathbb{C}$. Since $\pi_1(T)$ is Abelian this gives in turn that every element in $\rho_x^T(\pi_1(T))$ fixes 0. Thus ρ_x^T maps into the group of affine transformations of $\mathbb{C}$ that fix 0, which is canonically isomorphic to $\mathbb{C}^*$ through the (inverse of the) map which associates to $k \in \mathbb{C}^*$ the similarity $z \mapsto kz$. The map $\exp : \mathbb{C} \to \mathbb{C}^*$ induces an isomorphism from $\mathbb{C}/2\pi i\mathbb{Z}$ to $\mathbb{C}^*$. We put on $\mathbb{C}^*$ the complete metric for which this isomorphism is actually an isometry, where we consider $\mathbb{C}/2\pi i\mathbb{Z}$ endowed with the natural Euclidean metric inherited from $\mathbb{C}$. Then it is easily seen that $\mathrm{Isom}^+(\mathbb{C}^*) = \mathbb{C}^*$.

We now show that the developing map $D_x^T : \widetilde{T} \to \mathbb{C}$ maps into $\mathbb{C}^*$. If $\mathcal{L}_x = \{L_1, \ldots, L_s\}$, for $i = 1, \ldots, s$ choose a lift $\widetilde{L}_i \subset \widetilde{T}$ of L_i in such a way that each $\widetilde{L}_j$ shares an edge with some $\widetilde{L}_{j'}$ with $j \neq j'$ (thus the union $\bigcup_{i=1}^s \widetilde{L}_i$ provides a fundamental polygon for the action of $\pi_1(T)$ on $\widetilde{T}$). Let also $L_i' \subset \mathbb{C}$ be the image of $\widetilde{L}_i$ under D_x^T. Of course $D_x^T(\widetilde{T})$ is the union of the images of the L_i''s under the action of $\rho_x^T(\pi_1(T))$. Now it is easily seen that if x is sufficiently close to x_0, then $\rho_x^T(\mu)$ does not fix a point in any of the L_i''s. Thus 0 is not contained in any of these triangles, and $D_x^T(\widetilde{T}) \subset \mathbb{C}^*$ by $\rho_x^T(\pi_1(T))$-invariance of $\mathbb{C}^*$. Therefore x induces on T a $(\mathbb{C}^*, \mathbb{C}^*)$-structure. Now $\mathbb{C}^*$ acts on itself by isometries, so compactness of T implies completeness of this structure, in the sense of Definition 1.1.2. This gives in turn a complete $(\mathbb{C}^*, \mathbb{C}^*)$-structure on $\widetilde{T}$, and shows that $D_x^T : \widetilde{T} \to \mathbb{C}^*$ is a universal covering. Let now $\mathbb{C}$ be identified with the group of translations of $\mathbb{C}$. Since $\exp : \mathbb{C} \to \mathbb{C}^*$ is a Riemannian universal covering, the developing map $D_x^T : \widetilde{T} \to \mathbb{C}^*$ lifts to a map $\widetilde{D}_x^T : \widetilde{T} \to \mathbb{C}$ which is a developing map for a complete $(\mathbb{C}, \mathbb{C})$-structure on T.

Regard $\mathbb{C}$ as the boundary of the upper half-space model $\mathbb{C} \times (0,\infty)$ of $\mathbb{H}^3$. Let $D_x : \widetilde{N} \to \mathbb{C} \times (0,\infty)$ be the developing map for the structure $M(x)$ that maps $\widetilde{L}_i$

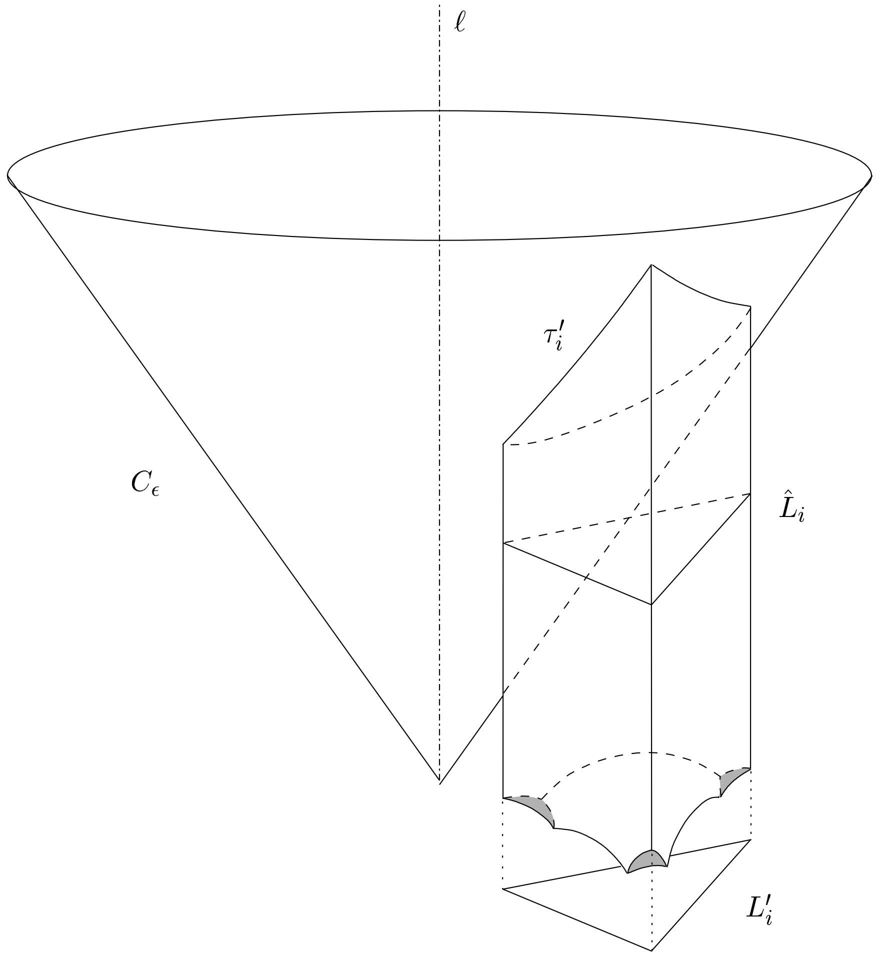

Figure 2.6: The tetrahedron $\Delta_i' \subset \mathbb{H}^3$ and the triangles $L_i', \hat{L}_i$ and τ_i'.

onto a horizontal triangle $\hat{L}_i$ directly above L_i'. Let $\rho_x : \pi_1(N) \to \mathrm{Isom}^+(\mathbb{H}^3)$ be the holonomy determined by D_x. For any $\gamma \in \mathrm{Isom}^+(\mathbb{C}^*) \subset \mathrm{Aff}^+(\mathbb{C})$ we denote by $j(\gamma) \in \mathrm{Isom}^+(\mathbb{H}^3)$ the unique isometry whose trace at the infinity coincide with γ, and by $i_* : \pi_1(T) \to \pi_1(N)$ the map induced by the inclusion. Of course we have $\rho_x \circ i_*(\gamma) = j \circ \rho_x^T(\gamma)$ for all $\gamma \in \pi_1(T)$.

Let Δ_i be the tetrahedron in $\mathcal{T}$ containing L_i and $\widetilde{\Delta}_i$ the lift of Δ_i in $\widetilde{N}$ containing $\widetilde{L}_i$. If $\Delta_i' = D_x(\widetilde{\Delta}_i)$, then Δ_i' is partially truncated tetrahedron with an ideal vertex at ∞ whose projection from ∞ to $\mathbb{C} \times \{0\} \subset \partial\mathbb{H}^3$ coincides with L_i'. Let ℓ be the geodesic with endpoints 0 and ∞ and for any $\epsilon > 0$ let C_ϵ be the ϵ-neighbourhood of ℓ in $\mathbb{H}^3$. If ϵ is sufficiently small, then the triangle $\hat{L}_i$ is outside of C_ϵ for all $i = 1, \ldots, s$. Then Δ_i' intersects ∂C_ϵ in a triangle τ_i' directly above Δ_i' (see Fig. 2.6). Let τ_i be the triangle in Δ_i corresponding to τ_i'. Since τ_i' is above Δ_i' for all $i = 1, \ldots, s$, the triangles $\tau_1, \ldots, \tau_s$ meet only along their boundaries in M. Furthermore, since $\rho_x \circ i_*(\pi_1(T)) = j \circ \rho_x^T(\pi_1(T))$ leaves ∂C_ϵ invariant, the triangles $\tau_1, \ldots, \tau_s$ fit together giving a triangulation of a torus S in M. This torus bounds an end of the form $S \times [0,\infty) \cong T \times [0,\infty)$ in N. We set $E = S \times [0,\infty)$. It is now easily seen that $M(x)$ induces on E a Dehn filling type singularity. $\square$

The very same proof of Proposition 1.3.8 gives the following:

Lemma 2.5.2. *Let $Y_j' = u_j^{-1}(0) \subset \Omega(\mathcal{T})$ be the set of structures which induce a complete metric on E_j and set $Y_j = \Omega(\mathcal{T}) \setminus Y_j'$. Then the limit*

$$\lim_{Y_j \ni x \to x_0} \frac{v_j(x)}{u_j(x)}$$

exists and has positive imaginary part.

For $x \in V$ we now set $(p_j(x), q_j(x)) \in \mathbb{R}^2 \cup \{\infty\} = S^2$ to be the unique point in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ with

$$\begin{array}{rl} (p_j(x), q_j(x)) = \infty & \text{if } u_j(x) = v_j(x) = 0, \\ p_j(x)u_j(x) + q_j(x)v_j(x) = 2\pi i & \text{otherwise.} \end{array}$$

We also set

$$d = (d_1, \ldots, d_k) : V \to \prod_{i=1}^{k} S^2, \quad d_j(x) = (p_j(x), q_j(x)) \in S^2.$$

An easy application of Corollary 2.4.3 and Lemma 2.5.2 gives the following:

Proposition 2.5.3. *The map d defines a diffeomorphism between a neighbourhood of x_0 in V and a neighbourhood of $(\infty, \ldots, \infty)$ in $S^2 \times \cdots \times S^2$.*

We have thus given a proof of Thurston's hyperbolic Dehn filling Theorem for 3-manifold with geodesic boundary, at least in the case that the manifold we start with admits a geometric decomposition into partially truncated tetrahedra:

Corollary 2.5.4. *For each $i = 1, \ldots, k$ there exists a finite set $\mathcal{S}_j$ of slopes on T_j with the following property: if $s_j \notin \mathcal{S}_j$ for $i = 1, \ldots, k$, then the filled manifold $N(s_1, \ldots, s_k)$ admits a complete finite-volume hyperbolic structure.*

Complex length of the added geodesics Let M be any $(\mathbb{H}^3, \mathrm{Isom}^+(\mathbb{H}^3))$-structure on N and take a closed curve $\ell \subset N$ which is a geodesic with respect to M. Then a well-defined *complex length* $\mathbb{C}L^M(\ell) \in \mathbb{C}/2\pi i\mathbb{Z}$ exists which depends on M and ℓ and is described as follows. Choose an orientation on ℓ, take a lift $\widetilde{\ell}$ of ℓ in $\widetilde{N}$ and let $D_M : \widetilde{N} \to \mathbb{C} \times (0,\infty) \cong \mathbb{H}^3$ be a developing map for M mapping the oriented geodesic $\widetilde{\ell}$ onto the oriented geodesic with endpoints 0 and ∞. Let now $\gamma \in \pi_1(N) \cong \mathrm{Aut}(\widetilde{N})$ be the element corresponding to the oriented curve ℓ which leaves $\widetilde{\ell}$ invariant. If $H_M : \pi_1(N) \to \mathrm{Isom}^+(\mathbb{C} \times (0,\infty))$ is the holonomy representation associated to D_M, the isometry $H_M(\gamma)$ obviously leaves the geodesic $D_M(\widetilde{\ell})$ invariant. Thus a complex number a exists such that

$$H_M(\gamma)(z,t) = (a \cdot z, |a| \cdot t), \quad (z,t) \in \mathbb{C} \times (0,\infty).$$

We set $\mathbb{C}L^M(\ell) = \ln a \in \mathbb{C}/2\pi i\mathbb{Z}$. It is easily seen that this is a good definition, *i.e.* that a only depends on the *unoriented* curve ℓ and the fixed hyperbolic structure M. Moreover, the usual length of ℓ as a regular curve in M is equal to $\Re(\mathbb{C}L^M(\ell))$. The complex length of a geodesic in M can also be related to the character of (a lift of) the holonomy representation by the equality

$$\text{trace } H_M(\gamma) = \pm 2\cosh \mathbb{C}L^M(\ell/2).$$

Let now x be a point in V, suppose $(p_j(x), q_j(x))$ is a pair of coprime integers, and let $\hat{N}$ be the manifold obtained by Dehn filling N along the slope $p_j(x)\mu_j + q_j(x)\lambda_j$ contained in T_j. By Proposition 2.5.1 the metric completion of the structure induced by $M(x)$ on E_j is obtained by adding to N a geodesic ℓ_j. This completion defines a non-singular hyperbolic structure on $\hat{N}$, which will be denote by $\hat{M}(x)$. We also set $\mathbb{C}L^x(\ell_j) := \mathbb{C}L^{\hat{M}(x)}(\ell_j)$. The following proposition is proved in [NZ85].

Proposition 2.5.5. *Choose integers $r_j(x), s_j(x)$ with $p_j(x)s_j(x) - q_j(x)r_j(x) = -1$. Then we have*

$$\mathbb{C}L^x(\ell_j) = r_j(x)u_j(x) + s_j(x)v_j(x).$$

PROOF: Let $\gamma_j \in \pi_1(\hat{N})$ be an element representing the geodesic ℓ_j endowed with an arbitrary orientation. Our hypothesis ensures that $(p_j(x)\mu_j + q_j(x)\lambda_j, r_j(x)\mu_j + s_j(x)\lambda_j)$ is a negatively oriented basis for $H_1(T_j;\mathbb{Z}) \cong \pi_1(T_j)$. This implies that if $r_j(x)\mu_j + s_j(x)\lambda_j \in \pi_1(T_j)$ projects onto $\gamma_j \in \pi_1(\hat{N})$, then $p_j(x)\mu_j + q_j(x)\lambda_j$ defines the positive orientation of small loops around ℓ_j. Put $a_j(x) = \exp(u_j(x)), b_j(x) = \exp(v_j(x))$. Then the isometry $H_{\hat{M}(x)}(\gamma_j) = H_{M(x)}(r_j(x)\mu_j + s_j(x)\lambda_j)$ is given by:

$$(z,t) \mapsto (a_j(x)^{r_j(x)} \cdot b_j(x)^{s_j(x)} \cdot z, |a_j(x)^{r_j(x)} \cdot b_j(x)^{s_j(x)}| \cdot t),$$

whence

$$\mathbb{C}L^x(\ell_j) = \ln(a_j(x)^{r_j(x)} \cdot b_j(x)^{s_j(x)}) = r_j(x)u_j(x) + s_j(x)v_j(x).$$

□

2.6 The Kojima decomposition

In this section we describe Kojima's canonical decomposition of a *complete* finite-volume hyperbolic 3-manifold with non-empty geodesic boundary. We restrict to the case of compact boundary and omit all proofs addressing the reader to [Koj90, Koj92].

Projective model and truncated polyhedra Let $\pi : \mathbb{M}^{3+1} \setminus \{0\} \to \mathbb{P}(\mathbb{M}^{3+1})$ be the canonical projection. We set $\Pi^3 = \{x \in \mathbb{M}^{3+1} : x_0 = 1\}$ and note that Π^3 can be viewed as a subset of $\mathbb{P}(\mathbb{M}^{3+1})$ via π. Moreover Π^3 is isometric to Euclidean 3-space $\mathbb{E}^3$, and π restricts to a bijection between $\mathcal{H}^3_-$ and the unit ball of Π^3. Giving this ball the metric that turns π into an isometry with $\mathbb{H}^3_{\text{hyp}}$, we get the projective model $\mathbb{H}^3_{\text{proj}}$ of hyperbolic space. This model is particularly suitable for describing partially truncated tetrahedra, as we will now show.

Take in Π^3 a tetrahedron Δ with distinct vertices outside $\mathbb{H}^3_{\text{proj}}$ or on its boundary, and call *ultra-ideal* the vertices not lying on $\partial\mathbb{H}^3_{\text{proj}}$ (those on $\partial\mathbb{H}^3_{\text{proj}}$ are called *ideal* as usual). Suppose also that the interior of every edge of Δ intersects $\mathbb{H}^3_{\text{proj}}$. Each ultra-ideal vertex v of Δ determines a hyperbolic plane H_v in $\mathbb{H}^3_{\text{proj}}$ obtained as the intersection of $\mathbb{H}^3_{\text{proj}}$ with the Lorentzian orthogonal to v. Let us now define Δ^* as the intersection of Δ with $\mathbb{H}^3_{\text{proj}}$ truncated by the planes H_v (see Fig. 2.3 above). It is easy to see that Δ^* is a geometric realization of the partially truncated tetrahedron Δ in which the ideal vertices are those on $\partial\mathbb{H}^3_{\text{proj}}$. This implies in particular that our notation Δ^* is consistent.

Before proceeding it is worth noting that the hyperbolic plane H_v described above can also be constructed by elementary Euclidean geometry on Π^3. Namely, if we take the cone in $\Pi^3 \cong \mathbb{E}^3$ with vertex v and tangent to $\partial\mathbb{H}^3_{\text{proj}} \cong \mathbb{S}^2$, then H_v is the Euclidean disc bounded by the circle where the cone intersects $\partial\mathbb{H}^3_{\text{proj}}$.

The above definition of Δ^* of course makes sense also for convex polyhedra Δ more complicated than tetrahedra, provided Δ only has ideal and ultra-ideal vertices, and the interior of every edge of Δ intersects $\mathbb{H}^3_{\text{proj}}$. Partially truncated polyhedra of this sort are the blocks of the Kojima decomposition described in the rest of this section.

Convex hull of lifted boundary components Let N be a complete finite-volume hyperbolic manifold with non-empty compact geodesic boundary. Identifying the universal covering of the double DN of N with $\mathbb{H}^3$, we can realize the universal covering of N itself as a closed convex region $\widetilde{N}$ of $\mathbb{H}^3$ bounded by a locally finite countable family $\mathcal{S}$ of pairwise disjoint planes. The group of deck transformations of the covering $\widetilde{N} \to N$, denoted henceforth by Γ, is the stabilizer of $\widetilde{N}$ in the group of deck transformations of $\mathbb{H}^3 \to DN$. In the rest of the section we will always use the $\mathbb{H}^3_{\text{proj}}$ model of $\mathbb{H}^3$.

Noting that $\widetilde{N}$ lies on a definite side of each plane $S \in \mathcal{S}$, we consider now the vector of $\mathcal{H}^3_+$ dual to the half-plane that contains $\widetilde{N}$ and is bounded by S. We denote by $\mathcal{B}$ the family of all these duals, and we define $\mathcal{C} \subset \mathbb{M}^{3+1}$ as the closure of $\mathrm{Conv}(\mathcal{B})$, where $\mathrm{Conv}(X)$ denotes from now on the convex hull of a set $X \subset \mathbb{M}^{3+1}$. Kojima has shown that $\mathcal{B}$ is a discrete subset of $\mathbb{M}^{3+1}$, that $0 \notin \mathcal{C}$ and that $\pi(\mathcal{C}) \supset \mathbb{H}^3_{\text{proj}}$. In particular $\mathcal{C}$ has non-empty interior, and it is Γ-invariant by construction. The idea is now to construct a Γ-equivariant tessellation of $\widetilde{N}$ that projects to a decomposition of N by intersecting $\widetilde{N}$ with the projections to $\mathbb{H}^3_{\text{proj}}$ of the 3-faces of $\partial\mathcal{C}$. However, it turns out that not all the faces should be projected, and that some faces have non-trivial stabilizer in Γ, so they must be subdivided. To explain the matter in detail we begin with the following:

Definition 2.6.1. Let X be a subset of $\mathbb{M}^{3+1}$ such that $0 \notin X$. A point $x \in X$ is called *almost-visible* with respect to X if the segment $[0, x]$ meets X in x only. The point is called *visible* if it is almost-visible and $\pi(x) \in \mathbb{H}^3_{\text{proj}}$.

Of course only the faces of $\partial\mathcal{C}$ containing visible points should contribute to the tessellation of $\widetilde{N}$. These faces are also called *visible*. It turns out that there are two quite different sorts of visible faces, called respectively *elliptic* and *parabolic* depending on whether the restriction of $\langle\,\cdot\,,\,\cdot\,\rangle$ to the hyperplane on which the face lies is positive-definite or positive-semi-definite.

Cut locus and elliptic faces The first type of visible faces of $\partial\mathcal{C}$ correspond to the vertices of the cut-locus of $\widetilde{N}$ relative to $\partial\widetilde{N}$, which we now define for an arbitrary manifold M.

Definition 2.6.2. Let M be hyperbolic with non-empty geodesic boundary. We define the *cut-locus* $\mathrm{Cut}(M,\partial M)$ of M relative to ∂M as the set of points of M that admit at least two different shortest paths to ∂M. A point is called a *vertex* of the cut-locus if it admits four different shortest paths to ∂M whose initial tangent vectors span the tangent space to M at the point as an affine space.

Proposition 2.6.3. $\mathrm{Cut}(N,\partial N)$ *has finitely many vertices. A point of* $\widetilde{N}$ *is a vertex of* $\mathrm{Cut}(\widetilde{N},\partial\widetilde{N})$ *if and only it projects in* N *to a vertex of* $\mathrm{Cut}(N,\partial N)$.

For every vertex v of $\mathrm{Cut}(\widetilde{N},\partial\widetilde{N})$ we define now $\mathcal{B}(v)\subset\mathcal{B}$ as the set of dual vectors to the hyperplanes in $\mathcal{S}$ having shortest distance from v. Since in $\mathbb{H}^3$ there is a unique shortest path joining a given point to a given plane, each $\mathcal{B}(v)$ contains at least four vectors. The next result describes the visible elliptic faces of $\partial\mathcal{C}$.

Proposition 2.6.4. *For every vertex* v *of* $\mathrm{Cut}(\widetilde{N},\partial\widetilde{N})$ *there exists a unique elliptic hyperplane* $E(v)\subset\mathbb{M}^{3+1}$ *such that* $E(v)\cap\mathcal{B}=\mathcal{B}(v)$. *Moreover* $E(v)$ *is a support hyperplane for* $\mathcal{C}$, *and:*

1. *The points in* $E(v)\cap\mathcal{C}$ *are almost-visible with respect to* $\mathcal{C}$;
2. $E(v)\cap\mathcal{C}$ *is a* 3*-dimensional compact polyhedron whose set of vertices is* $\mathcal{B}(v)$;
3. *The stabilizer of* $E(v)\cap\mathcal{C}$ *in* Γ *is trivial.*

Parabolic faces and subdivision Besides those corresponding to vertices of the cut-locus, $\partial\mathcal{C}$ has visible faces coming from toric cusps of N. Let us denote by $\widetilde{N}_\infty\subset\partial\mathbb{H}^3_{\mathrm{proj}}$ the set of points at infinity of $\widetilde{N}$. A point $q\in\widetilde{N}_\infty$ is said to *generate a toric cusp* if it is fixed under a $\mathbb{Z}\oplus\mathbb{Z}$ subgroup Γ_q of parabolic elements of Γ, *i.e.* if there is a horoball centred at q that projects to a toric cusp of N.

For $\tilde{q}$ in L^3_+ and $t\in\mathbb{R}$ we consider now the affine parabolic hyperplane

$$F(\tilde{q},t)=\{v\in\mathbb{M}^{3+1}\,:\,\langle v,\tilde{q}\rangle=-t\}.$$

Proposition 2.6.5. *Let* $q\in\widetilde{N}_\infty$ *generate a toric cusp of* N, *and take* $\tilde{q}\in L^3_+$ *such that* $\pi(\tilde{q})=q$. *Then there exists a unique* $t(\tilde{q})\in\mathbb{R}$ *such that* $F(\tilde{q},t(\tilde{q}))$ *is a support hyperplane for* $\mathcal{C}$. *Moreover* $F(\tilde{q},t(\tilde{q}))$ *depends only on* q, *not on* $\tilde{q}$, *and, setting* $F(q)=F(\tilde{q},t(\tilde{q}))$, *we have that:*

1. $t(\tilde{q})>0$ *and* $\mathcal{C}\subseteq\bigcup_{t\geqslant t(\tilde{q})}F(\tilde{q},t)$;
2. *The points in* $F(q)\cap\mathcal{C}$ *are almost-visible with respect to* $\mathcal{C}$;

3. *$F(q) \cap \mathcal{B}$ is infinite and $F(q) \cap \mathcal{C}$ is a non-compact 3-dimensional polyhedron whose set of vertices is $F(q) \cap \mathcal{B}$;*

4. *The 2-dimensional faces of $F(q) \cap \mathcal{C}$ are compact;*

5. *The stabilizer in Γ of $F(q) \cap \mathcal{C}$ coincides with the $\mathbb{Z} \oplus \mathbb{Z}$ stabilizer Γ_q of q.*

Point (5) of this proposition shows that $F(q) \cap \mathcal{C}$ must be subdivided before projecting to $\mathbb{H}^3_{\mathrm{proj}}$ and intersecting with $\widetilde{N}$. Continuing with the same notation, for every 2-dimensional face W of $F(q) \cap \mathcal{C}$ we define now $W^{(\tilde{q})}$ as the cone based on W with vertex in $\tilde{q}$. Note that $W^{(\tilde{q})}$ depends on $\tilde{q}$, and it meets $\mathcal{C}$ in W only. However one easily sees that $W^{(q)} = \pi(W^{(\tilde{q})})$ actually does not depend on $\tilde{q}$. Moreover the family of all $W^{(q)}$'s, as W varies among the 2-faces of $F(q) \cap \mathcal{C}$, gives a Γ_q-equivariant tessellation of $\{\tilde{q}\} \cup \pi(F(q) \cap \mathcal{C})$ in which every polyhedron has trivial stabilizer.

Canonical decomposition We begin with the following fact:

Proposition 2.6.6. *The faces of $\mathcal{C}$ described in Propositions 2.6.4 and 2.6.5 contain all the visible points of $\mathcal{C}$.*

In addition to this, one can easily show that if Δ is a visible face of $\mathcal{C}$ then the (partial) truncation $\pi(\Delta)^*$, defined earlier in this section, of $\pi(\Delta) \subset \Pi^3$ is obtained by intersection with $\widetilde{N}$.

We are now ready to summarize the construction. Let us denote by $\mathcal{V}$ the set of vertices of $\mathrm{Cut}(\widetilde{N}, \partial\widetilde{N})$, and by Q the family of all points of $\widetilde{N}_\infty$ that generate toric cusps of N. For each $q \in Q$ we fix an arbitrary $\tilde{q} \in L^3_+$ such that $\pi(\tilde{q}) = q$, and we denote by $\widetilde{Q}$ the family of all such $\tilde{q}$'s. We define $\mathcal{K}_{\mathbb{M}}(\widetilde{Q})$ to be the family of all visible elliptic faces $E(v) \cap \mathcal{C}$ and all faces $W^{(\tilde{q})}$ obtained from the visible parabolic faces. Here v varies in $\mathcal{V}$, $\tilde{q}$ varies in $\widetilde{Q}$, and W varies in the 2-dimensional faces of $F(\pi(\tilde{q})) \cap \mathcal{C}$. We also denote by $\mathcal{K}_{\mathbb{P}}$ the family of polyhedra obtained by projecting to Π^3 the elements of $\mathcal{K}_{\mathbb{M}}(\widetilde{Q})$, and by $\mathcal{K}^*_{\widetilde{N}}$ the family obtained by intersecting with $\widetilde{N}$ (or, equivalently, truncating) the elements of $\mathcal{K}_{\mathbb{P}}$. We know that indeed $\mathcal{K}_{\mathbb{P}}$ and $\mathcal{K}^*_{\widetilde{N}}$ are independent of $\widetilde{Q}$, and we have:

Theorem 2.6.7. *$\mathcal{K}^*_{\widetilde{N}}$ is a Γ-equivariant decomposition of $\widetilde{N}$ into partially truncated polyhedra. Every element of $\mathcal{K}^*_{\widetilde{N}}$ has trivial stabilizer in Γ, so $\mathcal{K}^*_{\widetilde{N}}$ projects to a canonical and finite decomposition $\mathcal{K}^*_N$ of N into partially truncated polyhedra.*

Some properties of the canonical decomposition Let us denote by $\mathrm{Isom}(N)$ the group of isometries of N, by $\mathcal{M}(N)$ the mapping class group of N, *i.e.* the group of homotopy classes of homeomorphisms of N onto itself, and by $\mathrm{Aut}(\mathcal{K}^*_N)$ the group of *combinatorial* automorphisms of the canonical decomposition of N. Mostow-Prasad's rigidity Theorem and naturality of $\mathcal{K}^*_N$ imply the following:

Theorem 2.6.8. *We have canonical isomorphisms*

$$\mathcal{M}(N) \cong \mathrm{Isom}(N) \cong \mathrm{Aut}(\mathcal{K}^*_N).$$

Definition 2.6.9. A *return path in* N is a geodesic segment in N intersecting ∂N at right angles in its endpoints.

Let $\Lambda = \{\ell \in \mathbb{R} : \ \ell$ is the length of a return path in $N\}$. Since the boundary of N is compact it is easily seen that Λ is discrete. We set $\ell_0 = \min \Lambda$. We observe that any compact edge of a decomposition of N into partially truncated polyhedra is a return path.

Proposition 2.6.10. *Let* $l \subset N$ *be a return path of minimal length* ℓ_0. *Then* l *is an edge of the canonical decomposition* $\mathcal{K}^*_N$.

The following result shows that any compact regular partially truncated tetrahedron which is contained in N and whose internal edges are of minimal length is a piece of the canonical decomposition of N.

Proposition 2.6.11. *Let* $\Delta^*_{\ell_0}$ *be a compact partially truncated tetrahedron such that the length of every internal edge of* $\Delta^*_{\ell_0}$ *is equal to* ℓ_0, *and let* $\varphi : \Delta^*_{\ell_0} \to N$ *be a local isometry which sends the truncation triangles of* $\Delta^*_{\ell_0}$ *into* ∂N. *Then* φ *lifts to an isometry* $\widetilde{\varphi} : \Delta^*_{\ell_0} \to \widetilde{N}$ *whose image is a piece of the tessellation* $\mathcal{K}^*_{\widetilde{N}}$.

We also have the following easy:

Remark 2.6.12. By construction, each polyhedron in $\mathcal{K}^*_N$ has at most one ideal vertex.

2.7 Recognizing the canonical decomposition

The general strategy to decide whether a given decomposition of a hyperbolic N is the canonical one is as in [Wee93], *i.e.* to lift the decomposition first to $\mathbb{H}^3_{\mathrm{proj}}$ and then to $\mathbb{M}^{3+1}$, and to make sure that the resulting polyhedra bound a convex set. In the setting of the construction described in the previous section, the lift of a polyhedron to $\mathbb{M}^{3+1}$ can be performed directly when N does not have toric cusps, because one only has to select which points of $\mathcal{B}$ are the vertices of the lift, and $\mathcal{B}$ depends on N only. When there are toric cusps, however, we have an arbitrariness for the lifts of the ideal vertices, corresponding to the arbitrariness of the choice of $\widetilde{Q}$. In addition, for some choices of $\widetilde{Q}$ it can happen that $\mathcal{K}_{\mathbb{M}}(\widetilde{Q})$ does *not* bound a convex set. The case of toric cusps must therefore be discussed with some care.

As in the preceding section, we skip all the proofs here, addressing the reader to [FP04] for more details.

Horospherical cross-sections We first show how to reduce the choice of $\widetilde{Q}$ to a choice that is more intrinsic, but still arbitrary at this stage.

Definition 2.7.1. A family $\mathcal{O}$ of disjoint tori in N that lift to horospheres in $\widetilde{N}$ and bound disjoint cusps of N will be called a *horospherical cross-section* of N. Given such an $\mathcal{O}$, we define $\widetilde{Q}(\mathcal{O})$ as the lift of Q where a point q is lifted to the only $\tilde{q} \in L^3_+$ such that $\pi(\tilde{q}) = q$ and the dual to $\tilde{q}$ projects in N to a component of $\mathcal{O}$. We also define $\mathcal{K}_{\mathbb{M}}(\mathcal{O})$ as $\mathcal{K}_{\mathbb{M}}(\widetilde{Q}(\mathcal{O}))$.

Remark 2.7.2. The cross-section $\mathcal{O}$ is determined by a sufficiently small positive number assigned to each toric cusp, namely the volume of the region between the cross-section at the cusp and infinity. Insisting all volumes to be equal to each other, one may also determine $\mathcal{O}$ by a single number. This number is naturally interpreted as the (inverse of the) *height* at which the cross-section should be taken —the smaller the volume, the higher the cross-section.

Augmented convex hull Before proceeding, we need to define a set of which $\mathcal{K}_{\mathbb{M}}(\mathcal{O})$ gives the visible boundary. Namely, we set:

$$\mathcal{C}'(\mathcal{O}) = \mathcal{C} \cup \bigcup \Big\{ \mathrm{Conv}\Big(\{\tilde{q}\} \cup (F(q) \cap \mathcal{C})\Big) \ : \ q \in Q, \ \tilde{q} = \widetilde{Q}(\mathcal{O}) \cap \pi^{-1}(q) \Big\}. \quad (2.11)$$

The fact that indeed the visible boundary of $\mathcal{C}'(\mathcal{O})$ is $\mathcal{K}_{\mathbb{M}}(\mathcal{O})$ easily follows from point (1) of Proposition 2.6.5.

Definition 2.7.3. Let P_1, P_2 be finite 3-dimensional convex polyhedra in $\mathbb{M}^{3+1} \setminus \{0\}$ which project injectively to $\mathbb{P}(\mathbb{M}^{3+1})$. Assume P_1 and P_2 share a 2-face F, and for $i = 1, 2$ let H_i be the half-hyperplane in $\mathbb{M}^{3+1}$ such that $H_i \supset P_i$ and $\partial H_i \supset F$. We say P_1 and P_2 form a *convex angle* at F if the connected component not containing 0 of $\mathbb{M}^{3+1} \setminus (H_1 \cup H_2)$ is convex. The angle is called *strictly convex* if it is convex and $H_1 \cup H_2$ is not a hyperplane.

Lemma 2.7.4. *The following facts are equivalent:*

1. $\mathcal{C}'(\mathcal{O}) \cap \pi^{-1}(\widetilde{N})$ *is convex;*
2. *The* 3*-faces of* $\mathcal{C}'(\mathcal{O})$ *meeting* $\pi^{-1}(\widetilde{N})$ *form convex angles with each other.*

Definition 2.7.5. We say that a horospherical cross-section $\mathcal{O}$ is *sufficiently small* if the set $\mathcal{C}'(\mathcal{O}) \cap \pi^{-1}(\widetilde{N})$ is convex.

Lemma 2.7.4 easily implies that if $\mathcal{O}$ is sufficiently small and $\mathcal{O}'$ is contained in $\mathcal{O}$, then $\mathcal{O}'$ is also sufficiently small.

Proposition 2.7.6. *There exists a sufficiently small* $\mathcal{O}$*.*

Recognizing the canonical decomposition Assume now that we have some decomposition $\mathcal{D}^*$ of N into partially truncated polyhedra, and let us ask ourselves whether $\mathcal{D}^*$ coincides with Kojima's canonical $\mathcal{K}^*_N$. We fix as above the universal covering $\widetilde{N} \to N$ with $\widetilde{N} \subset \mathbb{H}^3_{\mathrm{proj}}$ and a horospherical cross-section $\mathcal{O}$ of N. Now we can first lift $\mathcal{D}^*$ to a decomposition $\mathcal{D}^*_{\widetilde{N}}$ of $\widetilde{N}$, then we can consider the associated non-truncated family of polyhedra $\mathcal{D}_{\mathbb{P}}$ in Π^3, and finally we can lift $\mathcal{D}_{\mathbb{P}}$ to a $\mathcal{D}_{\mathbb{M}}(\mathcal{O})$ in $\mathbb{M}^{3+1}$ by lifting each ultra-ideal vertex to its representative in $\mathcal{B}$ and each ideal vertex $q \in Q$ to $\widetilde{Q}(\mathcal{O}) \cap \pi^{-1}(q)$.

Proposition 2.7.7. *Let* $\mathcal{D}^*$ *be a geometric decomposition of a hyperbolic* N *and let* $\mathcal{O}$ *be a sufficiently small horospherical cross-section for* N*. Let* $\mathcal{D}_{\mathbb{M}}(\mathcal{O})$ *be the lift of* $\mathcal{D}^*$ *to* $\mathbb{M}^{3+1}$ *just described. Then:*

- *$\mathcal{D}^*$ is the canonical decomposition $\mathcal{K}_N^*$ if and only if the 3-faces of $\mathcal{D}_{\mathbb{M}}(\mathcal{O})$ form strictly convex angles with each other;*
- *$\mathcal{D}^*$ is a subdivision of the canonical decomposition $\mathcal{K}_N^*$ if and only if the 3-faces of $\mathcal{D}_{\mathbb{M}}(\mathcal{O})$ form convex angles with each other.*

Proposition 2.7.8. *Let $\mathcal{D}^*$ be a geometric decomposition of a hyperbolic N and let $\mathcal{O}$ be a horospherical cross-section for N. Suppose the 3-faces of $\mathcal{D}_{\mathbb{M}}(\mathcal{O})$ form strictly convex angles with each other. Then $\mathcal{O}$ is sufficiently small and $\mathcal{D}^*$ is the canonical decomposition $\mathcal{K}_N^*$.*

Propositions 2.7.7 and 2.7.8 show that, to determine whether a geometric decomposition of a hyperbolic manifold is Kojima's canonical one, we must lift the decomposition to Minkowski 4-space and then check convexity of all the angles at the 2-faces of the lift. We will now provide the explicit formula that allows to check convexity. This formula, already obtained by Ushijima [Ush02] in more implicit terms, extends Weeks' tilt formula [Wee93, SW95].

Tilts and convexity Let $\widehat{\Delta}$ be a tetrahedron in $\mathbb{M}^{3+1}$ that projects in Π^3 to a tetrahedron Δ with vertices outside $\mathbb{H}^3_{\rm proj}$ or on its boundary, and edges meeting $\partial\mathbb{H}^3_{\rm proj}$ or tangent to it. Let $\widehat{F}$ be a face of $\widehat{\Delta}$ with image F in Δ. Let H be the unique half-space in $\mathbb{H}^3_{\rm proj}$ such that $H \supset \Delta \cap \mathbb{H}^3_{\rm proj}$ and $\partial H \supset F \cap \mathbb{H}^3_{\rm proj}$. Let $m \in \mathcal{H}^3_+$ be the dual to H, and let $p \in \mathbb{M}^{3+1}$ be the unique vector such that $\langle p, x\rangle = -1$ for every $x \in \widehat{\Delta}$. We define the *tilt* of $\widehat{\Delta}$ relative to $\widehat{F}$ as the real number $\langle m, p\rangle$. The next result shows how tilts relate to convexity. For a proof see [Ush02] or [Wee93].

Proposition 2.7.9. *Let $\widehat{\Delta}$ and $\widehat{\Delta}'$ be tetrahedra in $\mathbb{M}^{3+1}$ sharing a 2-face $\widehat{F}$. Assume that $\widehat{\Delta} \cup \widehat{\Delta}'$ projects injectively to Π^3, and let t and t' be the tilts of $\widehat{\Delta}$ and $\widehat{\Delta}'$ relative to $\widehat{F}$. Then the angle formed by $\widehat{\Delta}$ and $\widehat{\Delta}'$ at $\widehat{F}$ is convex (respectively, strictly convex) if and only if $t + t' \leqslant 0$ (respectively, $t + t' < 0$).*

From moduli to tilts Our task is now to compute the tilts of the lift of a partially truncated tetrahedron from the intrinsic geometry of the tetrahedron itself. Recall that the lift of a non-ideal vertex u is uniquely determined by the requirement that it should belong to $\mathcal{H}^3_+$. However, when u is ideal, to get uniqueness we must choose a horosphere at u and lift u to the dual in L^3_+ to this horosphere. We begin by fixing some notation and recalling how horospheres are encoded [Wee93].

Let Δ be an abstract partially truncated tetrahedron. Fix $\theta : \Delta^{(1)} \to [0, \pi)$ as in Theorem 2.2.1 and denote by $\Delta^{\theta,*}$ the corresponding geometric realization (up to isometry) of Δ^* in $\mathbb{H}^3_{\rm proj}$. Let Δ^θ be the associated tetrahedron with ideal and ultra-ideal vertices in Π^3.

Remark 2.7.10. If u is an ideal vertex of Δ^θ, the set of horospheres at u is parameterized by the positive reals, with a horosphere O corresponding to $r > 0$ if r is the radius of the smallest Euclidean disc on O containing $O \cap \Delta^\theta$.

So we fix a function $r : \mathcal{I} \to (0, \infty)$, where $\mathcal{I}$ is the set of ideal vertices of Δ, and we denote by $\mathcal{O}^{\theta,r}$ the associated family of horospheres at the ideal vertices of Δ^θ.

Now $\mathcal{O}^{\theta,r}$ determines a unique lift $\widehat{\Delta}^{\theta,r} \subset \mathbb{M}^{3+1}$ of Δ^θ, and $(\Delta^\theta, \widehat{\Delta}^{\theta,r})$ is well-defined up to isometry of pairs. Denoting by $u_1, \ldots, u_4$ the vertices of Δ and by F_i the face opposite to u_i, we can then define $t_i^{\theta,r}$ as the tilt of $\widehat{\Delta}^{\theta,r}$ relative to $\widehat{F}_i^{\theta,r}$. The tilt is unchanged under isometry, so $t_i^{\theta,r}$ is indeed well-defined.

To compute $t_i^{\theta,r}$ explicitly in terms of θ and r we must introduce certain positive real numbers $D_i^{\theta,r}$. To this end, we denote by H_i^θ the plane in $\mathbb{H}^3_{\mathrm{proj}}$ containing $F_i^{\theta,*}$. When u_i is non-ideal we denote by T_i^θ the truncation plane for $\Delta^{\theta,*}$ corresponding to u_i, and when u_i is ideal we denote by $O_i^{\theta,r}$ the horosphere at u_i determined by θ and r. Then we define $D_i^{\theta,r}$ as $D(H_i^\theta, T_i^\theta)$ or $D(H_i^\theta, O_i^{\theta,r})$ depending on the type of u_i, where D is the function defined as follows:

- If H and T are planes in $\mathbb{H}^3$ and $H \cap T = \emptyset$, we set $D(H,T) = \cosh d(H,T)$ where d is the usual distance in $\mathbb{H}^3$; if $H \cap T \neq \emptyset$ we set $D(H,T) = \cos\angle(H,T)$ where $\angle(H,T) \in (0, \pi/2]$ is the angle formed by H and T;

- If H is a plane in $\mathbb{H}^3$ and O is a horosphere not centred at a point of H, we set $D(H,O) = \exp \pm \ell$, where ℓ is the length of the unique geodesic arc that joins H to O and is orthogonal to both, with negative sign taken when $H \cap O \neq \emptyset$.

Remark 2.7.11. The choice of $\angle(H,T)$ in $(0, \pi/2]$ may look artificial at first sight, and indeed one could extend the definition to half-spaces rather than planes, and choose $\angle(H,T)$ in $(0,\pi)$. However it is easy to show that, in our situation, the dihedral angle at H and T that contains $\Delta^{\theta,*}$ is always the acute one, so the definition of D would remain the same.

The following result is proved in [Ush02].

Theorem 2.7.12. *Let θ and r determine the geometry and a family of horospheres at the ideal vertices of a partially truncated tetrahedron Δ with vertices $u_1, \ldots, u_4$. Let $\widehat{\Delta}^{\theta,r}$ be the tetrahedron in $\mathbb{M}^{3+1}$ determined by θ and r. Let $t_i^{\theta,r}$ be the tilt of $\widehat{\Delta}^{\theta,r}$ relative to the face opposite to u_i. Set $\theta_{ij} = \theta([u_i, u_j])$ and let $D_i^{\theta,r}$ be the number just introduced. Then:*

$$\begin{pmatrix} t_1^{\theta,r} \\ t_2^{\theta,r} \\ t_3^{\theta,r} \\ t_4^{\theta,r} \end{pmatrix} = \begin{pmatrix} 1 & -\cos\theta_{34} & -\cos\theta_{24} & -\cos\theta_{23} \\ -\cos\theta_{34} & 1 & -\cos\theta_{14} & -\cos\theta_{13} \\ -\cos\theta_{24} & -\cos\theta_{14} & 1 & -\cos\theta_{12} \\ -\cos\theta_{23} & -\cos\theta_{13} & -\cos\theta_{12} & 1 \end{pmatrix} \begin{pmatrix} 1/D_1^{\theta,r} \\ 1/D_2^{\theta,r} \\ 1/D_3^{\theta,r} \\ 1/D_4^{\theta,r} \end{pmatrix}.$$

To make the calculation of tilts explicit, we are left to compute the $D_i^{\theta,r}$'s. For ideal u_i we denote $r(u_i)$ by r_i, and we have:

Proposition 2.7.13. *If u_1 is ideal then:*

$$D_1^{\theta,r} = \frac{1}{2r_1} \cdot \frac{\sin\theta_{12}\cos\theta_{34} + \sin\theta_{13}\cos\theta_{24} + \sin\theta_{14}\cos\theta_{23}}{\sin\theta_{12}\sin\theta_{13}\sin\theta_{14}}.$$

For non-ideal u_i we denote $D_i^{\theta,r}$ by D_i^θ, because it is independent of r. To compute it we need to introduce the following constant $g^\theta > 0$:

$$\begin{aligned} g^\theta = -1 + & \sum \left\{ \cos^2\theta(e) : \ e \in \Delta^{(1)} \right\} \\ +2 & \sum \left\{ \cos\theta(e')\cos\theta(e'')\cos\theta(e''') : \ v \in \Delta^{(0)}, \ v = e' \cap \ e'' \cap e''' \right\} \\ +2 & \sum \left\{ \frac{\prod\{\cos\theta(e) : \ e \in \Delta^{(1)}\}}{\cos\theta(e') cos \theta(e'')} : \ \{e', e''\} \subset \Delta^{(1)}, \ e' \cap e'' = \emptyset \right\} \\ - & \sum \left\{ \cos^2\theta(e')\cos^2\theta(e'') : \ \{e', e''\} \subset \Delta^{(1)}, \ e' \cap e'' = \emptyset \right\}. \end{aligned} \tag{2.12}$$

Of course g^θ is well-defined. Moreover:

Proposition 2.7.14. *If u_i is non-ideal and d^θ is as in formula (2.3) of Section 2.2 then*

$$D_i^\theta = \sqrt{g^\theta / d^\theta(u_i)}.$$

Chapter 3

Similar fillings and isolation of cusps

This chapter is devoted to the description of certain classes $\mathcal{M}_{g,k}$ of hyperbolic 3-manifolds with geodesic boundary. Such classes were first introduced in [FMP03b], where they were studied in detail both from the topological and from the geometric point of view. In particular, using the technology of spines [Mat03] and results from [Gor99] we completely described in [FMP03b] the non-hyperbolic Dehn fillings of each M in $\mathcal{M}_{g,k}$: namely, we showed that on any cusp of any hyperbolic manifold obtained by partially filling M, there are precisely 3 boundary-reducing fillings, 3 annular fillings and no toroidal fillings, thus answering a question posed by Wu in [Wu98].

After recalling some results proved in [FMP03b], we concentrate here on small deformations of the complete hyperbolic structure defined on each manifold in $\mathcal{M}_{g,k}$. This approach will allow us to study some interesting geometric properties of hyperbolic manifolds obtained by filling elements in $\mathcal{M}_{g,k}$ along "sufficiently complicated" curves. Namely, we will prove that several elements in $\mathcal{M}_{g,k}$ admit different hyperbolic Dehn fillings sharing the same volume, homology, cusp volume, cusp shape, Heegaard genus, complex length of the shortest geodesic, length of the shortest return path, and Turaev-Viro invariants. Manifolds which share all these invariants are called *geometrically similar*, and were first studied in [HMW92], where it was shown that the Whitehead link complement admits an infinite sequence of pairs of non-homeomorphic geometrically similar Dehn fillings (the definition of geometric similarity introduced in [HMW92] is actually a bit different from ours, and regards cusped manifolds without geodesic boundary). The elements of any pair of geometrically similar manifolds described in [HMW92] are both obtained by filling one cusp of the Whitehead link complement, and they are commensurable to each other. We show here that if $M \in \mathcal{M}_{g,k}$ is generic, *i.e.* if it does not admit too many isometries, then for any $h \leqslant k$ we can construct different geometrically similar manifolds by filling M along slopes on any chosen set of h cusps of M. Moreover, the manifolds thus obtained are typically non-commensurable to each other (however, examples are also provided of non-homeomorphic geometrically similar commensurable Dehn fillings of the same element of $\mathcal{M}_{g,k}$).

Neumann and Reid introduced in [NR93] the notion of *geometric isolation* for cusps in a hyperbolic manifold: roughly speaking, if N is a complete finite-volume hyperbolic manifold with (possibly empty) geodesic boundary and cusps $C_1, \ldots, C_h, C_{h+1}, \ldots, C_k$, then $C_1, \ldots, C_h$ are geometrically isolated from $C_{h+1}, \ldots, C_k$ if any small deformation of the hyperbolic structure on N induced by Dehn filling $C_{h+1}, \ldots, C_k$ while keeping $C_1, \ldots, C_h$ complete does not affect the Euclidean structure at $C_1, \ldots, C_h$. Calegary described in [Cal01] several different ways of constructing manifolds with isolated cusps, also providing explanations for all the previously known examples of isolation phenomena. We show here that the cusps of any manifold $M \in \mathcal{M}_{g,k}$ are geometrically isolated from each other. Apparently, isolation of cusps in our examples arises for different reasons from those described in [Cal01].

In Section 3.5 we describe how the metric on the geodesic boundary of an element $M \in \mathcal{M}_{g,k}$ is affected by Dehn filling along sufficiently complicated slopes on the boundary tori of M. Examples of isolation of the geodesic boundary from the cusps in hyperbolic 3-manifolds were first provided in [NR93, Fuj93]. On the other hand, non-isolation phenomena were described in [Fuj92, FK97]. We show here that any element in $\mathcal{M}_{g,k}$ admits an infinite family of hyperbolic Dehn fillings inducing non-trivial deformations of the hyperbolic structure on the geodesic boundary.

3.1 Definition of $\mathcal{M}_{g,k}$, triangulations and hyperbolicity

Let Σ_g be the closed orientable surface of genus g. The following result [FMP03b] motivates our definition of $\mathcal{M}_{g,k}$.

Proposition 3.1.1. *An ideal triangulation of a manifold whose boundary is the union of Σ_g and k tori contains at least $g+k$ tetrahedra.*

We then define $\mathcal{M}_{g,k}$ for all $g \geqslant 2, k \geqslant 0$ as follows:

$$\begin{aligned} \mathcal{M}_{g,k} \;=\; \big\{ & \text{compact orientable manifolds } M \text{ having an ideal triangulation} \\ & \text{with } g+k \text{ tetrahedra, and } \partial M = \Sigma_g \sqcup \big(\bigsqcup_{i=1}^{k} T_i \big) \text{ with } T_i \cong \Sigma_1 \big\}. \end{aligned}$$

The sets $\mathcal{M}_g = \mathcal{M}_{g,0}$ were studied in [FMP03a].

Let N be a compact manifold with boundary. When this does not create ambiguities, in what follows we will denote by N also the (possibly non-compact) manifold obtained by removing the boundary tori from the original N. Thus the natural compactification of a hyperbolic manifold will be usually denoted by the same symbol as the manifold itself.

Growth of $\mathcal{M}_{g,k}$ The following result [FMP03b] shows that the cardinality of $\mathcal{M}_{g,k}$ grows very fast as g goes to infinity. We say that a numerical sequence $\big(a_n\big)_{n=1}^{\infty}$ has *growth type* n^n if there exist constants $C > c > 0$ such that $n^{c \cdot n} < a_n < n^{C \cdot n}$ for $n \gg 0$.

Theorem 3.1.2. *$\mathcal{M}_{g,k} \neq \emptyset$ if and only if either $g > k$ or $g = k$ is even. Moreover, for any fixed k the sequence $\big(\#\mathcal{M}_{g,k}\big)_{g=2}^{\infty}$ has growth type g^g.*

For any compact 3-manifold M, an $\mathbb{N}$-valued invariant $c(M)$ was defined by Matveev in [Mat90] and called the *complexity* of M. Matveev also proved that, when M is hyperbolic, $c(M)$ equals the minimal number of tetrahedra in an ideal triangulation of M. Theorem 3.1.2 and an easy upper bound also established in [FMP03b] readily imply the following:

Corollary 3.1.3. *The number of hyperbolic 3-manifolds of complexity c has growth type c^c.*

Triangulations In order to construct a hyperbolic structure on a manifold $M \in \mathcal{M}_{g,k}$, we choose a suitable triangulation of M and we solve the corresponding hyperbolicity equations. The following result is proved in [FMP03b] and implies Proposition 3.1.1. We recall that the *valence* of an edge in a triangulation is the number of tetrahedra incident to it (with multiplicity).

Proposition 3.1.4. *Let M be connected with $\partial M = \Sigma_g \sqcup \big(\bigsqcup_{i=1}^{k} T_i\big)$ and suppose that $\mathcal{T}$ is an ideal triangulation of M with $g+k$ tetrahedra. Then the following holds:*

- *$g \geqslant k$;*
- *For any $i = 1, \ldots, k$ there are exactly two tetrahedra of $\mathcal{T}$ with 3 vertices on Σ_g and one on T_i; the remaining $g-k$ tetrahedra have all 4 vertices on Σ_g;*
- *$\mathcal{T}$ has $k+1$ edges $e_0, \ldots, e_k$ such that e_0 has both its endpoints on Σ_g and valence $6g$, while e_i connects Σ_g to T_i and has valence 6 for any $i = 1, \ldots, k$.*

Geometric tetrahedra Let now M be a manifold in $\mathcal{M}_{g,k}$, and let $\mathcal{T}$ be a triangulation of M with $g+k$ tetrahedra. We try to give M a hyperbolic structure with geodesic boundary by solving the hyperbolicity equations described in Chapter 2. More precisely, we know from Proposition 3.1.4 that $\mathcal{T}$ consists of $2k$ tetrahedra with one vertex on the boundary tori and $g-k$ tetrahedra with all the vertices on Σ_g. If we suppose M to be hyperbolic and $\mathcal{T}$ to be geometric, than the edges of the tetrahedra with all the vertices on Σ_g should have all the same length. This would force the compact tetrahedra in $\mathcal{T}$ to be regular and isometric to each other. Moreover, all the finite edges of the tetrahedra with one vertex on the boundary tori should also have the same length, and this seems to suggest that the non-compact tetrahedra of $\mathcal{T}$ should be as symmetric as possible. These remarks are very helpful in order to guess the desired solutions of hyperbolicity equations.

Let us now fix some notation we will use extensively later on. Let $T_1, \ldots, T_k$ be the boundary tori of M. We denote by $\Delta_{2i-1}, \Delta_{2i}$ the tetrahedra of $\mathcal{T}$ incident to T_i and by $F^1_{2i-1}, F^2_{2i-1}, F^3_{2i-1}, F^1_{2i}, F^2_{2i}, F^3_{2i}$ the exceptional hexagons of $\Delta_{2i-1}, \Delta_{2i}$, in such a way that F^j_{2i-1} is glued to F^j_{2i} for $j = 1, 2, 3$. For $l = 1, \ldots, 2k$ we also suppose that F^1_l, F^2_l, F^3_l are positively arranged around the ideal vertex of Δ_l, and we call e^j_l the only internal finite edge of F^j_l, and f^j_l the edge of Δ_l opposite to e^j_l. We now consider a geometric realization θ of the tetrahedra of $\mathcal{T}$ such that compact tetrahedra are regular and isometric to each other, and for $l = 1, \ldots, 2k$, $j = 1, 2, 3$

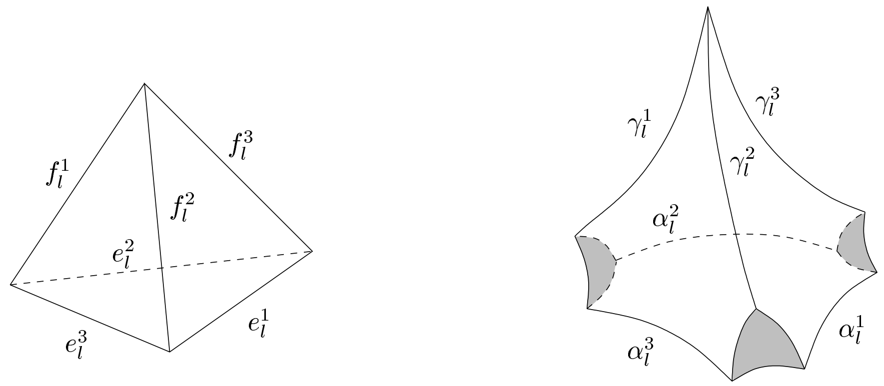

Figure 3.1: Dihedral angles along the edges of a non-compact tetrahedron of $\mathcal{T}$.

we set $\alpha_l^j = \theta(e_l^j)$, and $\gamma_l^j = \theta(f_l^j)$ (see Fig. 3.1). We finally set β to be the dihedral angle along the edges of the $g-k$ compact tetrahedra of $\mathcal{T}$.

We now refer to Theorem 2.2.12 and discuss hyperbolicity equations. Since $\gamma_l^1 + \gamma_l^2 + \gamma_l^3 = \pi$ for $l = 1, \ldots, 2k$, the total angle along any half-infinite edge of $\mathcal{T}$ is automatically forced to be equal to 2π, so conditions (1) actually translate into the following equation only:

$$6 \cdot (g-k) \cdot \beta + \sum_{l=1}^{2k} (\alpha_l^1 + \alpha_l^2 + \alpha_l^3) = 2\pi. \tag{3.1}$$

It is easily seen that conditions (2) of Theorem 2.2.12 are satisfied if and only if all the compact lateral hexagons of the tetrahedra in $\mathcal{T}$ are regular and isometric to each other, or equivalently if and only if the lengths of all the boundary edges on the compact lateral hexagons are equal to each other. Using equation (2.1) this condition translates into the following set of equations:

$$\frac{\cos\alpha_l^j \cdot \cos\alpha_l^{j+1} + \cos\alpha_l^{j+2}}{\sin\alpha_l^j \cdot \sin\alpha_l^{j+1}} = \frac{\cos^2\beta + \cos\beta}{\sin^2\beta}, \qquad l = 1, \ldots, 2k,\ j = 1,2,3. \tag{3.2}$$

(Actually, if $g = k$ then there are no compact tetrahedra, so β is not even defined, and equations (3.2) take the form $(\cos\alpha_l^j \cdot \cos\alpha_l^{j+1} + \cos\alpha_l^{j+2})/(\sin\alpha_l^j \cdot \sin\alpha_l^{j+1}) = (\cos\alpha_{l'}^{j'} \cdot \cos\alpha_{l'}^{j'+1} + \cos\alpha_{l'}^{j'+2})/(\sin\alpha_{l'}^{j'} \cdot \sin\alpha_{l'}^{j'+1})$, $l, l' \in \{1, \ldots, 2k\}$, $j, j' \in \{1,2,3\}$). Finally, Proposition 2.2.10 and Remark 2.2.4 imply that, under the assumption that equations (3.2) are in force, conditions (3) of Theorem 2.2.12 are equivalent to the following set of equations:

$$\begin{aligned} &\sin\alpha_{2i}^1 \sin\alpha_{2i-1}^1 \sin\gamma_{2i}^1 \sin\gamma_{2i-1}^1 = \sin\alpha_{2i}^2 \sin\alpha_{2i-1}^2 \sin\gamma_{2i}^2 \sin\gamma_{2i-1}^2 \\ &= \sin\alpha_{2i}^3 \sin\alpha_{2i-1}^3 \sin\gamma_{2i}^3 \sin\gamma_{2i-1}^3, \ i = 1, \ldots, k. \end{aligned} \tag{3.3}$$

Consistency Due to the symmetries of the the equations obtained, it seems quite reasonable to look for symmetric solutions. Let us fix constants $\overline{\alpha}, \overline{\beta} \in (0, \pi/3)$ and set

$$\beta = \overline{\beta}, \quad \alpha_l^j = \overline{\alpha}, \quad \gamma_l^j = \pi/3, \qquad l = 1, \ldots, 2k,\ j = 1,2,3. \tag{3.4}$$

Suppose first that $1 \leqslant k \leqslant g-1$. In this case the length conditions (3.2) translate into the equation $f(\overline{\alpha},\overline{\beta})=0$, where

$$f(\overline{\alpha},\overline{\beta}) = \frac{\cos^2\overline{\alpha}+1/2}{\sin^2\overline{\alpha}} - \frac{\cos^2\overline{\beta}+\cos\overline{\beta}}{\sin^2\overline{\beta}}.$$

The total dihedral angle condition (3.1) gives the equation

$$6k\cdot\overline{\alpha}+6(g-k)\cdot\overline{\beta}=2\pi.$$

Now let $\overline{\beta}(\overline{\alpha}) = \frac{\pi-3k\cdot\overline{\alpha}}{3(g-k)}$ be the solution of this equation. Setting $\phi(\overline{\alpha}) = f(\overline{\alpha},\overline{\beta}(\overline{\alpha}))$ we easily get that $\lim_{\overline{\alpha}\to 0}\phi(\overline{\alpha})=+\infty$, $\lim_{\overline{\alpha}\to\pi/3k}\phi(\overline{\alpha})=-\infty$. Moreover, ϕ is strictly monotonic on $(0,\pi/3k)$ so equations (3.1) and (3.2) have a unique solution $(\overline{\alpha}_{g,k},\overline{\beta}_{g,k})$ in $(0,\pi/3)\times(0,\pi/3)$. Since under our assumption (3.4) equations (3.3) are trivially satisfied, we have found a solution of the whole set of consistency equations, which determines a hyperbolic structure with geodesic boundary on M.

When $k=0$ or $k=g$ the situation is even simpler, and the scheme just described easily extends. More precisely, when $k=0$ only compact geometric polyhedra arise, so the shape of the tetrahedra of $\mathcal{T}$ is parameterized by $\overline{\beta}$ and the hyperbolicity condition is verified for $\overline{\beta}_{g,0}=\pi/3g$. On the other hand, when $k=g$ every tetrahedron of $\mathcal{T}$ has a vertex on a boundary torus, so the geometric realizations of $\mathcal{T}$ satisfying the assumption (3.4) are parameterized by $\overline{\alpha}$ and the hyperbolicity condition gives $\overline{\alpha}_{g,g}=\pi/3g$.

Completeness To check completeness of the hyperbolic structure just described we have to determine the similarity structure it induces on the boundary tori. By construction, each torus in ∂M is tiled by two equilateral Euclidean triangles. This shows that the structures on the boundary tori are indeed Euclidean, so the hyperbolic structure constructed in the previous paragraph is complete, and corresponds by Mostow-Prasad's rigidity Theorem to the *unique* complete finite-volume hyperbolic structure with geodesic boundary on the topological manifold M with the boundary tori removed. Of course the volume of M is the sum of volumes of the geometric realizations of the tetrahedra in $\mathcal{T}$, so it depends on g and k only. We have eventually proved the following:

Theorem 3.1.5. *Let $M\in\mathcal{M}_{g,k}$. Then M is hyperbolic, and its volume depends only on g and k.*

Remark 3.1.6. It follows from [Miy94] that $\mathcal{M}_{g,0}$ is precisely the set of hyperbolic 3-manifolds having minimal volume among those with boundary equal to Σ_g (see [FMP03a]). So $\mathcal{M}_{g,k}$ is a natural candidate for the set of minimal-volume hyperbolic manifolds with boundary given by $\Sigma_g \sqcup \left(\sqcup_{i=1}^k T_i\right)$.

3.2 Further topological properties

In this section we briefly describe some topological properties of the manifolds in $\mathcal{M}_{g,k}$. All the results listed here are taken from [FMP03b].

We first recall that a surface in a compact 3-manifold M is *essential* if it is properly embedded, connected, and either a reducing sphere, or a boundary-reducing disc, or an incompressible and ∂-incompressible surface not parallel to the boundary.

If M is a compact 3-manifold with $\partial M = \partial_0 M \sqcup \partial_1 M$, one can define the *Heegaard genus* of $(M, \partial_0 M, \partial_1 M)$ as the minimal genus of a surface that splits M as $C_0 \sqcup C_1$, where C_i is obtained by attaching 1-handles on one side of a collar of $\partial_i M$.

For any compact 3-manifold M and integer $r \geqslant 2$, after fixing in $\mathbb{C}$ a primitive $2r$-th root of unity, a real-valued invariant $\mathrm{TV}_r(M)$ for compact 3-manifolds with boundary was defined by Turaev and Viro in [TV92].

The following theorem is taken from [FMP03b]. A sketch of the proof will be given in the following paragraphs.

Theorem 3.2.1. *Let $M \in \mathcal{M}_{g,k}$ with $\partial M = \Sigma_g \sqcup \big(\bigsqcup_{i=1}^{k} T_i \big)$. Then the following holds:*

1. *M has a unique triangulation with $g+k$ tetrahedra, which gives the canonical Kojima decomposition of M;*
2. *Every essential surface in M has non-empty boundary which intersects Σ_g;*
3. *M has complexity $g+k$;*
4. *The Heegaard genus of $\big(M, \Sigma_g, \bigsqcup_{i=1}^{k} T_i\big)$ is $g+1$;*
5. *$H_1(M;\mathbb{Z}) = \mathbb{Z}^{g+k}$;*
6. *The Turaev-Viro invariant $\mathrm{TV}_r(M)$ depends only on r, g and k.*

Canonical decomposition We now establish Theorem 3.2.1-(1). Let $M \in \mathcal{M}_{g,k}$, let $\mathcal{T}$ be a geometric triangulation of M by $g+k$ partially truncated tetrahedra as described above and let $\mathcal{O}$ be a fixed neighbourhood of the cusps of M. It was shown in Remark 2.7.10 that $\mathcal{O}$ determines a real number $r(v) > 0$ for any ideal vertex v of any tetrahedron Δ in $\mathcal{T}$. This number $r(v)$ represents the "height" of the trace of $\partial\mathcal{O}$ near v in Δ (except that $r(v) \ll 1$ means that $\partial\mathcal{O}$ is "very" high).

Looking at the geometry of the gluings between the tetrahedra in $\mathcal{T}$, one can easily see that $r(v)$ has one and the same value $r(\mathcal{O})$ whenever v is an ideal vertex of some tetrahedron in $\mathcal{T}$. Using the formulae given in Section 2.7 we can now easily compute the tilts of the geometric blocks of $\mathcal{T}$.

- Let v_0 be the ideal vertex of some Δ in $\mathcal{T}$ and let $r = r(\mathcal{O}) = r(v_0)$ be the parameter associated to the intersection of Δ with $\mathcal{O}$. If F_0 is the face of Δ opposite to v_0 and F_1 is any other face of Δ, then

$$\begin{aligned} t(\Delta, F_0) &= r/(2\cos\overline{\alpha}_{g,k}) - \sqrt{4\cos^2\overline{\alpha}_{g,k} - 1}, \\ t(\Delta, F_1) &= -r/2; \end{aligned}$$

- If Δ is any compact regular tetrahedron of $\mathcal{T}$ and F is any face of Δ, then

$$t(\Delta, F) = -\sqrt{\frac{(3\cos\overline{\beta}_{g,k} - 1)(2\cos\overline{\beta}_{g,k} - 1)}{\cos\overline{\beta}_{g,k} + 1}}.$$

If r is small enough, we then get $t(\Delta_1, F_1) + t(\Delta_2, F_2) < 0$ for any pair (F_1, F_2) of matching faces of $\mathcal{T}$, and Proposition 2.7.8 implies that $\mathcal{T}$ is the Kojima canonical decomposition of M. Therefore there is only one triangulation of M with $g+k$ tetrahedra. We have proved Theorem 3.2.1-(1).

Normal surfaces Theorem 3.2.1-(2) can be proved using Haken's theory of normal surfaces [Mat90]. Since this technique is not used elsewhere in this thesis, we do not reproduce the strategy of the proof here, addressing the reader to [FMP03b].

Matveev complexity Theorem 3.2.1-(3), which states that $c(M) = g+k$ for $M \in \mathcal{M}_{g,k}$, is an easy consequence of Proposition 3.1.1 together with the fact [Mat90] that, because of hyperbolicity, $c(M)$ equals the minimal number of tetrahedra in an ideal triangulation of M.

Heegaard genus We need now to fix some notation which will be used extensively later on. For $g \geqslant 2$ we denote by H_g the handlebody of genus g and for $0 \leqslant k \leqslant g$ we introduce another manifold $H_{g,k}$. We do this noting that H_g can be viewed as the ∂-connected sum of g solid tori, and defining $H_{g,k}$ to be H_g minus open tubes around the cores of k of these solid tori. So $H_{g,k}$ is obtained from H_g by drilling out k tunnels along k different 1-handles. Of course $H_{g,k}$ is well-defined and $H_{g,0} = H_g$. Moreover, $H_{g,k}$ is not hyperbolic because it is ∂-reducible.

The genus of $\big(M, \Sigma_g, \bigsqcup_{i=1}^{k} T_i\big)$ is of course at least g, and it is actually at most $g+1$ because, if e is the only edge of the minimal triangulation of M having both ends in Σ_g, the boundary of a regular neighborhood of $\Sigma_g \cup e$ is easily seen to be a Heegaard surface. The next result [FMP03b], together with the fact that $H_{g,k}$ is not hyperbolic, shows that the genus is indeed $g+1$.

Lemma 3.2.2. *If M is compact with $\partial M = \Sigma_g \sqcup (\bigsqcup_{i=1}^{k} T_i)$ and $\big(M, \Sigma_g, \bigsqcup_{i=1}^{k} T_i\big)$ has genus g then $M = H_{g,k}$.*

Turaev-Viro invariants As pointed out in [MN94], Turaev-Viro invariants depend only on incidence numbers between edges and tetrahedra in a triangulation. In our context, let us consider the minimal triangulation $\mathcal{T}$ of some M in $\mathcal{M}_{g,k}$. If we assign the colour 0 to the edge having both endpoints in Σ_g, and colours $\{1, \ldots, k\}$ to the other edges, then the 6 edges of each of the $g+k$ tetrahedra in $\mathcal{T}$ are coloured. It is clear that this set of $g+k$ coloured tetrahedra is the same for each $M \in \mathcal{M}_{g,k}$. This implies that all such M's have the same Turaev-Viro invariants.

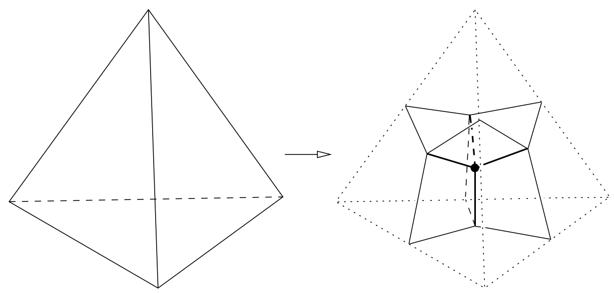

Figure 3.2: From an ideal triangulation to a special spine.

Spines and homology We prove here Theorem 3.2.1-(5). To do this, we switch from the viewpoint of ideal triangulations to the dual viewpoint of *special spines*, suggested in Fig. 3.2. Recall that a spine of a manifold is a subpolyhedron onto which the manifold collapses. A polyhedron is *special* if it is locally homeomorphic to that of Fig. 3.2-right and its natural stratification consists of 0-, 1-, and 2-cells.

Let us then fix $M \in \mathcal{M}_{g,k}$ and the spine P dual to the triangulation of M with $g+k$ tetrahedra. Note that P has a cellularization into vertices, edges, and faces corresponding to tetrahedra, faces, and edges of the triangulation. We denote in particular by $S(P)$ the 1-skeleton of P (a 4-valent graph). By Proposition 3.1.4 the spine P contains k (open) hexagonal faces $E_1, \ldots, E_k$ and one big face G with $6g$ vertices (with multiplicity). For $i = 1, \ldots, k$ the closure $\overline{E}_i$ of E_i is a torus which bounds a collar of the i-th toric component T_i of ∂M, and the rest of P lies outside this collar.

Suppose $k > 0$. Since M collapses onto P, we have $H_1(M;\mathbb{Z}) \cong H_1(P;\mathbb{Z})$, and we can use cellular homology to compute $H_1(P;\mathbb{Z})$. Since $\overline{E}_i$ intersects $S(P)$ in a θ-shaped graph, there is a maximal tree Y in the 4-valent graph $S(P)$ intersecting each $\overline{E}_i$ in an edge. Then $S(P) \setminus Y$ consists of $g+k+1$ edges $e_1, \ldots, e_{g+k+1}$, where e_{2i-1} and e_{2i} are contained in $\overline{E}_i$ for $i = 1, \ldots, k$, while $e_{2k+1}, \ldots, e_{g+k+1}$ are contained in $\overline{G}$. Choosing an orientation on each e_j, E_i, and G, we get a presentation for $H_1(P;\mathbb{Z})$ with generators $e_1, \ldots, e_{g+k+1}$ and relators given by the incidence numbers of G and the E_i's on the e_j's. Each E_i contributes with the trivial relator $e_{2i-1} + e_{2i} - e_{2i-1} - e_{2i}$, while G contributes with a big relator w containing $\pm e_1$ once. Therefore $H_1(P;\mathbb{Z}) = \mathbb{Z}^{g+k+1}/\langle w \rangle \cong \mathbb{Z}^{g+k}$, and we are done. The case in which $k = 0$ is simpler, and is treated in [FMP03a].

The following refinement of Theorem 3.2.1-(5) will prove useful later:

Proposition 3.2.3. *Suppose $g > k$. Then we have the exact sequence*

$$0 \longrightarrow H_1\left(\bigsqcup_{i=1}^{k} T_i; \mathbb{Z}\right) \xrightarrow{i_*} H_1(M;\mathbb{Z}) \longrightarrow \mathbb{Z}^{g-k} \longrightarrow 0,$$

where i_* is the map naturally induced by the inclusion $i : \bigsqcup_{i=1}^{k} T_i \to M$.

Proof: We keep the notation from the preceding paragraph. Since $g > k$ a vertex v of P exists which corresponds to a partially truncated tetrahedron without ideal

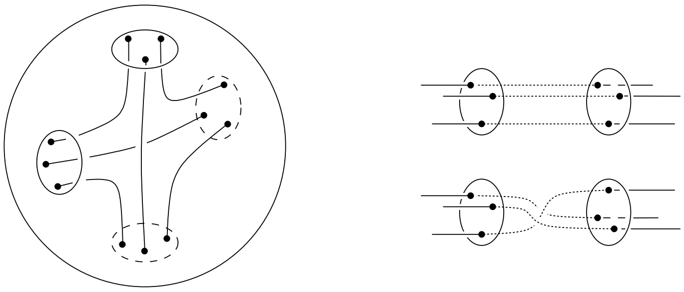

Figure 3.3: A tangle in a ball (left); a positive and a negative gluing of tangles (right).

vertices. Notice that G is the only face incident to v. Moreover, an easy analysis of the local structure of P near v shows that the number of edges emanating from v on which G passes three times with the same orientation is at most two. Let us also observe that any finite graph has an even number of vertices with odd valence, whence the number of connected components of $S(P) \setminus \{v\}$ is at most two. These facts easily imply that a maximal tree Y' in $S(P)$ exists with the following properties: $S(P) \setminus Y'$ consists of $g+k+1$ edges $e'_1, \ldots, e'_{g+k+1}$, where e'_{2i-1} and e'_{2i} represent generators of $H_1(T_i;\mathbb{Z})$ for $i = 1, \ldots, k$, and G passes three times on e'_{g+k+1} with different orientations. Therefore we get a presentation for $H_1(P;\mathbb{Z})$ with generators $e'_1, \ldots, e'_{g+k+1}$ and one relator w containing $\pm e'_{g+k+1}$ once. This implies in turn that the classes of $e'_1, \ldots, e'_{g+k}$ give a free basis of $\mathbb{Z}^{g+k+1}/\langle w\rangle \cong \mathbb{Z}^{g+k} \cong H_1(P;\mathbb{Z})$, whence the conclusion at once. $\square$

Remark 3.2.4. It can be shown that the requirement $g > k$ is essential for the conclusion of Proposition 3.2.3 to be true. Namely, if $g = k$ is even, then $\mathcal{M}_{g,g}$ contains only one manifold $M_{g,g}$, and a careful analysis of the minimal triangulation of $M_{g,g}$ shows that if $T_1, \ldots, T_g$ are the boundary tori of $M_{g,g}$ and $i : \bigsqcup_{i=1}^{g} T_i \to M_{g,g}$ is the natural inclusion then we have $H_1(M_{g,g};\mathbb{Z})/\mathrm{Im}\, i_* \cong \mathbb{Z}/_3$.

Knots giving $\mathcal{M}_{g,1}$ It turns out that the elements of $\mathcal{M}_{g,1}$ are knot exteriors in H_g, and the knots can be exhibited explicitly, as we now explain. Consider a ball B as in Fig. 3.3-left with the tangle $\tau \subset B$ as shown, the 12 ends of τ being arranged in four groups of three, each group contained in a disc. Now take $g-1$ copies of (B,τ) and glue together the $4(g-1)$ discs, matching the ends of the τ's. Each gluing should be of one of the two types suggested in Fig. 3.3-right. A gluing as in the top part of Fig. 3.3-right will be called *positive*, one as in the bottom part will be called *negative*. The result of the $2(g-1)$ gluings is a link in H_g, and one readily sees that if all the gluings are positive then the link is parallel to ∂H_g. The following result is proved in [FMP03b].

Proposition 3.2.5. *If a knot K in H_g is realized from $g-1$ copies of (B,τ) with $2g-3$ positive gluings and one negative gluing then the exterior of K belongs to $\mathcal{M}_{g,1}$. Every manifold in $\mathcal{M}_{g,1}$ arises like this.*

3.3 Dehn filling

Recall that a *slope* on a torus T is an isotopy class of simple closed essential curves, and that, after choosing a $\mathbb{Z}$-basis of $H_1(T;\mathbb{Z})$, a slope is represented by two pairs of coprime integers $(p,q),(-p,-q)\in\mathbb{Z}^2$, or, equivalently, by an element in $\mathbb{Q}\cup\{\infty\}$. In what follows we will often represent slopes as indivisible elements in $H_1(T;\mathbb{Z})$ without emphasizing the fact that each slope corresponds in fact to two such elements. If s,s' are two slopes on a torus, we denote by $\Delta(s,s')$ their *distance*, that is their geometric intersection number. Let s_1,s_2 be a basis of $H_1(T;\mathbb{Z})$. Then $\Delta(s_1,s_2)=1$ and, if $s=p\cdot s_1+q\cdot s_2$ and $s'=p'\cdot s_1+q'\cdot s_2$, then $\Delta(s_1,s_2)=|pq'-p'q|$. It is easily seen that s and s' represent the same slope if and only if $\Delta(s,s_1)=\Delta(s',s_1)$, $\Delta(s,s_2)=\Delta(s',s_2)$ and $\Delta(s,s_1+s_2)=\Delta(s',s_1+s_2)$. Note that the distances of a slope from s_1 and s_2 are not sufficient to determine the slope itself.

If M is a manifold with k boundary tori, and $s_1,\ldots,s_h$ are slopes in some $h\leqslant k$ of these tori, we denote by $M(s_1,\ldots,s_h)$ the result of Dehn filling M along $s_1,\ldots,s_h$.

Boundary slopes of manifolds in $\mathcal{M}_{g,k}$ Let now T be a fixed boundary torus of a manifold $M\in\mathcal{M}_{g,k}$, and recall that the unique complete finite-volume hyperbolic structure on M induces on T a Euclidean structure defined up to similarity. For the sake of simplicity, we endow T with a fixed Euclidean structure choosing the scale factor such that $\mathrm{Area}(T)=\sqrt{3}/2$. As described in the preceding section, this structure can be constructed by realizing T as the gluing of two isometric Euclidean equilateral triangles. This readily implies that T is isometric to $\mathbb{C}/\Gamma$, where $\mathbb{C}$ is endowed with the standard Euclidean metric, and $\Gamma=\langle\tau_1,\tau_2\rangle$ is the discrete additive subgroup of $\mathbb{C}$ with generators $\tau_1=1,\tau_2=-1/2+i\sqrt{3}/2$. Let an isometric identification $T\cong\mathbb{C}/\Gamma$ be fixed from now on, and let $*$ be the point of T corresponding to $\overline{0}$ in $\mathbb{C}/\Gamma$. We denote by $\mathrm{Isom}(T)$ the group of isometries of T, by $\mathrm{Isom}_0(T)$ the group of isometries of T fixing $*$, and by $\mathcal{M}(T)$ the group of isotopy classes of elements in $\mathrm{Isom}(T)$. Of course we have a canonical injection $i:\mathrm{Isom}_0(T)\to\mathrm{Isom}(T)$ and a canonical projection $p:\mathrm{Isom}(T)\to\mathcal{M}(T)$. Moreover each of these groups acts on the set of slopes on T, and the actions are preserved by the homomorphisms i and p. The following lemma is straight-forward, and does not depend on the particular isometry type of T.

Lemma 3.3.1. *The composition* $p\circ i:\mathrm{Isom}_0(T)\to\mathcal{M}(T)$ *is an isomorphism.*

Let now $H=\{e^{ik\pi/3}:\ k=0,\ldots,5\}\subset\mathbb{C}$ be the regular unitary Euclidean hexagon and let D_6 be the group of isometries of $\mathbb{C}$ which preserve H. Of course D_6 is the dihedral group with 12 elements, generated by the rotation $r:\mathbb{C}\to\mathbb{C},\ r(z)=e^{i\pi/3}\cdot z$ and the reflection $s:\mathbb{C}\to\mathbb{C},\ s(z)=\overline{z}$.

Proposition 3.3.2. *There is a canonical isomorphism* $\mathrm{Isom}_0(T)\cong D_6$.

PROOF: Let $\Psi:\mathrm{Isom}_0(T)\to\mathrm{Isom}(\mathbb{C})$ be the map which takes any $\varphi\in\mathrm{Isom}_0(T)$ to the unique lift of φ that fixes 0. It is easily seen that Ψ is an injective group homomorphism whose image is given by the isometries of $\mathbb{C}$ that fix 0 and leave Γ

invariant. Since $H = \Gamma \cap S^1 \subset \mathbb{C}$ it follows that $\Psi(\mathrm{Isom}_0(T)) = D_6$, and we are done. $\square$

Since the projection $\mathbb{C} \to \mathbb{C}/\Gamma = T$ is a universal covering, we have a natural isomorphism between $\Gamma = \langle \tau_1, \tau_2 \rangle$ and $H_1(T;\mathbb{Z}) \cong \pi_1(T, *)$. We denote by s_1, s_2 the elements in $H_1(T;\mathbb{Z})$ which correspond under this identification to τ_1, τ_2 respectively, and we choose s_1, s_2 as our preferred $\mathbb{Z}$-basis of $H_1(T;\mathbb{Z})$. For the sake of clarity, in the statement and the proof of the following corollary we denote by $\pm v$ the slope corresponding to an indivisible element v of $H_1(T;\mathbb{Z})$.

Corollary 3.3.3. *Let $s = \pm(p \cdot s_1 + q \cdot s_2)$ and $s' = \pm(p' \cdot s_1 + q' \cdot s_2)$ be slopes on T. Then s is $\mathrm{Isom}_0(T)$-equivalent to s' if and only if $\{|p|, |q|, |p-q|\} = \{|p'|, |q'|, |p'-q'|\}$.*

PROOF: Note that the set of slopes $\mathcal{S} = \{\pm s_1, \pm s_2, \pm(s_1 + s_2)\}$ is left invariant by the action of $\mathrm{Isom}_0(T)$. Moreover, $\mathrm{Isom}_0(T)$ acts on $\mathcal{S}$ as the full group of permutations on three elements. Now we have $|p| = \Delta(s, \pm s_2)$, $|q| = \Delta(s, \pm s_1)$, $|p-q| = \Delta(s, \pm(s_1 + s_2))$, and a slope is completely determined by its distances from $\pm s_1, \pm s_2$ and $\pm(s_1 + s_2)$, whence the conclusion. $\square$

Any slope s on T determines a well-defined isotopy class of geodesics on T, and we denote by $L(s)$ the Euclidean length of such geodesics. An elementary calculation gives the following:

Lemma 3.3.4. *Let $s = p \cdot s_1 + q \cdot s_2$ be a slope on T. Then $L(s) = \sqrt{p^2 + q^2 - pq}$.*

The following result is easily deduced from Lemma 3.3.4 and Corollary 3.3.3. Let $\{\ell_1 < \ell_2 < \ldots < \ell_n < \ldots\}$ be the set of lengths of slopes on T.

Proposition 3.3.5. *The following holds:*

- *There are exactly three slopes of length $\ell_1 = 1$. They are represented by s_1, s_2 and $s_1 + s_2$, and they are $\mathrm{Isom}_0(T)$-equivalent to each other.*
- *There are exactly three slopes of length $\ell_2 = \sqrt{3}$. They are are represented by $s_1 - s_2, s_1 + 2 \cdot s_2$ and $2 \cdot s_1 + s_2$, and they are $\mathrm{Isom}_0(T)$-equivalent to each other.*
- *There are exactly six slopes of length $\ell_3 = \sqrt{7}$. They are represented by $2 \cdot s_1 - s_2, s_1 - 2 \cdot s_2, 3 \cdot s_1 + s_2, s_1 + 3 \cdot s_2, 3 \cdot s_1 + 2 \cdot s_2, 2 \cdot s_1 + 3 \cdot s_2$, and they are $\mathrm{Isom}_0(T)$-equivalent to each other.*
- *If s is a slope with $L(s) \geqslant \ell_3$, then there exist exactly six slopes $\mathrm{Isom}_0(T)$-equivalent to s.*

Remark 3.3.6. Let s and s' be slopes on T. Of course if s' is $\mathrm{Isom}_0(T)$-equivalent to s then $L(s') = L(s)$, but the converse is not true. For example, the slopes $s = 19s_1 + 11s_2$ and $s' = 16s_1 - s_2$ have the same length $L(s) = L(s') = \sqrt{273}$, even if they are not $\mathrm{Isom}_0(T)$-equivalent as Corollary 3.3.3 shows.

Dehn fillings The following result, which is proved in [FMP03b], completely classifies the Dehn fillings of elements in $\mathcal{M}_{g,k}$. Recall that certain manifolds $H_{g',k'}$ have been defined on page 77.

Theorem 3.3.7. *Let $M \in \mathcal{M}_{g,k}$ with $\partial M = \Sigma_g \sqcup \big(\bigsqcup_{i=1}^{k} T_i\big)$, let $h \leqslant k$, let s_i be a slope on T_i for any $i = 1, \ldots, h$ and $N = M(s_1, \ldots, s_h)$. Then:*

- *If $L(s_i) = \ell_1 = 1$ for some i then $N = H_{g,k-h}$, so it is not hyperbolic;*
- *If $L(s_i) = \ell_2 = \sqrt{3}$ for some i then N contains a Möbius strip or non-separating annulus R with $\partial R \subset \Sigma_g$, and cutting N along R one gets $H_{g,k-h}$; also in this case N is not hyperbolic;*
- *If $L(s_i) > \ell_2 = \sqrt{3}$ for all i then N is hyperbolic, and the Heegaard genus of*
$$\big(N, \Sigma_g, T_{h+1} \sqcup \ldots \sqcup T_k\big)$$
is $g + 1$;
- *If $L(s_i) = \ell_3 = \sqrt{7}$ for all i then N belongs to $\mathcal{M}_{g,k-h}$.*

Moreover every essential surface in N has non-empty boundary intersecting Σ_g.

The following result is an easy consequence of Proposition 3.2.3.

Proposition 3.3.8. *Suppose $g > k$ and let N be as in the statement of Theorem 3.3.7. Then $H_1(N; \mathbb{Z}) \cong \mathbb{Z}^{g+k-h}$.*

Theorem 3.3.7 gives the following:

Corollary 3.3.9. *For any $g \geqslant 2$ there exist infinitely many hyperbolic manifolds N with $\partial N = \Sigma_g \sqcup \Sigma_1$ and with 6 slopes $s^1, \ldots, s^6$ on Σ_1, such that $N(s^i) = H_g$ for $i \in \{1, 2, 3\}$ and $N(s^i)$ is annular for $i \in \{4, 5, 6\}$. We have $\Delta(s^i, s^{i+3}) = 2$ for $i = \{1, 2, 3\}$, and $\Delta(s^j, s^{j'}) = 3$ for $j, j' \in \{4, 5, 6\}, j \neq j'$.*

Proof. Take $N = M(s)$ where $M \in \mathcal{M}_{g,2}$, $\partial M = \Sigma_g \sqcup T_1 \sqcup T_2$ and s is a slope on T_1 with $L(s) > \ell_2$. This gives infinitely many hyperbolic manifolds because the volume grows to $\mathrm{vol}(M)$ as $L(s)$ tends to infinity. Now let s^1, s^2, s^3 be the three slopes on T_2 of length 1 and s^4, s^5, s^6 the three slopes on T_2 of length $\sqrt{3}$ (we are considering lengths with respect to the Euclidean structure induced on T_2 by the hyperbolic metric on M). Then $N(s^i) = H_g$ for $i \in \{1, 2, 3\}$. For $i \in \{4, 5, 6\}$ the manifold $N(s^i)$ is not hyperbolic, so by Thurston's hyperbolization Theorem and the last assertion of Theorem 3.3.7 it contains either an essential disc or an essential annulus. Since a hyperbolic manifold admits at most 3 boundary-reducible fillings [Wu92], we conclude that $N(s_i)$ is annular for $i \in \{4, 5, 6\}$. □

Distances between exceptional slopes Let N be a hyperbolic 3-manifold, and $T_i \subset \partial N$ be a chosen boundary torus. A slope s on T_i is called *exceptional* if $N(s)$ is not hyperbolic. Assuming N has either some other cusp or non-empty geodesic boundary, s is exceptional if and only if $M(s)$ contains an essential sphere, disc, annulus, or torus. In these cases we say that s is respectively of type S, D, A, or T. For $X_1, X_2 \in \{S, D, A, T\}$, we define a number $\Delta^{\text{hyp}}(X_1, X_2)$ as the maximal distance of two slopes of type X_1 and X_2 in a boundary torus of a hyperbolic manifold [Gor99]. Wu pointed out [Wu98] that in most cases Δ^{hyp} is considerably lower when considering only *large* manifolds, *i.e.* manifolds with $H_2(N, \partial N \setminus T_i) \neq \{0\}$. He thus defined $\Delta^{\text{large}}(X_1, X_2)$ as the maximal distance of two slopes of type X_1 and X_2 in a boundary torus of a *large* hyperbolic manifold. Among other inequalities, he proved that $1 \leqslant \Delta^{\text{large}}(D, A) \leqslant 2$. Since every hyperbolic manifold N with $\chi(N) < 0$ is large, Corollary 3.3.9 implies that $\Delta^{\text{large}}(D, A) = 2$. This result leaves $\Delta^{\text{large}}(T, T)$ as the only unknown value for Δ^{hyp} and Δ^{large}, as shown in the following tables (which are taken from [Gor99] with the insertion of $\Delta^{\text{large}}(D, A) = 2$).

Δ^{hyp}	S	D	A	T
S	1	0	2	3
D		1	2	2
A			5	5
T				8

Δ^{large}	S	D	A	T
S	0	0	1	1
D		1	2	1
A			4	4
T				4–5

Knots in handlebodies Corollary 3.3.9 leads to infinitely many examples of knots in a handlebody H_g having non-meridinal surgeries that give back H_g (more precisely, two such surgeries) for all $g \geqslant 2$. Recall now that a knot K in a manifold X is 1-*bridge* if it can be isotoped to the form $\gamma_0 \cup \gamma_1$ where the γ_i's are simple arcs with common ends, γ_0 lies in ∂X and γ_1 is properly embedded and parallel to ∂X. For $g = 1$, *i.e.* for the solid torus H_1, knots in H_1 with non-meridinal surgeries giving H_1 were shown to be 1-bridge [Gab89] and then classified by Gabai [Gab90] and Berge [Ber91]. In particular, there is a unique knot in H_1 with two non-meridinal surgeries giving H_1. A knot in H_2 with one non-meridinal surgery giving H_2 was also shown in [Gab90], and in the same paper other examples in H_g for any g were attributed to Berge, together with the following question: if K is a knot in a ∂-reducible manifold M with ∂-irreducible exterior and a ∂-reducible non-meridinal surgery, is K boundary-parallel or 1-bridge in M? We showed in [FMP03b] that the answer is positive for all the hyperbolic examples one can construct from $\mathcal{M}_{g,k}$:

Proposition 3.3.10. *Every knot in H_g whose complement is a hyperbolic manifold obtained from $k-1$ Dehn fillings on a manifold in $\mathcal{M}_{g,k}$ is a 1-bridge knot.*

3.4 Isolation of cusps

Let M be a fixed manifold in $\mathcal{M}_{g,k}$ and let $\mathcal{T}$ be the unique partially truncated triangulation of M with $g+k$ tetrahedra. We now study small deformations of the complete hyperbolic structure of M by analyzing deformations of the shapes of the geometric tetrahedra of $\mathcal{T}$. To save some words, from now on we will also suppose that $g > k$, unless otherwise stated.

The deformation space Let $T_1, \ldots, T_k$ be the boundary tori of M and for $i = 1, \ldots, k$ let $\Delta_{2i-1}, \Delta_{2i}$ be the tetrahedra of $\mathcal{T}$ incident to T_i. As in Section 3.1, we also denote by $F^1_{2i-1}, F^2_{2i-1}, F^3_{2i-1}, F^1_{2i}, F^2_{2i}, F^3_{2i}$ the exceptional hexagons of $\Delta_{2i-1}, \Delta_{2i}$, in such a way that F^j_{2i-1} is glued to F^j_{2i} for $i = 1, \ldots, k$, $j = 1, 2, 3$. For each $l = 1, \ldots, 2k$, we suppose that F^1_l, F^2_l, F^3_l are positively arranged around the ideal vertex of Δ_l, and we call e^j_l the only finite edge of F^j_l, and f^j_l the edge of Δ_l opposite to e^j_l. We say that a geometric realization of the tetrahedra of $\mathcal{T}$ is *reasonable* if it assigns the same angle to all the edges of all the compact tetrahedra of $\mathcal{T}$, and note that any realization satisfying the consistency equations is automatically reasonable. So let θ be a reasonable realization of the tetrahedra of $\mathcal{T}$ and set $\alpha^j_l = \theta(e^j_l)$, $\gamma^j_l = \theta(f^j_l)$ for any $l = 1, \ldots, 2k$, $j = 1, 2, 3$. We also denote by β the dihedral angle along the edges of the compact tetrahedra of $\mathcal{T}$. Let $N = 12k + 1$ (resp. $N = 12k$ if $g = k$) and set

$$\begin{aligned}\mathcal{W} = \{x \in \mathbb{R}^N \,:\, 0 < x_i < \pi,\ & x_{6j+4} + x_{6j+5} + x_{6j+6} = \pi,\\ & i = 1, \ldots, N,\, j = 0, \ldots, 2k-1\}.\end{aligned}$$

Then the map

$$\theta \longmapsto (\alpha^1_1, \alpha^2_1, \alpha^3_1, \gamma^1_1, \gamma^2_1, \gamma^3_1, \ldots, \alpha^1_{2k}, \alpha^2_{2k}, \alpha^3_{2k}, \gamma^1_{2k}, \gamma^2_{2k}, \gamma^3_{2k}, \beta)$$

(resp. $\theta \mapsto (\alpha^1_1, \alpha^2_1, \alpha^3_1, \gamma^1_1, \gamma^2_1, \gamma^3_1, \ldots, \alpha^1_{2k}, \alpha^2_{2k}, \alpha^3_{2k}, \gamma^1_{2k}, \gamma^2_{2k}, \gamma^3_{2k})$ if $g = k$) defines a bijection between the space of reasonable realizations of the tetrahedra of $\mathcal{T}$ and $\mathcal{W}$. In what follows we will denote by

$$x_0 = (\overline{\alpha}_{g,k}, \overline{\alpha}_{g,k}, \overline{\alpha}_{g,k}, \pi/3, \pi/3, \pi/3, \ldots, \overline{\alpha}_{g,k}, \overline{\alpha}_{g,k}, \overline{\alpha}_{g,k}, \pi/3, \pi/3, \pi/3, \overline{\beta}_{g,k})$$

(resp. $x_0 = (\overline{\alpha}_{g,k}, \overline{\alpha}_{g,k}, \overline{\alpha}_{g,k}, \pi/3, \pi/3, \pi/3, \ldots, \overline{\alpha}_{g,k}, \overline{\alpha}_{g,k}, \overline{\alpha}_{g,k}, \pi/3, \pi/3, \pi/3)$ if $g = k$) the point of $\mathcal{W}$ corresponding to the unique solution of consistency and completeness equations and by $\Omega_{g,k} \subset \mathcal{W}$ the set corresponding to the solutions of consistency equations. Let us observe that $\Omega_{g,k}$ really depends only on g and k, since these natural numbers are the only relevant combinatorial data which are involved in the consistency equations.

Computing infinitesimal deformations In this paragraph we will give explicit equations for the tangent space $T_{x_0}\Omega_{g,k}$. These equations will prove useful later in order to discuss isolation of cusps in manifolds in $\mathcal{M}_{g,k}$. By Corollary 2.4.20, infinitesimal deformations of the complete structure only affect non-compact tetrahedra of $\mathcal{T}$, thus it makes sense to concentrate our attention on consistency equations involving this type of tetrahedra.

So fix a smooth arc $\varphi : (-\varepsilon, \varepsilon) \to \Omega_{g,k}$ and for any smooth $f : \Omega_{g,k} \to \mathbb{R}$ let us denote by $\dot{f}$ the derivative of $f \circ \varphi$ at $t = 0$. Consider the tetrahedra $\Delta_1, \Delta_2 \in \mathcal{T}$ with edges e^j_i, f^j_i, $i = 1, 2$, $j = 1, 2, 3$ as described above, and denote by b^j_i, $i =$

$1, 2$, $j = 1, 2, 3$ the boundary edge of the compact face of Δ_i which does not meet e_i^j. If $\theta(t)$ is the geometric realization of $\mathcal{T}$ parameterized by $\varphi(t) \in \Omega_{g,k}$, we set $\alpha_l^j(t) = \theta(t)(e_l^j)$, $\gamma_l^j(t) = \theta(t)(f_l^j)$. For any compact edge a of Δ_1, Δ_2 we also denote by $L(a) : \Omega_{g,k} \to \mathbb{R}$ the map which associates to $x \in \Omega_{g,k}$ the length of a in the geometric realization of the Δ_i's corresponding to x.

Recall that

$$L(b_i^j)(\varphi(t)) = \frac{\cos \alpha_i^{j+1}(t) \cdot \cos \alpha_i^{j+2}(t) + \cos \gamma_i^j(t)}{\sin \alpha_i^{j+1}(t) \cdot \sin \alpha_i^{j+2}(t)},$$

where apices are considered mod 3. Moreover, by Corollary 2.4.19 we have $\dot{L(b_i^j)} = 0$, so differentiating at 0 the equation above we easily get

$$\sqrt{3}(\dot\alpha_i^{j+1} + \dot\alpha_i^{j+2}) \cos \overline{\alpha}_{g,k} + \dot\gamma_i^j \sin \overline{\alpha}_{g,k} = 0.$$

Summing up these equations for $j = 1, 2, 3$ and observing that $\dot\gamma_i^1 + \dot\gamma_i^2 + \dot\gamma_i^3 = 0$ we obtain

$$\dot\alpha_i^1 + \dot\alpha_i^2 + \dot\alpha_i^3 = 0, \tag{3.5}$$

whence

$$\sqrt{3}\dot\alpha_i^j \cos \overline{\alpha}_{g,k} = \dot\gamma_i^j \sin \overline{\alpha}_{g,k}. \tag{3.6}$$

We now deal with equations arising from conditions (3) in Theorem 2.2.12, which in our setting take the form (3.3). Evaluating these equations along φ and differentiating at 0 we get

$$\begin{aligned}\sqrt{3}(\dot\alpha_1^1 + \dot\alpha_2^1) \cos \overline{\alpha}_{g,k} + (\dot\gamma_1^1 + \dot\gamma_2^1) \sin \overline{\alpha}_{g,k} &= \sqrt{3}(\dot\alpha_1^2 + \dot\alpha_2^2) \cos \overline{\alpha}_{g,k} + (\dot\gamma_1^2 + \dot\gamma_2^2) \sin \overline{\alpha}_{g,k} \\ &= \sqrt{3}(\dot\alpha_1^3 + \dot\alpha_2^3) \cos \overline{\alpha}_{g,k} + (\dot\gamma_1^3 + \dot\gamma_2^3) \sin \overline{\alpha}_{g,k}.\end{aligned}$$

By equations (3.6) these relations are equivalent to the following ones:

$$\dot\alpha_1^1 + \dot\alpha_2^1 = \dot\alpha_1^2 + \dot\alpha_2^2 = \dot\alpha_1^3 + \dot\alpha_2^3,$$

which by equations (3.5) imply in turn

$$\dot\alpha_1^1 = -\dot\alpha_2^1, \quad \dot\alpha_1^2 = -\dot\alpha_2^2, \quad \dot\alpha_1^3 = -\dot\alpha_2^3. \tag{3.7}$$

We can now summarize all these computations giving explicit equations for $T_{x_0}\Omega_{g,k}$. To this aim let us fix some more notation. Let Z be the linear subspace of $\mathbb{R}^{12}$ defined by the following equations:

$$\begin{cases} (\sqrt{3} \cos \overline{\alpha}_{g,k}) x_1 = (\sin \overline{\alpha}_{g,k}) x_4 \\ (\sqrt{3} \cos \overline{\alpha}_{g,k}) x_2 = (\sin \overline{\alpha}_{g,k}) x_5 \\ (\sqrt{3} \cos \overline{\alpha}_{g,k}) x_3 = (\sin \overline{\alpha}_{g,k}) x_6 \\ x_1 + x_7 = x_2 + x_8 = x_3 + x_9 = x_4 + x_{10} = x_5 + x_{11} = x_6 + x_{12} = 0 \\ x_1 + x_2 + x_3 = 0. \end{cases}$$

Observe that $\dim_{\mathbb{R}} Z = 2$. Let $N = 12k + 1$ (resp. $N = 12k$ if $g = k$) as before, and for $i = 1, \ldots, k$ let $r_i : \mathbb{R}^N \to \mathbb{R}^{12}$ be the map defined as follows: $r_i(x) = (x_{12i-11}, x_{12i-10}, \ldots, x_{12i-1}, x_{12i})$. Let

$$\overline{Z} = \{x \in \mathbb{R}^N \, : \, x_{12k+1} = 0 \text{ if } N = 12k + 1, \; r_i(x) \in Z \text{ for } i = 1, \ldots, k\}$$

be the product of one copy of the linear space Z for each cusp (so $\dim_{\mathbb{R}} Z = 2k$).

Proposition 3.4.1. *We have* $T_{x_0}\Omega_{g,k} = \overline{Z}$.

PROOF: If $g > k$, Corollary 2.4.20 ensures that a vector in $T_{x_0}\Omega_{g,k}$ has N-coordinate equal to 0. Together with equations (3.5), (3.6) and (3.7), this fact implies that $T_{x_0}\Omega_{g,k} \subseteq \overline{Z}$. But $\dim_{\mathbb{R}} \overline{Z} = 2k = \dim_{\mathbb{R}} T_{x_0}\Omega_{g,k}$, whence the conclusion. □

Isolation of cusps The following definition is taken from [NR93]:

Definition 3.4.2. Let N be a complete finite-volume hyperbolic 3-manifold with geodesic boundary and with cusps $C_1, \ldots, C_h, C_{h+1}, \ldots, C_k$. For any $i = 1, \ldots, k$ let T_i be the boundary torus of N corresponding to C_i. We say that $C_1, \ldots, C_h$ are *geometrically isolated* from $C_{h+1}, \ldots, C_k$ if any small deformation of the hyperbolic structure on N induced by Dehn filling $C_{h+1}, \ldots, C_k$ while keeping $C_1, \ldots, C_h$ complete does not affect the Euclidean structure at $C_1, \ldots, C_h$. More precisely, $C_1, \ldots, C_h$ are geometrically isolated from $C_{h+1}, \ldots, C_k$ if the following holds: for any $i = h+1, \ldots, k$ a finite set $\mathcal{S}_i$ of slopes on T_i exists such that if N' is obtained from N by Dehn filling $C_{h+1}, \ldots, C_k$ along slopes outside $\mathcal{S}_{h+1}, \ldots, \mathcal{S}_k$ then N' is complete hyperbolic with geodesic boundary and the Euclidean structures induced by N and N' on $T_1, \ldots, T_h$ are equal.

Let now $C_1, \ldots, C_k$ be the cusps of our fixed manifold $M \in \mathcal{M}_{g,k}$ corresponding to the boundary tori $T_1, \ldots, T_k$. The goal of this section is to prove the following:

Theorem 3.4.3. *Let* $h \leqslant k$. *Then the cusps* $C_1, \ldots, C_h$ *are geometrically isolated from the cusps* $C_{h+1}, \ldots, C_k$.

In order to prove Theorem 3.4.3 we look for nice equations defining the set of structures in $\Omega_{g,k}$ which are complete at $C_1, \ldots, C_h$. To this aim we denote by J_h the subspace of $\mathbb{R}^N$ defined by the following equations:

$$J_h = \{x \in \mathbb{R}^N \,:\, x_{12i+1} = x_{12i+2} = x_{12i+3} \text{ for all } i = 0, \ldots, h-1\}. \tag{3.8}$$

Lemma 3.4.4. *There exists a neighbourhood* V *of* x_0 *in* $\Omega_{g,k}$ *such that* $J_h \cap V$ *is a smooth submanifold of* V *of real dimension* $2(k-h)$.

PROOF: It is easily seen that $T_{x_0}J_h + T_{x_0}\Omega_{g,k} = J_h + \overline{Z} = T_{x_0}\mathbb{R}^N = \mathbb{R}^N$, so basic results about transverse intersections of submanifolds ensure that $J_h \cap V$ is a submanifold of V. Moreover $\dim_{\mathbb{R}}(J_h \cap V) = \dim_{\mathbb{R}} J_h + \dim_{\mathbb{R}} V - \dim_{\mathbb{R}} \mathbb{R}^N = (N-2h)+2k-N$.
□

Let Δ be a topological partially truncated tetrahedron with one ideal vertex v_0, and take $\vartheta \in (0, \pi/3)$. Then there exists, up to isometry, exactly one geometric realization of Δ with dihedral angles $\pi/3$ along the internal edges emanating from v_0, and angle ϑ along the other internal edges. We denote this geometric tetrahedron by Δ^ϑ.

Proposition 3.4.5. *For each* $l = 1, \ldots, 2h$ *let* $\Delta_l^*(p)$ *be the geometric realization of* Δ_l *parameterized by* $p \in J_h \cap \Omega_{g,k}$. *Then a real number* $\vartheta(p) \in (0, \pi/3)$ *exists such that* $\Delta_l^*(p)$ *is isometric to* $\Delta^{\vartheta(p)}$ *for* $l = 1, \ldots, 2h$.

Proof: For $l = 1, \ldots, 2h$, $j = 1,2,3$ let $T_l^j(p)$ be the truncation triangle of $\Delta_l^*(p)$ having a vertex on the edge f_l^j. Fix $i \in \{0, \ldots, h-1\}$ and consider the tetrahedron $\Delta_{2i+1}^*(p)$. The compact face of such tetrahedron is a *regular* right-angled hexagon, so condition $x_{12i+1}(p) = x_{12i+2}(p) = x_{12i+3}(p)$ implies that $T_{2i+1}^1(p), T_{2i+1}^2(p)$ and $T_{2i+1}^3(p)$ are isometric to each other. This gives $x_{12i+4}(p) = x_{12i+5}(p) = x_{12i+6}(p)$, whence Δ_{2i+1}^* is isometric to $\Delta^{\xi_{2i+1}}$ for some $\xi_{2i+1} = x_{12i+1}(p) \in (0, \pi/3)$. Moreover, since the non-compact faces of Δ_{2i+1}^* are isometrically glued to the non-compact faces of Δ_{2i+2}^* we easily see that the truncation triangles $T_{2i+2}^1(p), T_{2i+2}^2(p)$ and $T_{2i+2}^3(p)$ are isosceles and isometric to each other. This forces $x_{12i+7}(p) = x_{12i+8}(p) = x_{12i+9}(p) = \xi_{2i+2}$ and $x_{12i+10}(p) = x_{12i+11}(p) = x_{12i+12}(p) = \pi/3$, whence Δ_{2i+2}^* is isometric to $\Delta^{\xi_{2i+2}}$ for some $\xi_{2i+2} \in (0, \pi/3)$.

We have thus shown that for each $l = 1, \ldots, 2h$ a real number $\xi_l \in (0, \pi/3)$ exists such that $\Delta_l^*(p)$ is isometric to Δ^{ξ_l}. But the length of the compact internal edges of the Δ_l^*'s does not depend on l, whence $\xi_1 = \ldots = \xi_{2h}$. □

Corollary 3.4.6. *Let p be a point in $J_h \cap \Omega_{g,k}$ and denote by $M(p)$ the hyperbolic structure defined by p on M. Then for all $i = 1, \ldots, h$ the following holds:*

- *$M(p)$ induces a complete metric on the cusp C_i;*
- *The Euclidean structure induced on T_i by $M(p)$ is isometric to the regular hexagonal structure induced on T_i by the complete hyperbolic structure $M(x_0)$.*

The corollary just stated says that the Euclidean structures on $T_1, \ldots, T_h$ are not affected by deformations of the hyperbolic metric on M which correspond to points in $J_h \cap \Omega_{g,k}$. Therefore to conclude the proof of Theorem 3.4.3 we only need the following:

Proposition 3.4.7. *Let K_h be the subset of $\Omega_{g,k}$ corresponding to structures inducing* complete *metrics on cusps $C_1, \ldots, C_h$. Then there exists a neighbourhood V of x_0 in $\Omega_{g,k}$ with $K_h \cap V = J_h \cap V$.*

Proof: By Lemma 2.4.1, Corollary 2.4.3 and Lemma 3.4.4 there exists a neighbourhood W of x_0 in $\Omega_{g,k}$ such that both $K_h \cap W$ and $J_h \cap W$ are smooth submanifolds of $\Omega_{g,k}$ of real dimension $2(k-h)$. Moreover Corollary 3.4.6 shows that $J_h \cap W \subset K_h \cap W$, whence $J_h \cap V = K_h \cap V$ for some (maybe smaller) neighbourhood V of x_0 in $\Omega_{g,k}$. □

Remark 3.4.8. Theorem 3.4.3 shows that Dehn fillings along *sufficiently complicated* slopes on some boundary tori of M do not affect the Euclidean structure on the non-filled boundary tori. Using SnapPea we have checked in a number of cases that the same isolation phenomenon still holds when filling along *short* non-exceptional slopes. We are now going to explain why we expect the Euclidean structure on the non-filled tori to remain unchanged under *all* the partial hyperbolic Dehn fillings of M.

Let $\mathcal{T}_h^{\mathrm{geod}}(M)$ be the space of equivalence classes of *good* structures (see Sections 1.3, 1.4 for the definition) which induce complete metrics on cusps $C_1, \ldots, C_h$ and let $X_h^{\mathrm{geod}}(\pi_1(M)) \subset X^{\mathrm{geod}}(\pi_1(M))$ be the subset corresponding to characters of holonomies of structures in $\mathcal{T}_h^{\mathrm{geod}}(M)$. Let also $\mathrm{Teich}(T_1 \sqcup \ldots \sqcup T_h)$ be the Teichmüller space of (similarity classes of) the Euclidean structures on $T_1 \sqcup \ldots \sqcup T_h$. It is easily seen that the a well-defined algebraic map $B : X_h^{\mathrm{geod}}(\pi_1(M)) \to \mathrm{Teich}(T_1 \sqcup \ldots \sqcup T_h)$ exists which associates to $\chi \in X_h^{\mathrm{geod}}(\pi_1(M))$ the Euclidean structure on $T_1 \sqcup \ldots \sqcup T_h$ induced by a structure with character χ. We have shown that B is constant in a neighbourhood of the character $\chi_0 := \chi_{\rho_0}$ of the complete structure, and this implies that B is actually constant on the irreducible component of $X_h^{\mathrm{geod}}(\pi_1(M))$ containing χ_0. On the other hand it is conjectured in [HK98] that $X_h^{\mathrm{geod}}(\pi_1(M))$ is connected (with respect to the Euclidean topology) and that every point of $X_h^{\mathrm{geod}}(\pi_1(M))$ is smooth. If this were true, then $X_h^{\mathrm{geod}}(\pi_1(M))$ would be irreducible, the map B would be constant, and the Euclidean structure on $T_1 \sqcup \ldots \sqcup T_h$ would not change under *any* hyperbolic Dehn filling of $T_{h+1}, \ldots, T_k$.

3.5 Non-isolation of the boundary

We now study how deformations given by points in $\Omega_{g,k}$ close to x_0 affect the hyperbolic structure on the geodesic boundary.

Deforming the geodesic boundary Let $\mathrm{Teich}(\partial M)$ be the Teichmüller space of hyperbolic structures on ∂M, *i.e.* the space of equivalence classes of hyperbolic metrics on ∂M, where two such metrics are considered equivalent if they are isometric through a diffeomorphism isotopic to the identity of ∂M. Since ∂M is compact, it is well-known that for any $\gamma \in \pi_1(\partial M)$ and any metric $h \in \mathrm{Teich}(\partial M)$ there exists a unique closed h-geodesic in the free homotopy class of γ. We denote the h-length of this geodesic by $L^h(\gamma)$. An easy computation shows that if

$$\overline{\rho}_h : \pi_1(\partial M) \to \mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$$

is a holonomy representation for the hyperbolic structure h then for any $\gamma \in \pi_1(\partial M)$ we have

$$\mathrm{tr}\, \rho_h(\gamma) = \pm 2 \cosh(L^h(\gamma)/2), \tag{3.9}$$

where $\rho_h(\gamma)$ is a lift in $\mathrm{SL}(2, \mathbb{R})$ of $\overline{\rho}_h(\gamma) \in \mathrm{PSL}(2, \mathbb{R})$.

Let now $[M(x_0)] \in \mathcal{T}^{\mathrm{geod}}(M)$ be the point corresponding to the complete hyperbolic structure on M, let $\chi_0 \in X^{\mathrm{geod}}(\pi_1(M))$ be the character corresponding to $[M(x_0)]$, and $[\partial M(x_0)] \in \mathrm{Teich}(\partial M)$ be the equivalence class of the hyperbolic structure induced by $M(x_0)$ on ∂M. By Theorem 1.4.11, there exist a neighbourhood V of $[M(x_0)]$ in $\mathcal{T}^{\mathrm{geod}}(M)$ and a neighbourhood Z of χ_0 in $X^{\mathrm{geod}}(\pi_1(M))$ such that Z is naturally bianalytic to an open subset of $\mathbb{C}^k$ and the natural map $\mathrm{Hol}' : \mathcal{T}^{\mathrm{geod}}(M) \to X^{\mathrm{geod}}(\pi_1(M))$ restricts to a homeomorphism between V and Z. Thus V is naturally endowed with a differentiable structure. Let $\partial : \mathcal{T}^{\mathrm{geod}}(M) \to \mathrm{Teich}(\partial M)$ be the map which associates to any equivalence class of *good* hyperbolic structures the induced equivalence class of hyperbolic metrics on ∂M (we recall that a hyperbolic structure on M is good if it induces either a complete metric or a Dehn filling singularity on any end of M, see Sections 1.3, 1.4 for more details).

It is well-known that Teich(∂M) admits a structure of differentiable manifold with the following properties:

- Teich(∂M) is diffeomorphic to the Euclidean space $\mathbb{R}^{6g-6}$;
- For any $\gamma \in \pi_1(M)$ the map $L_\gamma : \text{Teich}(\partial M) \to \mathbb{R}$ given by $L_\gamma(h) = L^h(\gamma)$ is smooth;
- The restriction $\partial|_V$ is a smooth map.

The rest of this section is devoted to the proof of the following:

Theorem 3.5.1. *The map $\partial : \mathcal{T}^{\text{geod}}(M) \to \text{Teich}(\partial M)$ has non-trivial differential at the complete structure $[M(x_0)] \in \mathcal{T}^{\text{geod}}(M)$.*

Let us assume the theorem and draw some consequences.

Corollary 3.5.2. *There exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ of points in $\mathcal{T}^{\text{geod}}(M)$ with the following properties:*

- $\lim_{n\to\infty} y_n = [M(x_0)]$;
- *For any $n \in \mathbb{N}$ the metric completion of any good structure representing y_n is a non-singular hyperbolic manifold;*
- *For any $n \in \mathbb{N}$ we have $\partial(y_n) \neq \partial([M(x_0)])$ in* Teich(∂M).

Proof: Up to resizing V we can choose smooth coordinates $\varphi : V \to \mathbb{C}^k \cong \mathbb{R}^{2k}$ with $\varphi([M(x_0)]) = 0$. Let $v \in \mathbb{R}^{2k} \cong T_0\mathbb{R}^{2k}$ be a unit vector with $d(\partial \circ \varphi^{-1})_0(v) \neq 0$. We choose an infinite sequence $\{y_n\}_{n\in\mathbb{N}} \subset V$ such that the following conditions hold: y_n determines integral indivisible Dehn filling coefficients on any end of M for any $n \in \mathbb{N}$, and $\lim_{n\to\infty} \varphi(y_n)/||\varphi(y_n)|| = v$ in $S^{2k-1} \subset \mathbb{R}^{2k}$. □

Since the mapping class group of ∂M acts properly discontinuously on Teich(∂M) we also have the following:

Corollary 3.5.3. *There exists an infinite set $\{N_i\}_{i\in\mathbb{N}}$ of complete finite-volume hyperbolic manifolds with geodesic boundary such the the following holds: each N_i is obtained by Dehn filling M, and the hyperbolic surfaces $\partial M, \partial N_1, \ldots, \partial N_i, \ldots$ are pairwise non-isometric.*

Proof: We define N_i to be the metric completion of any good structure representing y_i, where $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{T}^{\text{geod}}(M)$ is the sequence described in the previous corollary. Let $\mathcal{MCG}(\partial M)$ be the mapping class group of ∂M and take a neighbourhood U of $\partial([M(x_0)])$ in Teich(∂M) such that the set $\{\psi \in \mathcal{MCG}(\partial M) : \psi(U) \cap U \neq \emptyset\}$ is finite. Up to passing to a subsequence, we may suppose that the equivalence classes of the ∂N_j's are pairwise distinct as elements in Teich(∂M), and that $\partial N_i \in U$ for all $i \in \mathbb{N}$. This readily implies that among the ∂N_j's there are infinitely many pairwise non-isometric hyperbolic surfaces. □

An alternative formulation We will now prove Theorem 3.5.1, providing first an alternative statement. Let $\pi : \Omega_{g,k} \to X^{\mathrm{geod}}(\pi_1(M))$ be the natural projection. Then there exists a neighbourhood U of x_0 in $\Omega_{g,k}$ such that $\pi|_U$ is a smooth branched covering of degree 2^k (see Corollary 2.4.4). Let $\partial' : \Omega_{g,k} \to \mathrm{Teich}(\partial M)$ be the map which associates to any $x \in \Omega_{g,k}$ the equivalence class of the hyperbolic structure induced by x on ∂M. Of course we have $\partial' = \partial \circ \pi$, so since $d\pi_{x_0} = 0$ we have

$$d\partial'_{x_0} = 0. \tag{3.10}$$

Thus in order to prove Theorem 3.5.1 we need to analyze the map ∂' up to the second order in a neighbourhood of x_0. More precisely, we obtain Theorem 3.5.1 as a consequence of Corollary 2.4.4 and the following:

Proposition 3.5.4. *There exist a smooth path $\overline{\varsigma} : (-\varepsilon, \varepsilon) \to \Omega_{g,k}$ and an element $\overline{\gamma} \in \pi_1(\partial M)$ with the following properties:*

- $\overline{\varsigma}(0) = x_0$*;*
- *If $\partial'(\overline{\varsigma}(t))$ is the hyperbolic structure induced by $\overline{\varsigma}(t)$ on ∂M, then the map $t \mapsto L^{\partial'(\overline{\varsigma}(t))}(\overline{\gamma})$ has non-zero second derivative at 0.*

So let $\varsigma : (-\varepsilon, \varepsilon) \to \Omega_{g,k} \subset \mathbb{R}^N$ be a fixed smooth path with $\varsigma(0) = x_0$ and for any smooth function $f : \Omega_{g,k} \to \mathbb{R}$ let $\dot{f}(t)$ (resp. $\ddot{f}(t)$) denote the first (resp. second) derivative of $f \circ \varsigma$ in t. To lighten a bit our notation we will denote simply by $\dot{f}$ (resp. $\ddot{f}$) the value of $\dot{f}(0)$ (resp. $\ddot{f}(0)$). For $x \in \Omega_{g,k}$ let $M(x)$ be the hyperbolic structure on M corresponding to x. For any $x \in \Omega_{g,k}$, all the boundary edges lying on *compact* faces of $\mathcal{T}$ have the same $M(x)$-length; such a length will be denoted by $\ell(x)$. We recall that $\dot{\ell} = \dot{x}_{12k+1} = 0$ by Corollaries 2.4.19, 2.4.20.

Lemma 3.5.5. *We have $\ddot{\ell} = \ddot{x}_{6l+1} + \ddot{x}_{6l+2} + \ddot{x}_{6l+3} = \ddot{x}_{12k+1} = 0$ for $l = 0, \ldots, 2k-1$.*

Proof: By Lemma 2.2.2 we have $\ell(x) = (\cos x_1 \cos x_2 + \cos x_6)/(\sin x_1 \sin x_2)$. Differentiating this equality along ς we get

$$\begin{aligned}(\sin^2 x_1(t) \sin^2 x_2(t))\dot{\ell}(t) &= -(\sin x_2(t) \cos x_2(t) + \cos x_1(t) \sin x_2(t) \cos x_6(t))\dot{x}_1(t) \\ &- (\sin x_1(t) \cos x_1(t) + \sin x_1(t) \cos x_2(t) \cos x_6(t))\dot{x}_2(t) \\ &- (\sin x_1(t) \sin x_2(t) \sin x_6(t))\dot{x}_6(t),\end{aligned}$$

so differentiating again and evaluating at 0 we get

$$\begin{aligned}(2\sin^4 \overline{\alpha}_{g,k})\ddot{\ell} &= (-3\cos \overline{\alpha}_{g,k} \sin \overline{\alpha}_{g,k})(\ddot{x}_1 + \ddot{x}_2) - \sqrt{3}\sin^2 \overline{\alpha}_{g,k}\ddot{x}_6 \\ &+ (5\cos^2 \overline{\alpha}_{g,k} + 1)(\dot{x}_1^2 + \dot{x}_2^2) - \sin^2 \overline{\alpha}_{g,k}\dot{x}_6^2 \\ &+ (2\cos^2 \overline{\alpha}_{g,k} + 4)\dot{x}_1\dot{x}_2 \\ &+ 2\sqrt{3}\cos \overline{\alpha}_{g,k} \sin \overline{\alpha}_{g,k}(\dot{x}_1\dot{x}_6 + \dot{x}_2\dot{x}_6).\end{aligned} \tag{3.11}$$

Observe now that on $\Omega_{g,k}$ we also have

$$\ell(x) = (\cos x_1 \cos x_3 + \cos x_5)/(\sin x_1 \sin x_3) = (\cos x_2 \cos x_3 + \cos x_4)/(\sin x_2 \sin x_3).$$

Differentiating two times these equalities along ς, evaluating at 0 as above and summing up with equality (3.11) we get

$$\begin{aligned}(6\sin^4\overline{\alpha}_{g,k})\ddot{\ell} &= -6\cos\overline{\alpha}_{g,k}\sin\overline{\alpha}_{g,k}(\ddot{x}_1+\ddot{x}_2+\ddot{x}_3) - \sqrt{3}\sin^2\overline{\alpha}_{g,k}(\ddot{x}_4+\ddot{x}_5+\ddot{x}_6)\\ &+ (10\cos^2\overline{\alpha}_{g,k}+2)(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2) - \sin^2\overline{\alpha}_{g,k}(\dot{x}_4^2+\dot{x}_5^2+\dot{x}_6^2)\\ &+ (2\cos^2\overline{\alpha}_{g,k}+4)(\dot{x}_1\dot{x}_2+\dot{x}_1\dot{x}_3+\dot{x}_2\dot{x}_3)\\ &+ 2\sqrt{3}\cos\overline{\alpha}_{g,k}\sin\overline{\alpha}_{g,k}(\dot{x}_1\dot{x}_5+\dot{x}_1\dot{x}_6+\dot{x}_2\dot{x}_4+\dot{x}_2\dot{x}_6+\dot{x}_3\dot{x}_4+\dot{x}_3\dot{x}_5).\end{aligned} \tag{3.12}$$

Since $\dot{x}$ belongs to $T_{x_0}\Omega_{g,k}$ we have $(\sin\overline{\alpha}_{g,k})\dot{x}_{i+3} = (\sqrt{3}\cos\overline{\alpha}_{g,k})\dot{x}_i$ for $i=1,2,3$ and $\dot{x}_1+\dot{x}_2+\dot{x}_3=0$. Substituting these relations in (3.12) we (rather strikingly) get

$$\ddot{\ell} = \frac{-\cos\overline{\alpha}_{g,k}(\ddot{x}_1+\ddot{x}_2+\ddot{x}_3)}{\sin^3\overline{\alpha}_{g,k}}. \tag{3.13}$$

By the very same argument it follows that

$$\ddot{\ell} = \frac{-\cos\overline{\alpha}_{g,k}(\ddot{x}_{6l+1}+\ddot{x}_{6l+2}+\ddot{x}_{6l+3})}{\sin^3\overline{\alpha}_{g,k}} \tag{3.14}$$

for $l=0,\ldots,2k-1$, whence $\ddot{x}_{6l+1}+\ddot{x}_{6l+2}+\ddot{x}_{6l+3} = \ddot{x}_{6h+1}+\ddot{x}_{6h+2}+\ddot{x}_{6h+3}$ for any $h,l\in\{0,\ldots,2k-1\}$. Thus the condition forcing the dihedral angle along the compact edge of $\mathcal{T}$ to be constantly equal to 2π gives $2k(\ddot{x}_1+\ddot{x}_2+\ddot{x}_3)+6(g-k)\ddot{x}_{12k+1}=0$, whence $\ddot{x}_1+\ddot{x}_2+\ddot{x}_3$ and $\ddot{x}_{12k+1}$ have opposite signs. Moreover condition $\ell(x) = \cos x_{12k+1}/(1-\cos x_{12k+1})$ implies $\ddot{\ell} = -(\sin\overline{\beta}_{g,k}/(1-\cos\overline{\beta}_{g,k})^2)\ddot{x}_{12k+1}$. Together with equation (3.13), this forces $\ddot{x}_1+\ddot{x}_2+\ddot{x}_3$ and $\ddot{x}_{12k+1}$ to have the same sign, so $\ddot{x}_1+\ddot{x}_2+\ddot{x}_3=0$, whence $\ddot{x}_{6l+1}+\ddot{x}_{6l+2}+\ddot{x}_{6l+3}=0$ for $l=0,\ldots,2k-1$ and $\ddot{\ell}=0$.

When $g=k$, the variable x_{12k+1} is not involved, and the same proof works with an obvious modification at the last step. □

The chosen curve Let us now concentrate our attention on a special curve in the deformation space $\Omega_{g,k}$. To this aim, let H be the subspace of $\mathbb{R}^N$ defined by the following equations:

$$H = \{x\in\mathbb{R}^N : x_2 = x_3,\ x_{12i+1} = x_{12i+2} = x_{12i+3} \text{ for all } i=1,\ldots,k-1\}.$$

Let v_0 be the tangent vector to $\Omega_{g,k}$ at x_0 defined as follows:

$$\begin{aligned} v_0 = {} & (2\sin\overline{\alpha}_{g,k}, -\sin\overline{\alpha}_{g,k}, -\sin\overline{\alpha}_{g,k}, 2\sqrt{3}\cos\overline{\alpha}_{g,k}, -\sqrt{3}\cos\overline{\alpha}_{g,k}, -\sqrt{3}\cos\overline{\alpha}_{g,k},\\ & -2\sin\overline{\alpha}_{g,k}, \sin\overline{\alpha}_{g,k}, \sin\overline{\alpha}_{g,k}, -2\sqrt{3}\cos\overline{\alpha}_{g,k}, \sqrt{3}\cos\overline{\alpha}_{g,k}, \sqrt{3}\cos\overline{\alpha}_{g,k},\\ & 0,\ldots,0).\end{aligned}$$

Using Proposition 3.4.1 one one can easily prove that $T_{x_0}\Omega_{g,k}\cap H = \mathrm{Span}\langle v_0\rangle$, so H intersects $\Omega_{g,k}$ transversely near x_0 in the support of a smooth curve $\overline{\varsigma} : (-\epsilon,\epsilon)\to\Omega_{g,k}$ with $\overline{\varsigma}(0)=x_0$ and $\dot{\overline{\varsigma}}(0)=v_0$. As before, for any smooth $f:\Omega_{g,k}\to\mathbb{R}$ we set $f(t) := f(\overline{\varsigma}(t))$ and we denote by $\dot{f}(t)$ (resp. $\ddot{f}(t)$) the first (resp. second) derivative of $f\circ\overline{\varsigma}$ in t. We also denote simply by $\dot{f}$, $\ddot{f}$ the values $\dot{f}(0)$, $\ddot{f}(0)$ respectively.

An immediate application of the chain rule for smooth maps gives the following:

Lemma 3.5.6. *Let $\psi : (-\epsilon', \epsilon') \to (-\epsilon, \epsilon)$ be a local diffeomorphism with $\dot{\psi}(0) = 1$. Then*
$$(\ddot{\overline{\varsigma} \circ \psi})(0) = \ddot{\overline{\varsigma}}(0) + \ddot{\psi}(0) \cdot v_0.$$

Thus, up to reparameterizing $\overline{\varsigma}$ without changing its tangent vector at 0 we can assume that the following condition holds:

$$\ddot{x}_1 = \ddot{x}_7. \tag{3.15}$$

A very similar argument to the proof of Proposition 3.4.5 gives the following:

Proposition 3.5.7. *For any $t \in (-\epsilon, \epsilon)$ we have*

$$x_2(t) = x_3(t), \quad x_5(t) = x_6(t), \quad x_8(t) = x_9(t), \quad x_{11}(t) = x_{12}(t),$$

and

$$\begin{array}{ll} x_{12i+1}(t) = x_{12i+2}(t) = x_{12i+3}(t), & x_{12i+4}(t) = x_{12i+5}(t) = x_{12i+6}(t) = \pi/3, \\ x_{12i+7}(t) = x_{12i+8}(t) = x_{12i+9}(t), & x_{12i+10}(t) = x_{12i+11}(t) = x_{12i+12}(t) = \pi/3 \end{array}$$

for $i = 1, \ldots, k-1$.

Proof: For $i = 1, \ldots, 2k$ let $\Delta_i^*(t)$ be the geometric realization of Δ_i parameterized by $\overline{\varsigma}(t) \in \Omega_{g,k}$. Let $T_1(t), T_1'(t)$ be the truncation triangles of $\Delta_1^*(t)$ having a vertex on the edge e_1^1, and $T_2(t), T_2'(t)$ be the truncation triangles of $\Delta_2^*(t)$ having a vertex on the edge e_2^1. Recall that the compact face of Δ_1^* and Δ_2^* is a *regular* right-angled hexagon, so condition $x_2(t) = x_3(t)$ implies that $T_1(t)$ and $T_1'(t)$ are isometric to each other, which gives $x_5(t) = x_6(t)$. Moreover the faces F_1^1 and F_2^1 are glued to each other, and this ensures that also $T_2(t)$ and $T_2'(t)$ are isometric to each other, so $x_8(t) = x_9(t)$ and $x_{11}(t) = x_{12}(t)$.

Since for all $i = 1, \ldots, k-1$ we have $x_{12i+1}(t) = x_{12i+2}(t) = x_{12i+3}(t)$, the argument just described applies to the pair of tetrahedra $\Delta_{2i+1}^*(t), \Delta_{2i+2}^*(t)$ giving

$$\begin{array}{r} x_{12i+4}(t) = x_{12i+5}(t) = x_{12i+6}(t), \quad x_{12i+7}(t) = x_{12i+8}(t) = x_{12i+9}(t), \\ x_{12i+10}(t) = x_{12i+11}(t) = x_{12i+12}(t). \end{array}$$

Now the conclusion follows since $x_{12i+4}(t) + x_{12i+5}(t) + x_{12i+6}(t) = x_{12i+10}(t) + x_{12i+11}(t) + x_{12i+12}(t) = \pi$. $\square$

We are now ready to prove the following:

Proposition 3.5.8. *We have*

$$\begin{array}{rcl} \ddot{\overline{\varsigma}}(0) & = & (8\cos\overline{\alpha}_{g,k}\sin\overline{\alpha}_{g,k}, -4\cos\overline{\alpha}_{g,k}\sin\overline{\alpha}_{g,k}, -4\cos\overline{\alpha}_{g,k}\sin\overline{\alpha}_{g,k}, 2\sqrt{3}, -\sqrt{3}, -\sqrt{3}, \\ & & 8\cos\overline{\alpha}_{g,k}\sin\overline{\alpha}_{g,k}, -4\cos\overline{\alpha}_{g,k}\sin\overline{\alpha}_{g,k}, -4\cos\overline{\alpha}_{g,k}\sin\overline{\alpha}_{g,k}, 2\sqrt{3}, -\sqrt{3}, -\sqrt{3}, \\ & & 0, \ldots, 0). \end{array}$$

Proof: Since $x_4(t) + x_5(t) + x_6(t) = x_{10}(t) + x_{11}(t) + x_{12}(t) = \pi$, by Proposition 3.5.7 we have $\ddot{x}_5 = \ddot{x}_6 = -(1/2)\ddot{x}_4$ and $\ddot{x}_{11} = \ddot{x}_{12} = -(1/2)\ddot{x}_{10}$, while by Lemma 3.5.5

and Proposition 3.5.7 we get $\ddot{x}_2 = \ddot{x}_3 = -(1/2)\ddot{x}_1$ and $\ddot{x}_8 = \ddot{x}_9 = -(1/2)\ddot{x}_7$. Substituting these relations in equation (3.11) and recalling that $\ddot{\ell} = 0$ we have after some computations

$$\sqrt{3}\cos\overline{\alpha}_{g,k}\ddot{x}_1 - \sin\overline{\alpha}_{g,k}\ddot{x}_4 = 2\sqrt{3}\sin\overline{\alpha}_{g,k}(4\cos^2\overline{\alpha}_{g,k} - 1). \tag{3.16}$$

The very same argument also gives

$$\sqrt{3}\cos\overline{\alpha}_{g,k}\ddot{x}_7 - \sin\overline{\alpha}_{g,k}\ddot{x}_{10} = 2\sqrt{3}\sin\overline{\alpha}_{g,k}(4\cos^2\overline{\alpha}_{g,k} - 1). \tag{3.17}$$

In the same way, differentiating two times the equality

$$\sin x_1(t)\sin x_4(t)\sin x_7(t)\sin x_{10}(t) = \sin x_2(t)\sin x_5(t)\sin x_8(t)\sin x_{11}(t),$$

evaluating at 0 and substituting in the result the relations $\ddot{x}_2 = -(1/2)\ddot{x}_1$, $\ddot{x}_5 = -(1/2)\ddot{x}_4$, $\ddot{x}_8 = -(1/2)\ddot{x}_7$ and $\ddot{x}_{11} = -(1/2)\ddot{x}_{10}$ we get

$$\sqrt{3}\cos\overline{\alpha}_{g,k}(\ddot{x}_1 + \ddot{x}_7) + \sin\overline{\alpha}_{g,k}(\ddot{x}_4 + \ddot{x}_{10}) = 4\sqrt{3}\sin\overline{\alpha}_{g,k}(4\cos^2\overline{\alpha}_{g,k} + 1). \tag{3.18}$$

Solving equations (3.15), (3.16), (3.17) and (3.18) we get the desired result for $\ddot{x}_1, \ldots, \ddot{x}_{12}$.

Let now $i = 1, \ldots, k-1$. By Proposition 3.5.7 and Lemma 3.5.5 we get

$$\ddot{x}_{12i+1} = \ddot{x}_{12i+2} = \ddot{x}_{12i+3} = \frac{\ddot{x}_{12i+1} + \ddot{x}_{12i+2} + \ddot{x}_{12i+3}}{3} = 0,$$
$$\ddot{x}_{12i+7} = \ddot{x}_{12i+8} = \ddot{x}_{12i+9} = \frac{\ddot{x}_{12i+7} + \ddot{x}_{12i+8} + \ddot{x}_{12i+9}}{3} = 0.$$

Moreover, Proposition 3.5.7 forces

$$\ddot{x}_{12i+4} = \ddot{x}_{12i+5} = \ddot{x}_{12i+6} = \ddot{x}_{12i+10} = \ddot{x}_{12i+10} = \ddot{x}_{12i+11} = \ddot{x}_{12i+12} = 0,$$

and if $g > k$ Lemma 3.5.5 also gives $\ddot{x}_{12k+1} = 0$. □

The final step The smooth path $\overline{\varsigma} : (-\varepsilon, \varepsilon) \to \Omega_{g,k}$ determines a smooth family of developing maps $D_t : \widetilde{\partial M} \to \mathbb{H}^2$, which gives in turn a smooth path of holonomy representations $\overline{\rho}_t : \pi_1(\partial M) \to \mathrm{PSL}(2,\mathbb{R})$ lifting to a smooth path of representations $\rho_t : \pi_1(\partial M) \to \mathrm{SL}(2,\mathbb{R})$. For any $\gamma \in \pi_1(\partial M)$ we set

$$\mathrm{tr}_\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}, \quad \mathrm{tr}_\gamma(t) = \mathrm{trace}\,\rho_t(\gamma).$$

Of course tr_γ is a smooth map for any $\gamma \in \pi_1(\partial M)$. Moreover, by equations (3.9) and (3.10) we have $\dot{\mathrm{tr}}_\gamma(0) = 0$. Together with equation (3.9), the following result readily implies Proposition 3.5.4, whence Theorem 3.5.1.

Proposition 3.5.9. *There exists an element $\overline{\gamma} \in \pi_1(\partial M)$ such that $\ddot{\mathrm{tr}}_{\overline{\gamma}}(0) \neq 0$.*

In order to find the element $\overline{\gamma} \in \pi_1(\partial M)$ mentioned in Proposition 3.5.9 we have to describe in some detail a particular portion of the triangulation induced on ∂M by the canonical decomposition $\mathcal{T}$ of M. So we fix our attention on the

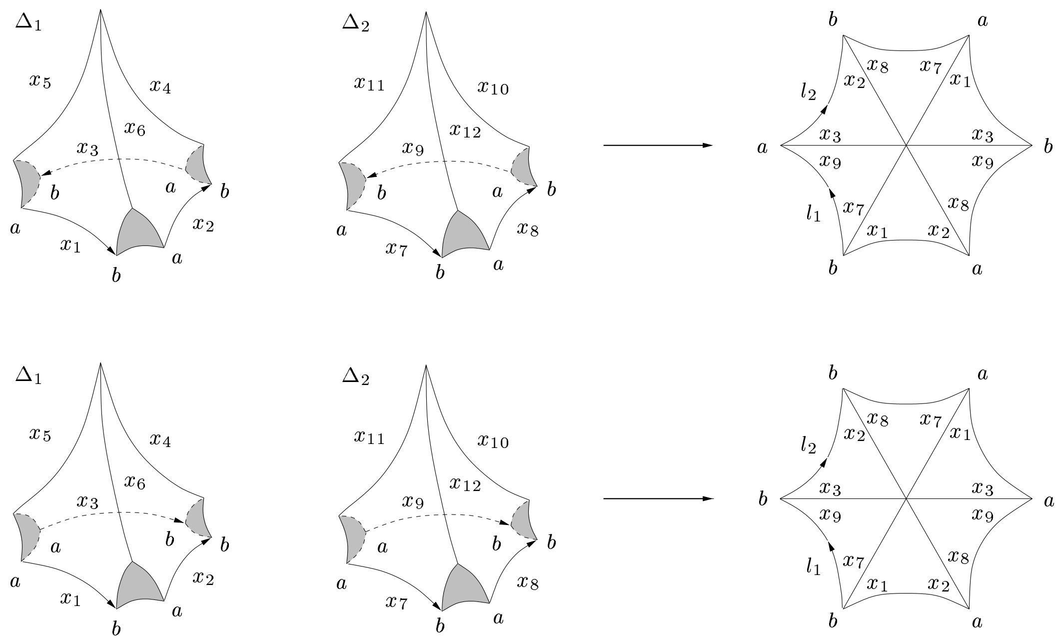

Figure 3.4: Up to cyclic reorderings of the internal edges e_1^1, e_1^2, e_1^3 of Δ_1, we show here the only possible identifications of the vertices of the hexagon tessellated by the truncation triangles of Δ_1 and Δ_2.

geodesic hexagon (with identifications on edges and vertices) which results from the gluing of the truncation triangles of $\Delta_1, \Delta_2 \in \mathcal{T}$. Let l_1, l_2 be the oriented edges of this hexagon described in Fig. 3.4 and observe that the starting point of l_1 and the endpoint of l_2 both coincide with the same point $b \in \partial M$. Thus the loop $l_2 * l_1$ defines an element $\overline{\gamma}$ of $\pi_1(\partial M, b)$. Our next aim is to give an explicit description of the isometry $\overline{\rho}_t(\overline{\gamma}) \in \mathrm{PSL}(2,\mathbb{R})$ in terms of angles and lengths of edges of the triangulation of ∂M. Let us fix two consecutive lifts $\widetilde{l}_1, \widetilde{l}_2$ of l_1, l_2 in $\widetilde{\partial M}$ and let $\widetilde{l}'_1 = \overline{\gamma}(\widetilde{l}_1) \subset \widetilde{\partial M}$ be the lift of l_1 starting at the endpoint of $\widetilde{l}_2$. Then it is easily seen that $\overline{\rho}_t(\overline{\gamma})$ is the unique orientation-preserving isometry of $\mathbb{H}^2$ taking the oriented geodesic segment $D_t(\widetilde{l}_1)$ onto the oriented geodesic segment $D_t(\widetilde{l}'_1)$.

Let $\eta(t)$ be the angle formed by $D_t(\widetilde{l}_1)$ and $D_t(\widetilde{l}_2)$ at the endpoint of $D_t(\widetilde{l}_1)$ and $\zeta(t)$ the angle formed by $D_t(\widetilde{l}_2)$ and $D_t(\widetilde{l}'_1)$ at the endpoint of $D_t(\widetilde{l}_2)$ (see Fig. 3.5). Of course we have $\eta(t) = x_3(t) + x_9(t)$, while

$$\zeta(t) = (x_2(t) + x_8(t)) + (x_1(t) + x_7(t)) + \delta(x_3(t) + x_9(t)) + r(t),$$

where $\delta \in \{0,1\}$ is determined by the combinatorics of $\mathcal{T}$ and $r(t) \in (0, 2\pi)$ is given by the sum (with multiplicity) of some of the $x_i(t)$'s with $i \geqslant 13$. The definition of the tangent vector v_0 of $\overline{\varsigma}(t)$ at 0 and Proposition 3.5.8 give the following:

Lemma 3.5.10. *We have* $\dot{\eta}(0) = \dot{\zeta}(0) = 0$ *and* $\ddot{\eta}(0) \neq 0$. *Moreover, if* $\delta = 0$ *then* $\ddot{\zeta}(0) = -\ddot{\eta}(0)$*; if* $\delta = 1$ *then* $\ddot{\zeta}(0) = 0$.

Let now $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ be the upper half-plane model of $\mathbb{H}^2$. Without loss of generality we may assume that for any $t \in (-\varepsilon, \varepsilon)$ the developing

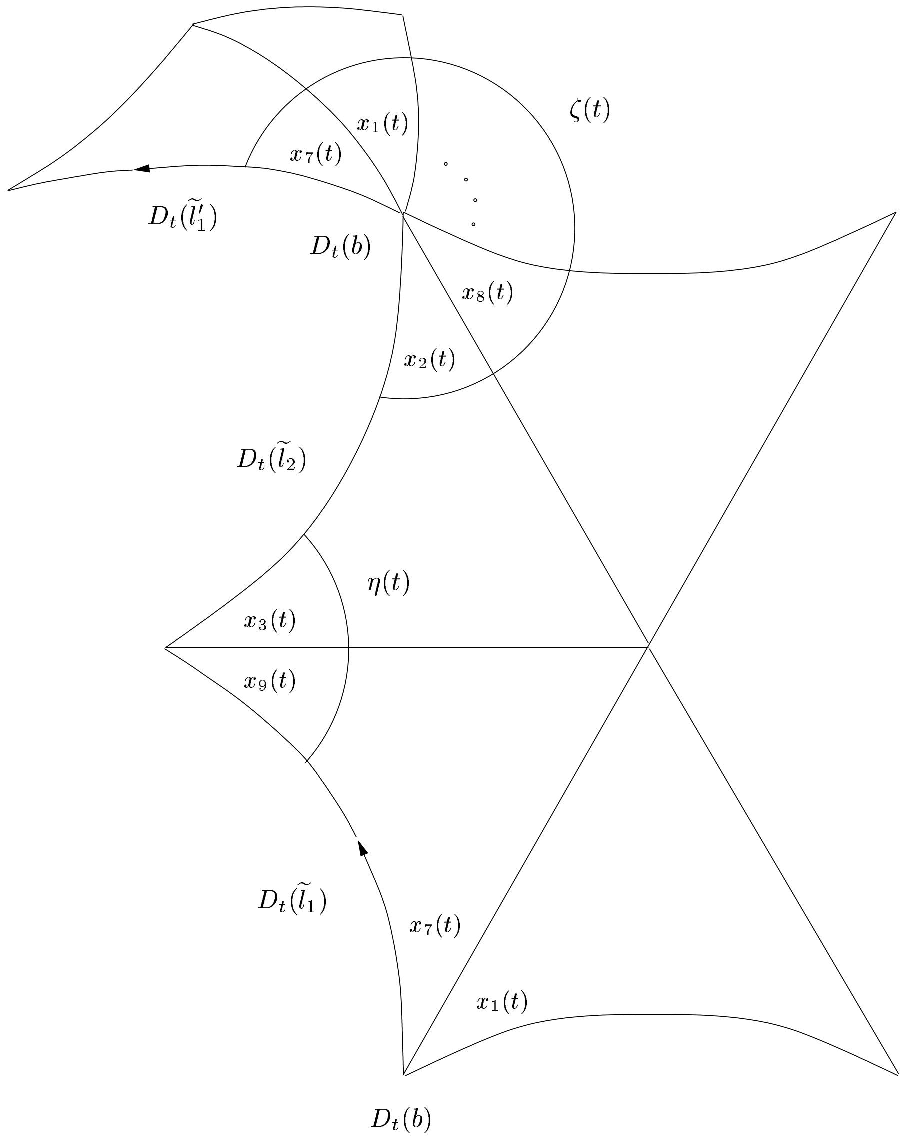

Figure 3.5: Notation for the proof of equation (3.21).

map $D_t : \widetilde{\partial M} \to \mathbb{H}$ sends $\widetilde{l}_1$ onto the geodesic segment starting at $i \in \mathbb{H}$ and ending at $\lambda(t)\cdot i \in \mathbb{H}$, where $\lambda(t)$ is a real number greater than 1. For $1 < \lambda \in \mathbb{R}$, $\theta \in (0, 2\pi)$ we now set

$$A(\lambda,\theta) = \left[\begin{pmatrix} \sqrt{\lambda}\sin(\theta/2) & -\sqrt{\lambda}\cos(\theta/2) \\ (1/\sqrt{\lambda})\cos(\theta/2) & (1/\sqrt{\lambda})\sin(\theta/2) \end{pmatrix}\right] \in \mathrm{PSL}(2,\mathbb{R}).$$

It is easily seen that $A(\lambda,\theta)$ sends the half-geodesic s starting at $i \in \mathbb{H}$ and ending at ∞ onto the half-geodesic s' starting at $\lambda \cdot i$ such that s and s' define at $\lambda \cdot i$ an angle equal to θ. Now the length of $D_t(\widetilde{l}_2)$ is equal to the length of $D_t(\widetilde{l}_1)$, and the isometry $\overline{\rho}_t(\overline{\gamma})$ takes the oriented geodesic segment $D_t(\widetilde{l}_1)$ onto the oriented geodesic segment $D_t(\widetilde{l'_1})$, so

$$\overline{\rho}_t(\overline{\gamma}) = A(\lambda(t),\eta(t)) \cdot A(\lambda(t),\zeta(t)). \tag{3.19}$$

Since

$$\lambda(t) = \exp \ell(t), \tag{3.20}$$

by Corollary 2.4.19 and Lemma 3.5.5 we have $\dot{\lambda}(0) = \ddot{\lambda}(0) = 0$. Moreover we have $\dot{\eta}(0) = \dot{\zeta}(0) = 0$, so differentiating two times equality (3.19), evaluating at 0 and taking the trace we obtain

$$\begin{array}{rcll} 2\ddot{\mathrm{tr}}_{\overline{\gamma}}(0) & = & \ddot{\eta}(0) & \cdot \;\; ((\lambda(0)+\lambda(0)^{-1})\cos(\eta(0)/2)\sin(\zeta(0)/2) \\ & & & \quad +2\sin(\eta(0)/2)\cos(\zeta(0)/2)) \\ & + & \ddot{\zeta}(0) & \cdot \;\; ((\lambda(0)+\lambda(0)^{-1})\sin(\eta(0)/2)\cos(\zeta(0)/2) \\ & & & \quad +2\cos(\eta(0)/2)\sin(\zeta(0)/2)). \end{array} \tag{3.21}$$

We now consider separately the cases $\ddot{\zeta}(0) = -\ddot{\eta}(0) \neq 0$ and $\ddot{\zeta}(0) = 0$. In the first case, from equation (3.21) we obtain

$$\ddot{\mathrm{tr}}_{\overline{\gamma}}(0) = (\ddot{\eta}(0)/2)(\lambda(0)+\lambda(0)^{-1}-2)(\sin((\zeta(0)/2)-(\eta(0)/2))) \neq 0.$$

In order to prove that $\ddot{\mathrm{tr}}_{\overline{\gamma}}(0) \neq 0$ also in the latter case we will show that

$$(\lambda(0)+\lambda(0)^{-1})\cos(\eta(0)/2)\sin(\zeta(0)/2) + 2\sin(\eta(0)/2)\cos(\zeta(0)/2) > 0. \tag{3.22}$$

First of all for any $g \geqslant 2$ we define $\overline{\beta}_{g,g}$ to be the unique real number $\overline{\beta}_{g,g} \in (0,\pi/2)$ such that

$$\frac{\cos\overline{\beta}_{g,g}}{1-\cos\overline{\beta}_{g,g}} = \frac{\cos^2\overline{\beta}_{g,g}+\cos\overline{\beta}_{g,g}}{\sin^2\overline{\beta}_{g,g}} = \frac{\cos^2\overline{\alpha}_{g,g}+1/2}{\sin^2\overline{\alpha}_{g,g}}. \tag{3.23}$$

We notice that this definition naturally extends to the case $g = k$ the usual definition of $\overline{\beta}_{g,k}$, $g > k \geqslant 1$. Solving equation (3.23) (or the corresponding consistency equation when $g > k \geqslant 1$) with respect to $\overline{\beta}_{g,k}$ we obtain

$$\begin{array}{l} \cos\overline{\beta}_{g,k} = (1+2\cos^2\overline{\alpha}_{g,k})/3 = (2+\cos 2\overline{\alpha}_{g,k})/3, \\ \sin\overline{\beta}_{g,k} = \left(2\sin\overline{\alpha}_{g,k}\sqrt{2+\cos^2\overline{\alpha}_{g,k}}\right)/3, \quad g \geqslant 2,\ k \geqslant 1,\ g \geqslant k. \end{array} \tag{3.24}$$

Lemma 3.5.11. *For any $g \geqslant 2, k \geqslant 1$ with $g \geqslant k$ we have*

$$\overline{\alpha}_{g,k} < \overline{\beta}_{g,k} < 2\overline{\alpha}_{g,k} \leqslant \pi/3.$$

Proof: By equation (3.24) we have $\cos\overline{\beta}_{g,k} = (2+\cos 2\overline{\alpha}_{g,k})/3 > \cos 2\overline{\alpha}_{g,k}$, whence $\overline{\beta}_{g,k} < 2\overline{\alpha}_{g,k}$. Moreover, since $\overline{\alpha}_{g,k} < \pi/3$ we have $2\cos^2\overline{\alpha}_{g,k} - 3\cos\overline{\alpha}_{g,k} + 1 < 0$, whence $\cos\overline{\beta}_{g,k} = (2\cos^2\overline{\alpha}_{g,k}+1)/3 < \cos\overline{\alpha}_{g,k}$ and $\overline{\alpha}_{g,k} < \overline{\beta}_{g,k}$. This inequality also implies $2\pi = 6(g-k)\cdot\overline{\beta}_{g,k} + 6k\cdot\overline{\alpha}_{g,k} \geqslant 6g\cdot\overline{\alpha}_{g,k}$, whence $\overline{\alpha}_{g,k} \leqslant \pi/6$. □

Since $\eta(0) = 2\overline{\alpha}_{g,k}$, if $\zeta(0) \leqslant \pi$ then inequality (3.22) is trivially satisfied, so we may assume $\pi \leqslant \zeta(0)$. By Lemma 3.5.11 we deduce $\pi \leqslant \zeta(0) \leqslant 2\pi - \overline{\beta}_{g,k}$, whence $\sin(\zeta(0)/2) \geqslant \sin(\overline{\beta}_{g,k}/2)$ and $\cos(\zeta(0)/2) \geqslant -\cos(\overline{\beta}_{g,k}/2)$. Thus we have

$$\begin{array}{rl}
 & (\lambda(0)+\lambda(0)^{-1})\cos(\eta(0)/2)\sin(\zeta(0)/2) + 2\sin(\eta(0)/2)\cos(\zeta(0)/2) \\
\geqslant & (\lambda(0)+\lambda(0)^{-1})\cos\overline{\alpha}_{g,k}\sin(\overline{\beta}_{g,k}/2) - 2\sin\overline{\alpha}_{g,k}\cos(\overline{\beta}_{g,k}/2) \\
> & \cos(\overline{\beta}_{g,k}/2)\cdot((\lambda(0)+\lambda(0)^{-1})\cos\overline{\alpha}_{g,k}\sin(\overline{\beta}_{g,k}/2) - 2\sin\overline{\alpha}_{g,k}\cos(\overline{\beta}_{g,k}/2)) \\
= & (1/2)(\lambda(0)+\lambda(0)^{-1})\cos\overline{\alpha}_{g,k}\sin\overline{\beta}_{g,k} - \sin\overline{\alpha}_{g,k}(1+\cos\overline{\beta}_{g,k}).
\end{array} \tag{3.25}$$

By equation (3.20) we have

$$\lambda(0) + \lambda(0)^{-1} = 2\cosh\ell(0) = (2\cos^2\overline{\alpha}_{g,k}+1)/(\sin^2\overline{\alpha}_{g,k}). \tag{3.26}$$

Substituting equalities (3.24), (3.26) in (3.25) and recalling that $\overline{\alpha}_{g,k} \leqslant \pi/6$ we now deduce, after some computations:

$$\begin{array}{rl}
 & (1/2)(\lambda(0)+\lambda(0)^{-1})\cos\overline{\alpha}_{g,k}\sin\overline{\beta}_{g,k} - \sin\overline{\alpha}_{g,k}(1+\cos\overline{\beta}_{g,k}) \\
= & \left((2\cos^2\overline{\alpha}_{g,k}+1)\cos\overline{\alpha}_{g,k}\sqrt{2+\cos^2\overline{\alpha}_{g,k}} + 2\cos^4\overline{\alpha}_{g,k} + 2\cos^2\overline{\alpha}_{g,k} - 4\right)/(3\sin\overline{\alpha}_{g,k}) \\
\geqslant & (5\sqrt{33}-11)/12 > 0.
\end{array}$$

This finally proves inequality (3.22), thus concluding the proof of Proposition 3.5.9 and Theorem 3.5.1.

3.6 Symmetries of $\Omega_{g,k}$ and return paths

We now prove some results which will prove useful in the last section. We begin by analyzing in detail the symmetries of the deformation space $\Omega_{g,k}$, and explaining how these symmetries act on the space of Dehn filling coefficients. After that we will describe how the length of the shortest return path (see Definition 2.6.9) changes under deformation of the complete structure.

Symmetries of $\Omega_{g,k}$ Let $\mathrm{Aut}(\Omega_{g,k})$ denote the set of diffeomorphisms of $\Omega_{g,k}$ onto itself, fix $i \in \{1,\ldots,k\}$ and take an element σ of the symmetric group $\mathfrak{S}_3$. Since the group of combinatorial automorphisms of a partially truncated tetrahedron with exactly one ideal vertex is isomorphic to $\mathfrak{S}_3$, we can make σ act on the dihedral angles of Δ_{2i-1} and of Δ_{2i}. In this way σ defines an automorphism $\hat{\sigma}_i \in \mathrm{Aut}(\Omega_{g,k})$ which acts on these angles according to σ and leaves all the other angles unchanged. More formally, if $x \in \Omega_{g,k}$ we set:

$$(\hat{\sigma}_i(x))_j = \begin{cases} x_{12(i-1)+\sigma^{-1}(l)} & \text{if } j = 12(i-1)+l,\ l=1,2,3; \\ x_{12(i-1)i+3+\sigma^{-1}(l)} & \text{if } j = 12(i-1)+3+l,\ l=1,2,3; \\ x_{12(i-1)+6+\sigma^{-1}(l)} & \text{if } j = 12(i-1)+6+l,\ l=1,2,3; \\ x_{12(i-1)+9+\sigma^{-1}(l)} & \text{if } j = 12(i-1)+9+l,\ l=1,2,3; \\ x_j & \text{otherwise.} \end{cases} \tag{3.27}$$

The fact that $\hat{\sigma}_i$ takes indeed $\Omega_{g,k}$ into itself is a consequence of the symmetries of the consistency equations (3.1), (3.2), (3.3), which are clearly invariant under the permutation of indices described in (3.27). Another symmetry $\zeta_i : \Omega_{g,k} \to \Omega_{g,k}$ exists which corresponds to interchanging the rôles of the tetrahedra Δ_{2i-1} and Δ_{2i}. This symmetry is defined by the following formulae:

$$(\hat{\sigma}_i(x))_j = \begin{cases} x_{12(i-1)+6+l} & \text{if } j = 12(i-1)+l,\ l = 1,\dots,6; \\ x_{12(i-1)+l} & \text{if } j = 12(i-1)+6+l,\ l = 1,\dots,6; \\ x_j & \text{otherwise.} \end{cases} \tag{3.28}$$

Also in this case the fact that $\zeta_i(\Omega_{g,k}) = \Omega_{g,k}$ easily follows from a straight-forward analysis of the consistency equations. We can now define a map

$$\varphi_i : \mathfrak{S}_3 \times \mathbb{Z}/_2 \to \mathrm{Aut}(\Omega_{g,k}),\ \varphi_i(\sigma,\epsilon) = \hat{\sigma}_i \circ \zeta_i^\epsilon, \quad \sigma \in \mathfrak{S}_3,\ \epsilon = 0,1. \tag{3.29}$$

Using that ζ_i commutes with $\hat{\sigma}_i$ for all $\sigma \in \mathfrak{S}_3$ it is easily seen that φ_i is an injective homomorphism with image a certain subgroup $\mathrm{Sym}_i(\Omega_{g,k})$ of $\mathrm{Aut}(\Omega_{g,k})$.

Let now κ be an element of the symmetric group $\mathfrak{S}_k$ on k elements. We define a symmetry $\widetilde{\kappa} \in \mathrm{Aut}(\Omega_{g,k})$ which induces the permutation of cusps corresponding to κ. Namely for any $x \in \Omega_{g,k}$ we set

$$(\hat{\kappa}(x))_j = x_{12(\kappa^{-1}(h)-1)+l} \qquad j = 12(h-1)+l,\ h = 1,\dots,k,\ l = 1,\dots,12. \tag{3.30}$$

It is easily seen that the map $\kappa \mapsto \hat{\kappa}$ defines an injective homomorphism $\nu : \mathfrak{S}_k \to \mathrm{Aut}(\Omega_{g,k})$.

Let us denote by $\mathrm{Sym}(\Omega_{g,k})$ the subgroup of $\mathrm{Aut}(\Omega_{g,k})$ generated by $\nu(\mathfrak{S}_k) \cup \left(\bigcup_{i=1}^k \mathrm{Sym}_i(\Omega_{g,k})\right)$. It is easily seen that elements in $\mathrm{Sym}_i(\Omega_{g,k})$ commute with elements in $\mathrm{Sym}_j(\Omega_{g,k})$ whenever $i \neq j$, thus the group generated by the $\mathrm{Sym}_i(\Omega_{g,k})$'s is actually isomorphic to the product $\prod_{i=1}^k \mathrm{Sym}_i(\Omega_{g,k})$. Moreover if $\pi : \mathrm{Sym}(\Omega_{g,k}) \to \mathfrak{S}_k$ is the natural homomorphism which maps each symmetry to the corresponding permutation of cusps we have $\mathrm{Ker}\ \pi = \prod_{i=1}^k \mathrm{Sym}_i(\Omega_{g,k})$ and $\pi \circ \nu = \mathrm{Id} : \mathfrak{S}_k \to \mathfrak{S}_k$. Thus

$$\mathrm{Sym}(\Omega_{g,k}) = \nu(\mathfrak{S}_k) \ltimes (\mathrm{Sym}_1(\Omega_{g,k}) \times \cdots \times \mathrm{Sym}_k(\Omega_{g,k})) \cong \mathfrak{S}_k \ltimes \left(\prod_{i=1}^k \mathfrak{S}_3 \times \mathbb{Z}/_2\right). \tag{3.31}$$

Our next task is to investigate how symmetries in $\mathrm{Sym}(\Omega_{g,k})$ act on the space of Dehn filling coefficients parameterizing a small neighbourhood of x_0 in $\Omega_{g,k}$.

Action on Dehn filling coefficients Recall that for $j = 1,\dots,k$ and $x \in \Omega_{g,k}$ the hyperbolic structure on M parameterized by x induces on the torus T_j a well-defined similarity structure, which will be denoted by $T_j(x)$. As described in Section 3.4, the structure $T_j(x_0)$ can by realized as the gluing of two isometric Euclidean equilateral triangles, and is therefore similar to $\mathbb{C}/\Gamma$, where $\mathbb{C}$ is endowed with the standard Euclidean metric, and $\Gamma = \langle \tau_1, \tau_2 \rangle$ is the discrete additive subgroup of $\mathbb{C}$ with generators $\tau_1 = 1, \tau_2 = -1/2 + i\sqrt{3}/2$. Once an isometric identification $T_j(x_0) \cong \mathbb{C}/\Gamma$ is fixed, we have a natural isomorphism between Γ and $H_1(T_j;\mathbb{Z}) \cong \pi_1(T_j)$. We denote by μ_j, λ_j the elements in $\pi_1(T_j)$ which correspond under this identification to τ_1, τ_2 respectively, and we choose them also as our preferred positively oriented $\mathbb{Z}$-basis of $H_1(T;\mathbb{Z})$. With notation as in Section 3.3, we denote by $\mathcal{M}(T_j)$ the group of isotopy classes of isometries of $T_j(x_0)$, and by $\mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ the group of isotopy classes of isometries of $T_1(x_0) \sqcup \ldots \sqcup T_k(x_0)$. For any $\sigma \in \mathfrak{S}_k$ we now define an element $\eta(\sigma) \in \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ which permutes the *marked* tori $T_1, \ldots, T_k$ according to σ. Namely, $\eta(\sigma)$ is the isotopy class of any element $\sigma' \in \mathrm{Isom}^+(T_1(x_0) \sqcup \ldots \sqcup T_k(x_0))$ with the following properties: $\sigma'(T_j) = T_{\sigma(j)}$, $\sigma'_*(\mu_j) = \mu_{\sigma(j)}$, $\sigma'_*(\lambda_j) = \lambda_{\sigma(j)}$ for $i = 1, \ldots, k$. It is easily seen that the map $\eta : \mathfrak{S}_k \to \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ is a well-defined injective homomorphism. Let now $\pi' : \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k) \to \mathfrak{S}_k$ be the natural projection which associates to any element in $\mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ the induced permutation of the T_j's. Of course we have $\pi' \circ \eta = \mathrm{Id} : \mathfrak{S}_k \to \mathfrak{S}_k$, and the kernel of π' is canonically isomorphic to $\mathcal{M}(T_1) \times \ldots \times \mathcal{M}(T_k)$. Therefore we have

$$\mathcal{M}(T_1 \sqcup \ldots \sqcup T_k) = \eta(\mathfrak{S}_k) \ltimes (\mathcal{M}(T_1) \times \ldots \times \mathcal{M}(T_k)) \cong \mathfrak{S}_k \ltimes \left(\prod_{i=1}^{k} D_6 \right), \quad (3.32)$$

where $D_6 \cong \mathfrak{S}_3 \times \mathbb{Z}/_2$ is the dihedral group with 12 elements.

Recall now that functions $u_j, v_j : \Omega_{g,k} \to \mathbb{C}$ were defined in Section 2.4 by formulae (2.10) with the property that $u_j(x)$ (resp. $v_j(x)$) is equal to the complex logarithm of the dilation coefficient of $\rho^j_x(\mu_j)$ (resp. $\rho^j_x(\lambda_j)$), where $\rho^j_x : \pi_1(T_j) \to \mathrm{Aff}(\mathbb{C})$ is a holonomy representation for the structure $T_j(x)$ on T_j. For any x in a sufficiently small neighbourhood U of x_0 in $\Omega_{g,k}$ we also denote by $(p_j(x), q_j(x))$ the Dehn filling coefficient for x at the end E_j, *i.e.* the unique point in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ with

$$\begin{array}{rl} (p_j(x), q_j(x)) = \infty & \text{if } u_j(x) = v_j(x) = 0, \\ p_j(x)u_j(x) + q_j(x)v_j(x) = 2\pi i & \text{otherwise.} \end{array}$$

We set

$$d = (d_1, \ldots, d_k) : U \to \prod_{i=1}^{k} S^2, \quad d_j(x) = (p_j(x), q_j(x)) \in S^2. \quad (3.33)$$

By Corollary 2.4.3 and Proposition 2.5.3 there exists a small neighbourhood $V \subset U$ of x_0 in $\Omega_{g,k}$ with the following properties:

- The map $u = (u_1, \ldots, u_k) : \Omega_{g,k} \to \mathbb{C}^k$ restricts to a diffeomorphism of V onto an open neighbourhood of 0 in $\mathbb{C}^k$;

- The restriction of d to V defines a diffeomorphism of V onto an open neighbourhood of $\{\infty\} \times \ldots \times \{\infty\}$ in $S^2 \times \ldots \times S^2$;

- $\psi(V) = V$ for any $\psi \in \mathrm{Sym}(\Omega_{g,k})$.

We now observe that $\mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ naturally acts on small neighbourhoods of $\{\infty\} \times \ldots \times \{\infty\}$ in $S^2 \times \ldots \times S^2$ in the following way. Any element $h \in \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ decomposes into isotopy classes of isometries $h_j : T_j \to T_{\pi(h)(j)}$, $i = 1, \ldots, k$, where $\pi' : \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k) \to \mathfrak{S}_k$ is the projection introduced above.

Thus we have isomorphisms $(h_j)_* : H_1(T_j;\mathbb{R}) \to H_1(T_{\pi(h)(j)};\mathbb{R})$ and we can define $h_* : S^2 \times \ldots \times S^2 \to S^2 \times \ldots \times S^2$ as follows:

$$h_*((p_1,q_1),\ldots,(p_k,q_k)) = ((p'_1,q'_1),\ldots,(p'_k,q'_k)),$$

where

$$\begin{array}{rll} p'_{\pi(h)(j)}\mu_{\pi(h)(j)} + q'_{\pi(h)(j)}\lambda_{\pi(h)(j)} = (h_j)_*(p_j\mu_j + q_j\lambda_j) & \text{if } (p_j,q_j) \neq \infty, \\ (p'_{\pi(h)(j)}, q'_{\pi(h)(j)}) = \infty & \text{if } (p_j,q_j) = \infty. \end{array}$$

Our next aim is to prove the following:

Proposition 3.6.1. *For any $\psi \in \mathrm{Sym}(\Omega_{g,k})$ there exists a unique $h(\psi) \in \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ such that $d(\psi(x)) = h(\psi)_*(d(x))$ for all $x \in V$. Moreover the map*

$$\mathrm{Sym}(\Omega_{g,k}) \to \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k), \quad \psi \mapsto h(\psi)$$

is a group isomorphism (which preserves decompositions (3.31), (3.32)). Thus the action of $\mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ on $S^2 \times \ldots \times S^2$ restricts to an action on $d(V)$.

PROOF: We need to describe as explicitly as possible how the maps $d_j : V \to S^2$, $i = 1,\ldots,k$ change under precompositions with elements in $\mathrm{Sym}(\Omega_{g,k})$. First of all we fix $x \in V$, $i \in \{1,\ldots,k\}$ and compute $u_j(x)$, $v_j(x)$. In order to clarify a bit our argument we come back to the notation described at the beginning of the section, *i.e* we set

$$\gamma^l_{2j-1} = x_{12(j-1)+3+l}, \ \gamma^l_{2j} = x_{12(j-1)+9+l}, \quad l = 1,2,3.$$

To compute $u_j(x)$ and $v_j(x)$ we look at the decomposition of $T_j(x)$ into Euclidean triangles defined by x and use formulae (2.10) (see Fig. 3.6 where the tessellation of T_1 is developed on $\mathbb{C}$). Also recalling that $\gamma^1_{2j-1}+\gamma^2_{2j-1}+\gamma^3_{2j-1} = \gamma^1_{2j}+\gamma^2_{2j}+\gamma^3_{2j} = \pi$ we obtain

$$\begin{array}{l} u_j(x) = \ln((\sin\gamma^1_{2j-1}\sin\gamma^2_{2j})/(\sin\gamma^2_{2j-1}\sin\gamma^1_{2j})) + i(\gamma^3_{2j-1} - \gamma^3_{2j}), \\ v_j(x) = \ln((\sin\gamma^2_{2j-1}\sin\gamma^3_{2j})/(\sin\gamma^3_{2j-1}\sin\gamma^2_{2j})) + i(\gamma^1_{2j-1} - \gamma^1_{2j}). \end{array}$$

We now consider the action of $\mathfrak{S}_3 \times \mathbb{Z}/_2$ on $\Omega_{g,k}$ via the representation $\varphi_j : \mathfrak{S}_3 \times \mathbb{Z}/_2 \to \mathrm{Sym}_j(\Omega_{g,k})$ defined in equation (3.29). By definition the element $(\sigma,0) \in \mathfrak{S}_3 \times \mathbb{Z}/_2$ acts on the γ^l_{2j-1}'s and the γ^l_{2j}'s just by applying σ to the apices, while the action of $(\mathrm{Id},1) \in \mathfrak{S}_3 \times \mathbb{Z}/_2$ interchanges indices $2j-1, 2j$. Let now $r = ((123),1)$, $s = ((12),0)$ be fixed elements of $\mathfrak{S}_3 \times \mathbb{Z}/_2$. The description of the action of $\varphi_j(r), \varphi_j(s)$ given above implies (after some computations) that

$$u_j(\varphi_j(r)(x)) = -v_j(x), \quad v_j(\varphi_j(r)(x)) = u_j(x) + v_j(x), \tag{3.34}$$
$$u_j(\varphi_j(s)(x)) = -\overline{u_j(x)}, \quad v_j(\varphi_j(s)(x)) = \overline{u_j(x)} + \overline{v_j(x)}. \tag{3.35}$$

We can now compute the action of $h(\varphi_j(r))$ and $h(\varphi_j(s))$ on Dehn filling coefficients. Using equations (3.34), (3.35) and the very definition of Dehn filling coefficients we get

$$\begin{array}{rcl} 2\pi i & = & p_j(\varphi_j(r)(x))u_j(\varphi_j(r)(x)) + q_j(\varphi_j(r)(x))v_j(\varphi_j(r)(x)) \\ & = & p_j(\varphi_j(r)(x))(-v_j(x)) + q_j(\varphi_j(r)(x))(u_j(x)+v_j(x)) \\ & = & q_j(\varphi_j(r)(x))u_j(x) + (q_j(\varphi_j(r)(x)) - p_j(\varphi_j(r)(x)))v_j(x), \end{array}$$

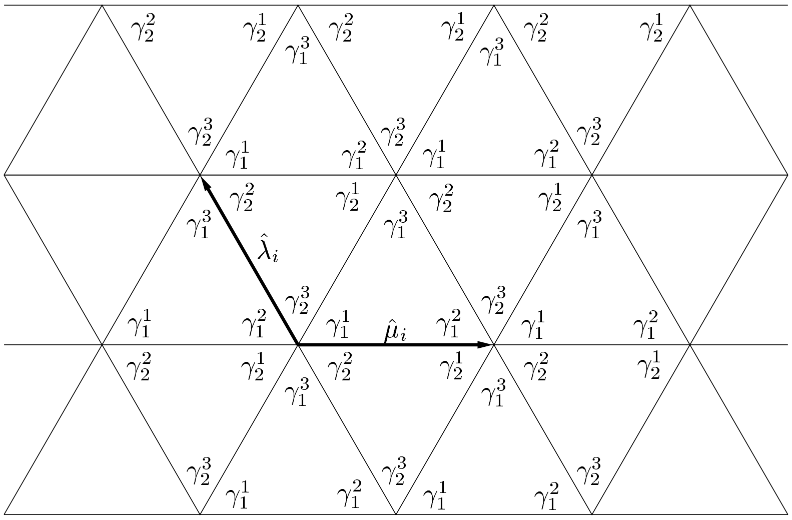

Figure 3.6: If $\hat{\mu}_j$ (resp. $\hat{\lambda}_j$) is the image of a lift of μ_j (resp. λ_j) under a developing map, then $-u_j(x)$ (resp. $-v_j(x)$) is the product of the moduli of triangles lying on the right side of of $\hat{\mu}_j$ (resp. $\hat{\lambda}_j$).

whence $p_j(x) = q_j(\varphi_j(r)(x))$, $q_j(x) = q_j(\varphi_j(r)(x)) - p_j(\varphi_j(r)(x))$ by uniqueness of the Dehn filling coefficients, and

$$p_j(\varphi_j(r)(x)) = p_j(x) - q_j(x), \quad q_j(\varphi_j(r)(x)) = p_j(x). \tag{3.36}$$

A similar computation also gives

$$p_j(\varphi_j(s)(x)) = p_j(x) - q_j(x), \quad q_j(\varphi_j(s)(x)) = -q_j(x). \tag{3.37}$$

This easily implies that $h(\varphi_j(r))$ and $h(\varphi_j(r))$ act on Dehn filling coefficients at the j-th end of M respectively as a rotation of angle $\pi/3$ and a reflection with respect to the line $\mathbb{R} \cdot \mu_j$ (see Proposition 3.3.2). This gives in turn that h restricts to an isomorphism

$$\mathrm{Sym}(\Omega_{g,k}) \supset \mathrm{Sym}_j(\Omega_{g,k}) \cong \mathcal{M}(T_j) \subset \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k).$$

Moreover, with notation as in formulae (3.31), (3.32), any permutation of cusps in $\nu(\mathfrak{S}_k) \subset \mathrm{Sym}(\Omega_{g,k})$ is taken by h into the corresponding permutation in $\eta(\mathfrak{S}_k) \subset \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$, and this concludes the proof. $\square$

Remark 3.6.2. Just as the space $\Omega_{g,k}$, also the groups $\mathrm{Sym}(\Omega_{g,k})$, $\mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$, the maps $u, v : \Omega_{g,k} \to \mathbb{C}^k$, $d : \Omega_{g,k} \to S^2 \times \ldots \times S^2$, the neighbourhood V of x_0 in $\Omega_{g,k}$ and the homomorphism $h : \mathrm{Sym}(\Omega_{g,k}) \to \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ depend only on g and k, and not on the fixed manifold $M \in \mathcal{M}_{g,k}$.

Return paths We are now interested in describing the shortest return paths (see Definition 2.6.9) in small deformations of the complete structure on M. Let us define three real numbers $\mu_{g,k}, \nu_{g,k}, \eta_{g,k}$ as follows: $\mu_{g,k}$ is the distance between a truncation triangle and the opposite lateral hexagon in a non-compact tetrahedron of the canonical decomposition of M; $\nu_{g,k}$ is the distance between a truncation triangle and the opposite lateral hexagon in a compact tetrahedron of the canonical decomposition of M; $\eta_{g,k}$ is the length of the compact edge of the canonical decomposition of M.

Lemma 3.6.3. *We have* $2\mu_{g,k} > \eta_{g,k}$, $2\nu_{g,k} > \eta_{g,k}$.

PROOF: By Proposition 2.7.14 and equations (2.3), (2.12) we have, after some computations:

$$\begin{aligned}\cosh \mu_{g,k} &= (3\cos\overline{\alpha}_{g,k})/\sqrt{(2\cos\overline{\alpha}_{g,k}-1)(2\cos\overline{\alpha}_{g,k}+1)},\\ \cosh \nu_{g,k} &= \sqrt{((\cos\overline{\beta}_{g,k}+1)(3\cos\overline{\beta}_{g,k}-1))/(2\cos\overline{\beta}_{g,k}-1)},\end{aligned}$$

whence

$$\begin{aligned}\cosh 2\mu_{g,k} &= (14\cos^2\overline{\alpha}_{g,k}+1)/((2\cos\overline{\alpha}_{g,k}-1)(2\cos\overline{\alpha}_{g,k}+1)),\\ \cosh 2\nu_{g,k} &= (6\cos^2\overline{\beta}_{g,k}+2\cos\overline{\beta}_{g,k}-1)/(2\cos\overline{\beta}_{g,k}-1).\end{aligned}$$

Moreover by Proposition 2.2.6 and equations (2.2), (2.3) we also obtain

$$\begin{aligned}\cosh \eta_{g,k} &= 2(\cos\overline{\alpha}_{g,k}+1)^2/(3(2\cos\overline{\alpha}_{g,k}-1)(2\cos\overline{\alpha}_{g,k}+1))\\ &= \cos\overline{\beta}_{g,k}/(2\cos\overline{\beta}_{g,k}-1).\end{aligned}$$

We now observe that since $\cos\overline{\beta}_{g,k} \geqslant 1/2 > 1/3$ we have

$$6\cos^2\overline{\beta}_{g,k}+2\cos\overline{\beta}_{g,k}-1 > \cos\overline{\beta}_{g,k},$$

whence $2\mu_{g,k} > \eta_{g,k}$. Moreover we always have

$$3(14\cos^2\overline{\alpha}_{g,k}+1) > 2(\cos\overline{\alpha}_{g,k}+1)^2,$$

whence $2\nu_{g,k} > \eta_{g,k}$. □

Let us set

$$\begin{aligned}I\Omega_{g,k} = \ &\{x \in \Omega_{g,k} :\ \text{for } i = 1,\ldots,k \text{ the } i-\text{th Dehn filling coefficient associated}\\ &\ \text{to } x \text{ is equal either to } \infty \text{ or to an indivisible element in } H_1(T_i;\mathbb{Z})\}.\end{aligned}$$

For any $x \in \Omega_{g,k}$ we denote by $M(x)$ the hyperbolic structure induced on M by x, and by $\hat{M}(x)$ the metric completion of $M(x)$. If $x \in I\Omega_{g,k}$ then $\hat{M}(x)$ is a complete finite-volume hyperbolic manifold with geodesic boundary. In this case the unique compact edge in the geometric triangulation of $M(x)$ defines a return path l_x in $\hat{M}(x)$. Of course we have

$$\lim_{y \in I\Omega_{g,k},\, y \to x_0} L^y(l_y) = L^{x_0}(l_{x_0}) = \eta_{g,k},$$

where L^y denotes the length with respect to the hyperbolic metric on $\hat{M}(y)$ for any $y \in I\Omega_{g,k}$. The following lemma can be thought as an extension of Proposition 2.6.10 to manifolds obtained by Dehn filling M along sufficiently long slopes.

Lemma 3.6.4. *There exist $\delta > 0$ and a neighbourhood $V' \subset V$ of x_0 in $\Omega_{g,k}$ such that for every $x \in I\Omega_{g,k} \cap V'$ the following holds: if l is a return path in $\hat{M}(x)$ having length less than $L^{x_0}(l_{x_0}) + \delta = \eta_{g,k} + \delta$, then $l = l_x$.*

PROOF: We suppose by contradiction that a sequence $\{y_n\}_{n\in\mathbb{N}} \subset I\Omega_{g,k}$ exists with the following properties:

- $\lim_{n\to\infty} y_n = x_0$;
- A return path l'_n in $\hat{M}(y_n)$ exists such that $l'_n \neq l_{y_n}$ for any $n \in \mathbb{N}$ and $\limsup_{n\to\infty} L^{y_n}(l'_n) \leqslant \eta_{g,k}$.

Since the distance between the added geodesics $\hat{M}(y)\setminus M(y)$ and the geodesic boundary of $M(y)$ approaches ∞ as y tends to x_0, we can suppose $l'_n \subset M(y_n)$ for any $n \in \mathbb{N}$. Let now p_n, q_n be the endpoints of l'_n and $\Delta^{p_n}, \Delta^{q_n}$ the geometric tetrahedra containing p_n, q_n respectively. We denote by l'_{p_n} (resp. l'_{q_n}) the connected component of $l'_n \cap \Delta^{p_n}$ (resp. $l'_n \cap \Delta^{q_n}$) containing p_n (resp. q_n). Since $l'_n \neq l_{y_n}$ and l'_n intersect $\partial M(y_n)$ at right angles, by Lemma 3.6.3 we easily get

$$\begin{aligned} \limsup_{n\to\infty} L^{y_n}(l'_n) \geqslant\ & \liminf_{n\to\infty} L^{y_n}(l'_n) \geqslant \liminf_{n\to\infty} L^{y_n}(l'_{p_n}) + \liminf_{n\to\infty} L^{y_n}(l'_{q_n}) \\ \geqslant\ & 2\min\{\mu_{g,k}, \nu_{g,k}\} > \eta_{g,k}, \end{aligned}$$

a contradiction. □

Let V' be a neighbourhood of x_0 in $\Omega_{g,k}$ as in the statement of Lemma 3.6.4.

Corollary 3.6.5. *For every $x \in I\Omega_{g,k} \cap V'$ the arc l_x is the unique shortest return path in $\hat{M}(x)$.*

For any $x \in I\Omega_{g,k} \cap V$ we denote by $U(x)$ the universal covering of the hyperbolic manifold $\hat{M}(x)$. We have the following easy:

Corollary 3.6.6. *The minimal distance between distinct connected components of $\partial U(x)$ is equal to $L^x(l_x)$ for all $x \in I\Omega_{g,k} \cap V'$. Moreover, if S_1, S_2 are distinct connected components of $\partial U(x)$, then the distance between S_1 and S_2 equals $L^x(l_x)$ if and only if the shortest path joining S_1 with S_2 projects onto $l_x \subset \hat{M}(x)$.*

The following proposition relates the intrinsic geometry of $U(x)$ to properties of the geometric triangulation of $M(x)$, when $x \in I\Omega_{g,k}$.

Proposition 3.6.7. *There exists a neighbourhood $V'' \subset V'$ of x_0 in $\Omega_{g,k}$ with the following property. Let $x \in I\Omega_{g,k} \cap V''$ and $S_1, \ldots, S_4$ be pairwise distinct connected components of $\partial U(x)$. Then the distance between S_i and S_j equals $L^x(l_x)$ for all $i \neq j$ if and only if there exists a lift of a compact tetrahedron in the geometric triangulation of $M(x) \subset \hat{M}(x)$ whose truncation triangles lie on $S_1, \ldots, S_4$.*

PROOF: We concentrate our attention on the "only if" part of the statement, the "if" part being obvious.

Let $x \in I\Omega_{g,k} \cap V'$. Since $M(x), M(x_0)$ are both homeomorphic to M, the local isometry $i_x : \widetilde{M(x)} \to \widetilde{\hat{M}(x)} = U(x)$ induced by the inclusion $M(x) \hookrightarrow \hat{M}(x)$ defines a (neither injective nor locally isometric) immersion $p_x : U(x_0) = \widetilde{M(x_0)} \to U(x)$ with $p_x(\partial U(x_0)) = \partial U(x)$, $p_x^{-1}(\partial U(x)) = \partial U(x_0)$. (Note that p_x is not uniquely defined, since it depends on the identification of $U(x_0)$ with the Riemannian universal covering of $M(x)$). Let δ be as in the statement of Lemma 3.6.4. Then there exist $\varepsilon > 0$ and a small neighbourhood $V'' \subset V'$ of x_0 in $\Omega_{g,k}$ such that for every $y \in I\Omega_{g,k} \cap V''$ we have

$$(1+\varepsilon)L^y(l_y) < L^{x_0}(l_{x_0}) + \delta.$$

Recall that the distance between the added geodesics $\hat{M}(y) \setminus M(y)$ and the geodesic boundary of $M(y)$ approaches ∞ as y tends to x_0, so, up to resizing V'', for every $y \in I\Omega_{g,k} \cap V''$ a compact set $K_y \subset M(y)$ exists with the following properties:

1. K_y contains all the points of $\hat{M}(y)$ whose distance from $\partial\hat{M}(y)$ is smaller than the diameter of the regular truncated tetrahedron with edge-length equal to $L^y(l_y)$;
2. If $\widetilde{K}_y$ is the preimage of K_y under the covering map $U(y) \to \hat{M}(y)$, then we can choose the map $p_y : U(x_0) \to U(y)$ in such a way that $p_y|_{p_y^{-1}(\widetilde{K}_y)} : p_y^{-1}(\widetilde{K}_y) \to \widetilde{K}_y$ is locally $(1+\varepsilon)$-biLipschitz.

Let now $x \in I\Omega_{g,k} \cap V''$ and $S_1, \ldots, S_4$ be pairwise distinct connected components of $\partial U(x)$ such that the distance between S_i and S_j equals $L^x(l_x)$ for all $i \neq j$. Let Δ be a topological partially truncated tetrahedron with truncation triangles B_1, B_2, B_3, B_4 and internal edges e_{ij} joining B_i with B_j, $i,j \in \{1,2,3,4\}$, $i \neq j$. We consider a geometric realization $r : \Delta \hookrightarrow U(x) \subset \mathbb{H}^3$ with $r(B_i) \subset S_i$, $i = 1,2,3,4$, and we observe that $r(e_{ij})$ is the shortest geodesic arc joining S_i with S_j. Since $r(\Delta) \subset K_x \subset M(x)$, it is easily seen that r lifts to a map $r' : \Delta \to U(x_0)$ with $p_x \circ r' = r$. (Notice that since p_x is not a local isometry, the map r' does not necessarily define a geometric realization of Δ.) For $i = 1,2,3,4$ let S'_i be the component of $\partial U(x_0)$ containing $r'(B_i)$, so that $r'(e_{ij})$ is a (not necessarily geodesic) arc joining S'_i with S'_j for $i,j = 1,2,3,4$, $i \neq j$. By Corollary 3.6.6 we know that $r(e_{ij})$ projects onto the return path $l_x \subset \hat{M}(x)$, and this easily implies that the arc $r'|_{e_{ij}} : e_{ij} \to U(x_0)$ is not null-homotopic relatively to $\partial M(x_0)$. Thus $S'_i \neq S'_j$ for $i \neq j$. Moreover by property (2) above we have $L^{x_0}(r'(e_{ij})) \leqslant (1+\varepsilon)L^x(l_x)$, whence

$$d_{x_0}(S'_i, S'_j) \leqslant (1+\varepsilon)L^x(l_x) \leqslant L^{x_0}(l_{x_0}) + \delta,\ i,j \in \{1,2,3,4\},\ i \neq j,$$

where we denote by d_{x_0} the hyperbolic metric on $U(x_0)$. Thus by Lemma 3.6.4 and Proposition 2.6.11 the hyperbolic planes S'_1, S'_2, S'_3, S'_4 bound a compact geodesic regular partially truncated tetrahedron which projects onto a piece of the Kojima decomposition of $M(x_0)$. This easily implies that $r(\Delta)$ projects onto a compact partially truncated tetrahedron in the geometric triangulation of $M(x)$, whence the conclusion. □

3.7 Similar fillings

Before stating our results about fillings which yield geometrically similar manifolds some definitions are needed.

Definition 3.7.1. Let N be a complete finite-volume hyperbolic 3-manifold with geodesic boundary and k cusps. The *cusp shape* of N is the set of Euclidean structures (up to a scale factor) induced on the boundary tori of N. A *regular horocusp neighbourhood* for N is a set $O_1 \sqcup \ldots \sqcup O_k \subset N$, where O_i is an open embedded horospherical neighbourhood of the i-th cusp of N, $O_i \cap O_j = \emptyset$ for $i \neq j$ and $\mathrm{vol}(O_i) = \mathrm{vol}(O_j)$ for $i, j \in \{i, \ldots, k\}$. The *cusp volume of* N is the volume of a maximal regular horocusp neighbourhood for N (where this volume is intended to be 0 if N is compact).

Definition 3.7.2. Let N be a complete finite-volume hyperbolic 3-manifold with geodesic boundary and cusps $C_1, \ldots, C_k$. A *set of slopes* $\mathcal{S}$ for N is a set $\mathcal{S} = \{s_{i_1}, \ldots, s_{i_h}\}$ of either 0 or 1 slope per cusp. If $\mathcal{S} = \{s_{i_1}, \ldots, s_{i_h}\}$ is a set of slopes for N we denote by $N(\mathcal{S})$ the manifold obtained by filling N along $s_{i_1}, \ldots, s_{i_h}$.

The following definition is taken (with minor modifications) from [HMW92].

Definition 3.7.3. Let N, N' be complete finite-volume hyperbolic 3-manifolds with geodesic boundary and the same number of cusps. We say that N and N' are *geometrically similar* if the following conditions hold:

- N and N' have the same volume;
- $H_1(N; \mathbb{Z}) \cong H_1(N'; \mathbb{Z})$;
- N and N' have the same cusp volume;
- N and N' have the same cusp shape;
- The shortest return paths of N and N' have the same length;
- The shortest closed geodesics of N and N' have the same complex length;
- N and N' have the same Turaev-Viro invariants;
- Manifolds obtained by sufficiently complicated Dehn fillings on N can be paired to manifolds obtained by sufficiently complicated Dehn fillings on N' in such a way that the elements in each pair share the same volume, first homology group, cusp volume, cusp shape, length of the shortest return path, complex length of the shortest geodesic and Turaev-Viro invariants.

Let now M and M' be elements in $\mathcal{M}_{g,k}$ (we do not exclude the case $M = M'$) with boundary tori $T_1, \ldots, T_k$ and $T'_1, \ldots, T'_k$. We endow each of these tori with the Euclidean metric defined on them by the hyperbolic structures on M, M' together with the requirement that $\mathrm{Area}(T_i) = \mathrm{Area}(T'_i) = \sqrt{3}/2$ for $i = 1, \ldots, k$. We say that a set of slopes $\mathcal{S}$ for M is *equivalent* to the set of slopes $\mathcal{S}'$ for M' if there exists an orientation-preserving isometry $\psi : T_1 \sqcup \ldots \sqcup T_k \to T'_1 \sqcup \ldots \sqcup T'_k$ taking $\mathcal{S}$ onto $\mathcal{S}'$. Of course if $\mathcal{S}$ is equivalent to $\mathcal{S}'$ then the lengths of the slopes in $\mathcal{S}$ are equal to the lengths of the slopes in $\mathcal{S}'$. We recall however that the converse is not true (see Remark 3.3.6).

Theorem 3.7.4. *Suppose $g > k$, let M, M' be elements of $\mathcal{M}_{g,k}$ and $\mathcal{S}$ (resp. $\mathcal{S}'$) be a set of slopes for M (resp. M'). Then there exists a positive constant C depending only on M and M' such that the following holds: if all the slopes of $\mathcal{S}$ are longer than C and $\mathcal{S}$ is equivalent to $\mathcal{S}'$, then $M(\mathcal{S})$ is geometrically similar to $M'(\mathcal{S}')$.*

Proof: Let V' be a neighbourhood of x_0 in $\Omega_{g,k}$ as in the statement of Lemma 3.6.4 and $d : V' \to S^2 \times \ldots \times S^2$ the map defined in equation (3.33). We can choose a positive constant C depending only on g and k such that the following holds: if $\mathcal{S} = \{s_{i_1}, \ldots, s_{i_h}\}$ is a set of slopes for M with $L(s_{i_l}) > C$ for $l = 1, \ldots, h$, then any k-uple of Dehn filling coefficients corresponding to $\mathcal{S}$ lie in $d(V')$ (due to the choice of signs, there exist exactly 2^h such k-uples).

Let now $\mathcal{S}$ be a set of slopes for M whose elements are longer than C and let $\mathcal{S}'$ be a set of slopes for M' which is equivalent to $\mathcal{S}$. Choose also points $x, x' \in V' \subset \Omega_{g,k}$ such that $d(x)$ (resp. $d(x')$) gives a k-uple of Dehn filling coefficients corresponding to $\mathcal{S}$ (resp. $\mathcal{S}'$). By Proposition 3.6.1 it follows that a symmetry $\varphi \in \mathrm{Sym}(\Omega_{g,k})$ exists with $\varphi(x) = x'$. Let $M(x)$ (resp. $M'(x')$) be the hyperbolic structure defined by x on M (resp. by x' on M'). Recall that $M(\mathcal{S})$ (resp. $M'(\mathcal{S}')$) is isometric to the metric completion of $M(x)$ (resp. of $M'(x')$), and that $M(\mathcal{S}) \setminus M(x)$ (resp. $M'(\mathcal{S}') \setminus M'(x')$) is the union of h disjoint geodesics in $\mathcal{M}(\mathcal{S})$ (resp. in $\mathcal{M}(\mathcal{S}')$). Notice that since x is $\mathrm{Sym}(\Omega_{g,k})$-equivalent to x', the geometric partially truncated tetrahedra in the decomposition of $M(x)$ are isometric to the geometric tetrahedra in the decomposition of $M'(x')$. Together with Corollary 3.6.5, this readily implies that the shortest return paths of $M(\mathcal{S})$ and $M(\mathcal{S}')$ have the same length. Moreover, $M(x)$ and $M'(x')$ have the same volume, whence $\mathrm{volume}(M(\mathcal{S})) = \mathrm{volume}(M'(\mathcal{S}'))$.

The fact that $H_1(M(\mathcal{S}); \mathbb{Z})$ is isomorphic to $H_1(M'(\mathcal{S}'); \mathbb{Z})$ is an immediate consequence of Proposition 3.3.8.

By Theorem 3.4.3 the bases of the cusps of $M(\mathcal{S})$ and $M'(\mathcal{S}')$ are all isometric to regular hexagonal tori, whence $M(\mathcal{S})$ and $M'(\mathcal{S}')$ share the same cusp shape.

Consider now the shape of the geometric tetrahedra in the triangulations $\mathcal{T} = \{\Delta_1, \ldots, \Delta_{g+k}\}$, $\mathcal{T}' = \{\Delta'_1, \ldots, \Delta'_{g+k}\}$ of M, M' respectively. Without loss of generality we can order the tetrahedra of these triangulations in such a way that Δ_l, Δ'_l are asymptotic to the cusps of $M(\mathcal{S}), M'(\mathcal{S}')$ for $l = 2h+1, \ldots, 2k$ (this is equivalent to requiring that the slopes in $\mathcal{S}$ and $\mathcal{S}'$ lie on tori $T_1, \ldots, T_h$ and $T'_1, \ldots, T'_h$). Then by Proposition 3.4.5 a real number $\vartheta(x) = \vartheta(x') \in (0, \pi/3)$ exists such that Δ_l and Δ'_l are isometric to $\Delta^{\vartheta(x)}$ for $l = 2h+1, \ldots, 2k$. For $l = 2h+1, \ldots, 2k$ let now v_l, v'_l be the ideal vertices of Δ_l, Δ'_l respectively. Due to the symmetric shape of $\Delta^{\vartheta(x)}$, a unique horocusp neighbourhood H_l of v_l in Δ_l exists which is tangent to the truncation triangles of Δ_l. Moreover H_{2i-1} and H_{2i} glue up in $M(\mathcal{S})$ giving a horocusp neighbourhood O_i of the i-th cusp for $i = h+1, \ldots, k$. Also notice that the total horocusp neighbourhood $O_{h+1} \sqcup \ldots \sqcup O_k$ is regular (since the H_l's are isometric to each other) and maximal (since each O_i is tangent to the boundary of $M(\mathcal{S})$). The very same construction also leads to a horocusp neighbourhood $O'_{h+1} \sqcup \ldots \sqcup O'_k$ for $M'(\mathcal{S}')$. Since Δ_l is isometric to Δ'_l for $l = 2h+1, \ldots, 2k$ we have $\mathrm{vol}(O_{h+1} \sqcup \ldots \sqcup O_k) = \mathrm{vol}(O'_{h+1} \sqcup \ldots \sqcup O'_k)$, whence $M(\mathcal{S})$ and $M'(\mathcal{S}')$ share the same cusp volume.

Recall now that Thurston's hyperbolic Dehn filling Theorem ensures that if for all $l = 1, \ldots, h$ we have $L(s_l) > C' > 0$ for some sufficiently large C', then the shortest geodesics of $M(\mathcal{S})$ and $M'(\mathcal{S}')$ are exactly the geodesics added to $M(x)$ and $M'(x')$. Thus under the hypothesis that $L(s_l) > C'$ for all $l = 1, \ldots, h$, in order to prove that the shortest geodesics of $M(\mathcal{S})$ and $M'(\mathcal{S}')$ have the same complex length we only have to compute the complex length of these added geodesics. Up to reordering $T_1, \ldots, T_h$ and $T'_1, \ldots, T'_h$ we can assume that the orientation preserving isometry $\psi : T_1 \sqcup \ldots \sqcup T_k \to T'_1 \sqcup \ldots \sqcup T'_k$ such that $\psi(\mathcal{S}) = \mathcal{S}'$ takes T_i onto T'_i for $i = 1, \ldots, h$. We denote by ℓ_i, ℓ'_i the added geodesics corresponding to T_i, T'_i respectively, and by $\mathbb{C}L^x(\ell_i), \mathbb{C}L^{x'}(\ell'_i)$ the complex lengths (with respect to the complete structure on $M(\mathcal{S}), M'(\mathcal{S}')$) of ℓ_i, ℓ'_i. Recall that on $H_1(T_i;\mathbb{Z})$, $H_1(T'_i;\mathbb{Z})$ we have fixed positively oriented bases (μ_i, λ_i), (μ'_i, λ'_i) which provide isomorphisms $H_1(T_i;\mathbb{C}) \cong \mathbb{C}^2$, $H_1(T'_i;\mathbb{C}) \cong \mathbb{C}^2$. Let $(p_i(x), q_i(x))$ (resp. $(p'_i(x'), q'_i(x'))$) be the coefficients of a representative for s_i (resp. s'_i) in $H_1(T_i;\mathbb{Z})$ (resp. in $H_1(T'_i;\mathbb{Z})$) with respect to these bases and take integers $r_i(x), s_i(x), r'_i(x'), s'_i(x')$ with

$$p_i(x)s_i(x) - q_i(x)r_i(x) = p'_i(x')s'_i(x') - q'_i(x')r'_i(x') = -1.$$

Let $R_i \in \mathcal{M}(T'_i)$ be the isotopy class of isometries corresponding to the rotation of angle $\pi/3$ according to the isomorphism $\mathcal{M}(T'_i) \cong D_6$.

If f_i is the only isotopy class of isometries between T_i and T'_i with $(f_i)_*(\mu_i) = \mu'_i$, $(f_i)_*(\lambda_i) = \lambda'_i$, then since ψ and f_i are orientation-preserving we have $\psi_i := \psi|_{T_i} = R_i^n \circ f_i$ for some $n \in \{0, \ldots, 5\}$. Let us consider the natural action $(R_i^n)_* : \mathbb{C}^2 \cong H_1(T'_i;\mathbb{C}) \to H_1(T'_i;\mathbb{C}) \cong \mathbb{C}^2$. By the very definitions we have $(R_i^n)_*(p_i(x), q_i(x)) = (p'_i(x'), q'_i(x'))$. Moreover, since $\det (R_i^n)_* = 1$ we can choose $r'_i(x'), s'_i(x')$ in such a way that $(R_i^n)_*(r_i(x), s_i(x)) = (r'_i(x'), s'_i(x'))$. Let now $u_i, v_i, u'_i, v'_i : V' \to \mathbb{C}$ be the Neumann-Zagier parameters corresponding to the bases (μ_i, λ_i), (μ'_i, λ'_i) respectively. By equations (3.34), (3.36) we easily deduce that

$$\begin{pmatrix} u'_i(x') \\ v'_i(x') \end{pmatrix} = {}^t(R_i^{-n})_* \begin{pmatrix} u_i(x) \\ v_i(x) \end{pmatrix},$$

whence

$$\mathbb{C}L^{x'}(\ell'_i) = \left\langle \begin{pmatrix} u'_i(x') \\ v'_i(x') \end{pmatrix}, \begin{pmatrix} r'_i(x') \\ s'_i(x') \end{pmatrix} \right\rangle = \left\langle {}^t(R_i^{-n})_* \begin{pmatrix} u_i(x) \\ v_i(x) \end{pmatrix}, (R_i^n)_* \begin{pmatrix} r_i(x) \\ s_i(x) \end{pmatrix} \right\rangle$$
$$= \left\langle \begin{pmatrix} u_i(x) \\ v_i(x) \end{pmatrix}, \begin{pmatrix} r_i(x) \\ s_i(x) \end{pmatrix} \right\rangle = \mathbb{C}L^x(\ell_i).$$

In order to prove our statement about Turaev-Viro invariants we need to construct special spines for $M(\mathcal{S})$ and $M'(\mathcal{S}')$. Let $P \subset M$ be the special spine of M dual to the canonical decomposition $\mathcal{T}$ of M, and recall that for $j = 1, \ldots, k$ a hexagonal face E_j of P exists which is parallel to the j-th boundary torus of M. Let $E_1, \ldots, E_h$ be the faces corresponding to the filled tori in $M(\mathcal{S})$ and for $l = 1, \ldots, h$ let m_l be a loop on E_l which represents the slope $s_l \in \mathcal{S}$ and is in general position with respect to the singular locus $S(P)$ of P. The complement of $P \subset M \subset M(\mathcal{S})$ inside $M(\mathcal{S})$ consists of the disjoint union of an open collar of $\partial M(\mathcal{S})$ and h open solid tori. Take meridinal discs $D_1, \ldots, D_h$ of these solid tori with $\partial D_l = m_l$ for $l = 1, \ldots, h$. The complement of $P \cup D_1 \cup \ldots \cup D_h$ is as above, with h open balls instead of the h open solid tori. Fix now $l \in \{1, \ldots, h\}$. The loop $\partial D_l \subset E_l$ cuts E_l into several discal open faces of $P \cup D_1 \cup \ldots \cup D_h$ (these faces are indeed homeomorphic to open discs because the loop m_l is sufficiently complicated with respect to the graph $S(P) \cap \overline{E}_l)$. Each such face separates $\Sigma_g \subset \partial M(\mathcal{S})$ from the open ball corresponding to the l-th added solid torus, so, if we remove from $P \cup D_1 \cup \ldots \cup D_h$ a face for each added solid torus we end up with a special spine $P(\mathcal{S})$ of $M(\mathcal{S})$. The very same procedure also provides a special spine $P'(\mathcal{S}')$ for $M'(\mathcal{S}')$. Let $\mathcal{T}(\mathcal{S}), \mathcal{T}'(\mathcal{S}')$ be the triangulations of $M(\mathcal{S}), M'(\mathcal{S}')$ dual to $P(\mathcal{S}), P'(\mathcal{S}')$ respectively. It is not difficult to show that since $\mathcal{S}$ is equivalent to $\mathcal{S}'$ the loops representing slopes in $\mathcal{S}$ and in $\mathcal{S}'$ can be chosen so that the incidence numbers between edges and tetrahedra are the same for $\mathcal{T}(\mathcal{S})$ and for $\mathcal{T}'(\mathcal{S}')$. This implies that $M(\mathcal{S})$ and $M'(\mathcal{S}')$ share the same Turaev-Viro invariants.

Recall that $\psi : T_1 \sqcup \ldots \sqcup T_k \to T'_1 \sqcup \ldots \sqcup T'_k$ is the orientation-preserving isometry taking $\mathcal{S}$ onto $\mathcal{S}'$. Let also $\mathcal{S}_*$ be a set of sufficiently complicated slopes for $M(\mathcal{S})$. Of course we can think of $\mathcal{S}_*$ as of a set of slopes for M too. Let $\mathcal{S}'_*$ be the set of slopes for M' obtained by applying ψ to $\mathcal{S}_*$. Of course $\mathcal{S}'_*$ is a set of slopes for $M'(\mathcal{S}')$, and the manifolds $M(\mathcal{S})(\mathcal{S}_*) \cong M(\mathcal{S} \cup \mathcal{S}_*)$ and $M'(\mathcal{S}')(\mathcal{S}'_*) \cong M'(\mathcal{S}' \cup \mathcal{S}'_*)$ share the same volume, homology, cusp volume, cusp shape, length of the shortest return path, complex length of the shortest geodesic and Turaev-Viro invariants. $\square$

Remark 3.7.5. Let M, M' be elements of $\mathcal{M}_{g,k}$ and suppose that s, s' are sufficiently long slopes on the tori $T \subset \partial M, T' \subset \partial M'$. Let us give to T, T' the usual Euclidean structures induced by the hyperbolic structures on M, M' and suppose that there exists an *orientation-reversing* isometry of T onto T' taking s into s'. Then the complex length of the added geodesic in $M(s)$ is equal to the conjugate of the complex length of the added geodesic in $M'(s')$.

Non-homeomorphic fillings This paragraph is devoted to the proof of the following:

Theorem 3.7.6. *For any $k > 0$ there exists an orientable complete finite-volume hyperbolic 3-manifold Y_k with geodesic boundary and with k toric cusps having the following property. A positive constant C exists such that if $\mathcal{S} = \{s_{j_1}, \ldots, s_{j_h}\}$ is a set of slopes for Y_k with $L(s_{j_l}) > C$ for $l = 1, \ldots, h$, then $Y_k(\mathcal{S})$ is hyperbolic and at least $(k! \cdot 3^h)/(h! \cdot (k-h)!)$ pairwise non-homeomorphic hyperbolic Dehn fillings of Y_k are geometrically similar to $Y_k(\mathcal{S})$.*

In order to prove Theorem 3.7.6 we concentrate our attention on a specific manifold $X_k \in \mathcal{M}_{k+1,k}$ which is defined as follows. Let P_k be the special polyhedron whose 1-skeleton has the regular neighbourhood described in Fig. 3.7. It is easily seen that P_k is the spine of a manifold X_k. Computing the boundary of X_k as explained in [BP95] one can easily prove that X_k belongs to $\mathcal{M}_{k+1,k}$. Recall that P_k is the unique special spine of X_k with $k+1$ vertices, so any topological property of P_k readily translates into an intrinsic property of X_k itself. In particular we can fix an ordering on the cusps $C_1, \ldots, C_k$ of X_k which is well-defined up to exchanging C_i with C_{k-i} for all $i = 1, \ldots, k$ (see Fig. 3.7).

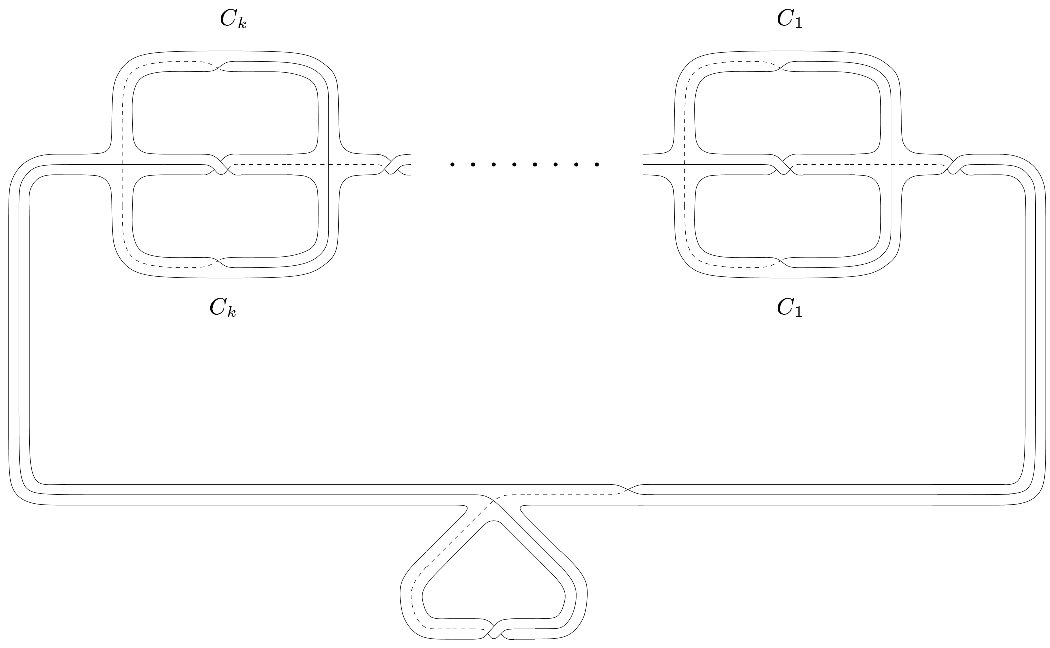

Figure 3.7: The regular neighbourhood of the 1-skeleton $S(P_k)$ of P_k. Each pair of vertices joined by three edges in $S(P_k)$ gives rise to a toric cusp in X_k.

Proposition 3.7.7. *If k is odd, then X_k admits no non-trivial isometries. If k is even, then X_k admits exactly one non-trivial isometry ψ_k. Moreover ψ_k is orientation-preserving and $\psi_k(C_i) = C_{k-i}$ for $i = 1, \ldots, k$, where $C_1, \ldots, C_k$ is the ordering on the cusps of X_k described in Fig. 3.7.*

Proof: By Theorems 2.6.8 and 3.2.1-(1) we only have to compute the group $\mathrm{Aut}(\mathcal{T}_k)$ of the combinatorial automorphisms of the triangulation $\mathcal{T}_k$ dual to P_k. So let Δ be the unique tetrahedron of $\mathcal{T}_k$ without ideal vertices and $v_1, \ldots, v_4$ be the vertices of Δ. We denote by F_{ijk} the face of Δ with vertices v_i, v_j, v_k. Without loss of generality we can suppose that the unique pairing between faces of Δ is given by the map $\varphi : F_{234} \to F_{132}$ with $\varphi(v_2) = v_1$, $\varphi(v_3) = v_2$, $\varphi(v_4) = v_3$. Let now ψ be an element in $\mathrm{Aut}(\mathcal{T}_k)$. Of course ψ leaves Δ invariant, and we have $\psi(\{v_1, v_4\}) = \{v_1, v_4\}$, $\psi(\{v_2, v_3\}) = \{v_2, v_3\}$. Moreover, if $\psi(v_1) = v_1$ we should have $(\varphi \circ \psi)|_{\{v_2,v_3,v_4\}} = (\psi \circ \varphi)|_{\{v_2,v_3,v_4\}}$, while if $\psi(v_1) = v_4$ we should have $(\varphi^{-1} \circ \psi)|_{\{v_2,v_3,v_4\}} = (\psi \circ \varphi)|_{\{v_2,v_3,v_4\}}$. This readily gives that ψ can act non-trivially on Δ only if $\psi(v_1) = v_4$, $\psi(v_2) = v_3$, $\psi(v_3) = v_2$, $\psi(v_4) = v_1$. An easy analysis of the combinatorics of $\mathcal{T}_k$ shows that the permutation of $\{v_1, v_2, v_3, v_4\}$ just described extends to an automorphism of $\mathcal{T}_k$ if and only if k is even, whence the conclusion. $\square$

For any $k \geqslant 2$ let us choose a slope s_k on the boundary torus T_k of X_k such that the manifold Y_{k-1} obtained by Dehn filling X_k along s_k is hyperbolic, and the core of the added solid torus gives the shortest geodesic of Y_{k-1} (these conditions hold as long as s_k is complicated enough).

Proposition 3.7.8. *For every $k \geqslant 1$ we have* $\mathrm{Isom}(Y_k) = \{1\}$.

Proof: Let ψ be a non-trivial element in $\mathrm{Isom}(Y_k)$. Of course ψ leaves the shortest geodesic of Y_k invariant, so it induces a homeomorphism $\psi' : X_{k+1} \to X_{k+1}$ leaving invariant the $(k+1)$-th cusp of X_{k+1}. Since ψ is not homotopic to the identity of Y_k it is easily seen that ψ' is not homotopic to the identity of X_{k+1}. By Mostow-Prasad's rigidity Theorem this gives in turn that ψ' is homotopic to a non-trivial isometry of X_{k+1} which leaves the $(k+1)$-th cusp of X_{k+1} invariant, against Proposition 3.7.7. $\square$

Let now $\mathcal{S} = \{s_{j_1}, \ldots, s_{j_h}\}$, $\mathcal{S}' = \{s'_{j_1}, \ldots, s'_{j_h}\}$ be sets of slopes for Y_k. We observe that the natural inclusion $\partial Y_k \subset \partial X_k$ allows us to regard $\mathcal{S}, \mathcal{S}'$ also as sets of slopes for X_{k+1}, and we say that $\mathcal{S}$ is *equivalent* to $\mathcal{S}'$ if $\mathcal{S}$ and $\mathcal{S}'$ are equivalent as sets of slopes for X_{k+1}. Thurston's hyperbolic Dehn filling Theorem and Theorem 3.7.4 imply that for each $i = 1, \ldots, k$ we can choose a finite set $\mathcal{S}_i$ of slopes on T_i with the following property. Fix $h \leqslant k$ and let $\mathcal{S} = \{s_{j_1}, \ldots, s_{j_h}\}$ be a set of slopes for Y_k, where s_{j_l} is a slope on T_{j_l} with $s_{j_l} \notin \mathcal{S}_{j_l}$, $l = 1, \ldots, h$. Let also $\mathcal{S}' = \{s'_{j'_1}, \ldots, s'_{j'_h}\}$ be a set of slopes equivalent to $\mathcal{S}$, where $s'_{j'_l}$ is a slope on $T_{j'_l}$, $l = 1, \ldots, h$. Then:

1. $s'_{j_l} \notin \mathcal{S}_{j'_l}$ for $l = 1, \ldots, h$;
2. $Y_k(\mathcal{S})$, $Y_k(\mathcal{S}')$ are geometrically similar hyperbolic 3-manifolds;
3. The cores of the solid tori added to Y_k give the h shortest geodesics of $Y_k(\mathcal{S})$, $Y_k(\mathcal{S}')$.

In order to conclude the proof of Theorem 3.7.6 we are now left to show the following:

Proposition 3.7.9. *Let $\mathcal{S}$ be a set of slopes for Y_k whose elements do not belong to $\mathcal{S}_i$, $i = 1, \ldots, k$. Then the number of sets of slopes equivalent to $\mathcal{S}$ is greater than or equal to $(k! \cdot 3^h)/(h! \cdot (k-h)!)$. Moreover, if $\mathcal{S}'$ is a set of slopes equivalent to $\mathcal{S}$ and $Y_k(\mathcal{S})$ is homeomorphic to $Y_k(\mathcal{S}')$, then $\mathcal{S} = \mathcal{S}'$.*

Proof: Let $\mathcal{S} = \{s_{j_1}, \ldots, s_{j_h}\}$, where $s_{j_l} \notin \mathcal{S}_{j_l}$ is a slope on T_{j_l}. It is easily seen that the number of sets of slopes equivalent to $\mathcal{S}$ can vary from a minimum of $(k! \cdot 3^h)/(h! \cdot (k-h)!)$ (when for all $l, m \in \{1, \ldots, h\}$ there exists an orientation-preserving isometry between T_{j_l} and T_{j_m} taking s_{j_l} into s_{j_m}) to a maximum of $(k! \cdot 3^h)/(k-h)!$ (when for any $l, m \in \{1, \ldots, h\}, l \neq m$ there does not exist any orientation-preserving isometry between T_{j_l} and T_{j_m} taking s_{j_l} into s_{j_m}).

Suppose now $\mathcal{S}'$ is equivalent to $\mathcal{S}$ and let $\psi : Y_k(\mathcal{S}) \to Y_k(\mathcal{S}')$ be a homeomorphism. By rigidity ψ is homotopic to an isometry ψ', which by property (3) above must take the added geodesics of $Y_k(\mathcal{S})$ to the added geodesics of $Y_k(\mathcal{S}')$. This gives in turn a homeomorphism of Y_k onto itself taking $\mathcal{S}$ onto $\mathcal{S}'$. By rigidity again, up to homotopy such a homeomorphism restricts to an isometry of Y_k, whence $\mathcal{S} = \mathcal{S}'$ by Proposition 3.7.8.

Remark 3.7.10. Our proof makes clear that in Theorem 3.7.6 we could choose as Y_k any element of $\mathcal{M}_{g,k}$ without non-trivial isometries. We are quite confident that for any $k > 0$ there should exist $g > k$ and an element in $\mathcal{M}_{g,k}$ with this property (the manifold X_k provides such an element for any even k).

Commensurability of similar Dehn fillings Let M, M' be elements in $\mathcal{M}_{g,k}$ with canonical decompositions $\mathcal{T}, \mathcal{T}'$ respectively. Let N (resp. N') be a hyperbolic manifold obtained by Dehn filling M (resp. M') along sufficiently complicated slopes, and let $x \in I\Omega_{g,k}$ (resp. $x' \in I\Omega_{g,k}$) be such that $N \cong \hat{M}(x)$, $N' \cong \hat{M}'(x')$. In this paragraph we describe an explicit criterion which allows us to determine if N is commensurable with N' just by looking at x, x' and at the combinatorics of $\mathcal{T}, \mathcal{T}'$. We will show that geometrically similar manifolds constructed through the procedure described above in this section are usually non-commensurable. However, examples of commensurable geometrically similar non-homeomorphic Dehn fillings of a specific element of $\mathcal{M}_{g,k}$ are also provided.

Definition 3.7.11. Two complete hyperbolic n-manifolds with geodesic boundary M_1, M_2 are *commensurable* if a hyperbolic manifold with geodesic boundary M_3 exists which is the total space of a finite Riemannian covering both of M_1 and of M_2.

Proposition 3.7.12. *Let N_1, N_2 be complete finite-volume hyperbolic n-manifolds with non-empty geodesic boundary and denote by $\widetilde{N}_1, \widetilde{N}_2$ the universal coverings of N_1, N_2 respectively. Then N_1 is commensurable with N_2 if and only if $\widetilde{N}_1$ is isometric to $\widetilde{N}_2$.*

Proof: We concentrate our attention on the "if" part of the statement, the "only if" part being obvious.

For $j = 1, 2$ we have isometric inclusions $i_j : \widetilde{N}_j \to \mathbb{H}^n$, and our hypothesis implies that we can assume $i(\widetilde{N}_1) = i(\widetilde{N}_2) = P$, where where P is a closed geodesic polyhedron whose boundary is given by the union of a countable set of hyperbolic hyperplanes. Let $\Gamma = \mathrm{Isom}(P) < \mathrm{Isom}(\mathbb{H}^n)$. Using the fact that Γ permutes the connected components of ∂P it is easily seen that Γ is discrete in $\mathrm{Isom}(\mathbb{H}^n)$, so P/Γ is a hyperbolic orbifold of positive volume. For $j = 1, 2$ let $\Gamma_j \cong \pi_1(N_j)$ be the subgroup of Γ such that P/Γ_j is isometric to N_j. We have $[\Gamma : \Gamma_j] = \mathrm{vol}(P/\Gamma_j)/\mathrm{vol}(P/\Gamma) < \infty$, whence $[\Gamma_1 : (\Gamma_1 \cap \Gamma_2)] < \infty$, $[\Gamma_2 : (\Gamma_1 \cap \Gamma_2)] < \infty$. Thus the natural projections $P/(\Gamma_1 \cap \Gamma_2) \to P/\Gamma_1 \cong N_1$, $P/(\Gamma_1 \cap \Gamma_2) \to P/\Gamma_2 \cong N_2$ define the desired finite Riemannian coverings of N_1, N_2. $\square$

In the previous paragraph we have defined a hyperbolic manifold X_k with k cusps and with a connected geodesic boundary of genus $k+1$. We choose one out of the two natural orderings on the cusps of X_k, we denote by T_i the boundary torus of X_k corresponding to its i-th cusp, we let μ_i, λ_i be a fixed basis of $H_1(T_i; \mathbb{Z})$ with the usual properties and $u_i, v_i : \Omega_{k+1,k} \to \mathbb{C}$ be the corresponding Neumann-Zagier parameters.

Let $\Delta_1, \ldots, \Delta_{2k+1}$ be the partially truncated tetrahedra of the canonical decomposition $\mathcal{T}_k$ of X_k, and suppose as usual that for $i = 1, \ldots, k$ the tetrahedra $\Delta_{2i-1}, \Delta_{2i}$ are non-compact and glue up to a neighbourhood of the i-th cusp of X_k, while Δ_{2k+1} is compact regular. As in Sections 3.1, 3.4 we denote by $F^1_{2i-1}, F^2_{2i-1}, F^3_{2i-1}, F^1_{2i}, F^2_{2i}, F^3_{2i}$ the exceptional hexagons of $\Delta_{2i-1}, \Delta_{2i}$, in such a way that F^j_{2i-1} is glued to F^j_{2i} for $i = 1, \ldots, k$, $j = 1, 2, 3$. For $l = 1, \ldots, 2k$ we also call e^j_l the only finite edge of F^j_l, and f^j_l the edge of Δ_l opposite to e^j_l. We emphasize that here we do *not* require that F^1_l, F^2_l, F^3_l are positively arranged around the ideal vertex of Δ_l. Recall that a point $x \in \Omega_{k+1,k}$ determines the geometric realization of $\mathcal{T}_k$ with dihedral angle x_{6l-6+j} along e_l^j, angle x_{6l-3+j} along f_l^j, and angle x_{12k+1} along the compact edges of the unique compact tetrahedron. It is easily seen the exceptional lateral hexagons of the non-compact tetrahedra can be ordered around the ideal vertices in such a way that the following condition holds:

- For $i = 1, \ldots, k-1$ $j = 1, 2, 3$ the isometry which glues the compact face of Δ_{2i} to the compact face of Δ_{2i+1} sends e_{2i}^j to e_{2i+1}^j. Moreover, if i is odd (resp. even) then $F_{2i-1}^1, F_{2i-1}^2, F_{2i-1}^3$ and $F_{2i}^1, F_{2i}^2, F_{2i}^3$ are positively (resp. negatively) arranged around the ideal vertices of Δ_{2i-1} and Δ_{2i}.

(The fact that these conditions are coherent with each other depends on the combinatorial properties of $\mathcal{T}_k$. The second condition will be taken into account when we will explicitly consider the action of $\mathrm{Sym}^+(\Omega(\mathcal{T}_k)) = \mathrm{Sym}^+(\Omega_{k+1,k})$ on $\Omega(\mathcal{T}_k)$). Let $l \subset X_k$ be the compact edge of $\mathcal{T}_k$. A straight-forward analysis of the combinatorics of $\mathcal{T}_k$ shows that the dihedral angles of the tetrahedra of $\mathcal{T}_k$ are arranged along l according to the following cyclic ordering:

$$\begin{array}{ll} x_{12k+1}, & x_1, x_7, x_{13} \ldots, x_{6l+1}, \ldots, x_{12k-5}, \\ x_{12k+1}, x_{12k+1}, & x_2, x_8, x_{14} \ldots, x_{6l+2}, \ldots, x_{12k-4}, \\ x_{12k+1}, x_{12k+1}, x_{12k+1}, & x_3, x_9, x_{15} \ldots, x_{6l+3}, \ldots, x_{12k-3}. \end{array}$$

For $i = 1, \ldots, k$ let $a_i, b_i, c_i : \Omega_{k+1,k} \to \mathbb{R}$ be the functions defined as follows:

$$\begin{aligned} a_i(x) &= x_{12(j-1)+1} + x_{6(j-1)+7}, \qquad b_i(x) = x_{12(j-1)+2} + x_{6(j-1)+8}, \\ c_i(x) &= x_{12(j-1)+3} + x_{6(j-1)+9}, \end{aligned}$$

and set

$$a, b, c : \Omega_{k+1,k} \to \mathbb{R}, \qquad a(x) = \sum_{i=1}^{k} a_i(x), \quad b(x) = \sum_{i=1}^{k} b_i(x), \quad c(x) = \sum_{i=1}^{k} c_i(x).$$

We now recall that $I\Omega_{k+1,k}$ is the subset of $\Omega_{k+1,k}$ corresponding to structures with integral indivisible (or infinite) Dehn filling coefficients. For any $x \in \Omega_{k+1,k}$ we denote by $X_k(x)$ the hyperbolic structure defined on X_k by x, and by $\hat{X}_k(x)$ the metric completion of $X_k(x)$. If $x \in I\Omega_{k+1,k}$ then $\hat{X}_k(x)$ is a complete finite-volume hyperbolic manifold with geodesic boundary. Let also V'' be a neighbourhood of x_0 in $\Omega_{k+1,k}$ as in the statement of Proposition 3.6.7 and for any $x \in V'' \cap I\Omega_{k+1,k}$ let us denote by $U(x)$ the universal covering of $\hat{X}_k(x)$. We now show that the real numbers $a(x), b(x), c(x)$ completely determine the isometry type of the universal covering $U(x)$ of $\hat{M}(x)$, whence the commensurability class of $\hat{M}(x)$.

Proposition 3.7.13. *Let x, x' be points in $I\Omega_{k+1,k} \cap V''$. Then $\hat{X}_k(x)$ is commensurable to $\hat{X}_k(x')$ if and only if*

$$a(x) = a(x'), \quad b(x) = b(x'), \quad c(x) = c(x').$$

PROOF: Let $L^x(l_x)$ be the minimal distance between different connected components of $\partial U(x)$, and let $t \subset U(x)$ be a geodesic arc of length $L^x(l_x)$ joining two such components S_1, S_2. By Corollary 3.6.6, if $t' \subset U(x)$ is any other geodesic arc of length $L^x(l_x)$ connecting different components of $\partial U(x)$, then there exists an isometry of $U(x)$ taking t to t'. Let us consider the set $R \subset U(x)$ given by the union of all the compact regular truncated tetrahedra whose truncation triangles lie on $S_1 \cup S_2 \cup S' \cup S''$ for some connected components S', S'' of $\partial U(x)$. Let $N_\epsilon(t)$ be the ϵ-neighbourhood of t, and consider the sets $A = N_\epsilon(t) \cap R$ and $B = N_\epsilon(t) \setminus R$. Both A and B are unions of germs of dihedral sectors whose number, amplitude and cyclic order (up to the choice of a positive orientation around t) only depend on the isometry type of $U(x)$. We will call such sectors A-sectors or B-sectors, according to the fact that they are contained in A or in B. Corollary 3.6.5 implies that t is a lift in $U(x)$ of the unique compact edge of the geometric triangulation of $X_k(x)$, while by Proposition 3.6.7 the set R is the union of the lifts containing t of the geometric tetrahedron $\Delta_{12k+1} \subset X_k(x)$. Thus A-sectors are in number of three and have angles $x_{12k+1}, 2x_{12k+1}$ and $3x_{12k+1}$. Moreover, the B-sector between the A-sectors with angles $x_{12k+1}, 2x_{12k+1}$ has angle $a(x)$; the B-sector between the A-sectors with angles $2x_{12k+1}, 3x_{12k+1}$ has angle $b(x)$; the B-sector between the A-sectors with angles $3x_{12k+1}, x_{12k+1}$ has angle $c(x)$. This shows that $a(x), b(x), c(x)$ can be recovered solely from the isometry type of $U(x)$, so if $\hat{X}_k(x)$ is commensurable with $\hat{X}_k(x')$ we have $a(x) = a(x'), b(x) = b(x'), c(x) = c(x')$.

Suppose now that $a(x) = a(x'), b(x) = b(x'), c(x) = c(x')$. Since $a(x) + b(x) + c(x) + 6x_{12k+1} = a(x') + b(x') + c(x') + 6x'_{12k+1}$ we have $x_{12k+1} = x'_{12k+1}$, so the compact tetrahedron in the decomposition of $X_k(x)$ is isometric to the compact tetrahedron in the decomposition of $X_k(x')$. Let now $U_*(x)$ (resp. $U_*(x')$) be the complement in $U(x)$ (resp. in $U(x')$) of the preimage of the added geodesics $\hat{X}_k(x) \setminus X_k(x)$ (resp. $\hat{X}_k(x') \setminus X_k(x')$). The geometric decomposition of X_k parameterized by x (resp. x') naturally lifts to a tessellation $\mathcal{D}_*(x)$ of $U_*(x)$ (resp. $\mathcal{D}_*(x')$ of $U_*(x')$). Let $K(x) \subset U(x)$ (resp. $K(x') \subset U(x')$) be the union of the compact pieces of $\mathcal{D}_*(x)$ (resp. $\mathcal{D}_*(x')$). Let now $U(x), U(x')$ be isometrically identified with suitable polyhedra in $\mathbb{H}^3$. Since the compact tetrahedra of $\mathcal{D}_*(x)$ and $\mathcal{D}_*(x')$ are all isometric to each other and $a(x) = a(x'), b(x) = b(x'), c(x) = c(x')$ it is easily seen that an element $\psi \in \mathrm{Isom}(\mathbb{H}^3)$ exists which takes $K(x) \subset U(x) \subset \mathbb{H}^3$ onto $K(x') \subset U(x') \subset \mathbb{H}^3$. Since any component of $\partial U(x)$ (resp. $\partial U(x')$) meets $K(x)$ (resp. $\partial K(x')$) in a non-empty open subset of a hyperbolic plane, this readily implies that $\psi(\partial U(x)) = \psi(\partial U(x'))$. Now $U(x), U(x')$ are the hyperbolic convex hulls of $\partial U(x), \partial U(x')$ respectively, so $\psi(U(x)) = U(x')$. By Proposition 3.7.12, this implies that $\hat{X}_k(x)$ and $\hat{X}_k(x')$ are commensurable with each other. □

Let $\mathrm{Sym}^+(\Omega_{k+1,k})$ be the subgroup of $\mathrm{Sym}(\Omega_{k+1,k})$ corresponding to the group of isotopy classes of *orientation-preserving* isometries of $T_1 \sqcup \ldots \sqcup T_k$, under the isomorphism $\mathrm{Sym}(\Omega_{k+1,k}) \cong \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$. In order to determine if geometrically similar manifolds obtained by Dehn filling X_k are commensurable to each other, we are now reduced to understand when the functions a, b and c introduced above take different values on $\mathrm{Sym}^+(\Omega_{k+1,k})$-equivalent points in $\Omega_{k+1,k}$. Let $i \in \{1, \ldots, k\}$ and suppose $x' \in \Omega_{k+1,k}$ is obtained from $x \in \Omega_{k+1,k}$ just by exchanging the rôles of Δ_{2i-1} and Δ_{2i} (*i.e.* $x' = \varphi_i(r^3)(x)$ according to the notation of equation (3.34)).

Then by equation (3.34) we have $u_i(x') = -u_i(x)$ and $u_j(x') = u_j(x)$ for $j = 1, \dots, k$, $j \neq i$. By Corollary 2.4.4, this ensures that $a_i, b_i, c_i : \Omega_{k+1,k} \to \mathbb{R}$ induce real maps on a small neighbourhood of the complete structure in the character variety $X^{\text{geod}}(\pi_1(X_k))$. More precisely, if $\chi_0 := \chi_{\rho_0}$ is the character of the holonomy of the complete structure on X_k, then there exist an open neighbourhood Z of χ_0 in $X^{\text{geod}}(\pi_1(X_k))$ and analytic maps $\overline{a}_i, \overline{b}_i, \overline{c}_i : Z \to \mathbb{R}$ with $a_i = \overline{a}_i \circ \pi, b = \overline{b}_i \circ \pi, c = \overline{c}_i \circ \pi$.

For later purpose we slightly modify the definition given in equation (3.8) and set

$$H'_h = \{x \in \mathbb{R}^{12k+1} : \ x_{12i+1} = x_{12i+2} = x_{12i+3} \text{ for all } i = h-1, \dots, k-1\}.$$

We recall that in a neighbourhood of x_0 in $\Omega_{k+1,k}$ the set $\Omega^h_{k+1,k} := H'_h \cap \Omega_{k+1,k}$ is a smooth manifold of dimension $2h$ whose points correspond to those structures which induce a complete metric on the last h cusps of X_k (see Lemma 3.4.4 and Proposition 3.4.7). We also set $X^{\text{geod}}_h(\pi_1(X_k)) = \pi(\Omega^h_{k+1,k}) \subset X^{\text{geod}}(\pi_1(X_k))$, where $\pi : \Omega_{k+1,k} \to X^{\text{geod}}(\pi_1(X_k))$ is the natural projection to the character variety.

Lemma 3.7.14. *There exists a tangent vector $v \in T_{\chi_0} X^{\text{geod}}_h(\pi_1(X_k))$ such that*

$$(d\overline{a}_i)_{\chi_0}(v) > (d\overline{b}_i)_{\chi_0}(v) > (d\overline{c}_i)_{\chi_0}(v)$$

for $i = 1, \dots, h$.

PROOF: Let $\overline{\varsigma} : (-\varepsilon, \varepsilon) \to \Omega_{k+1,k}$ be the chosen curve at page 91, Section 3.5. By Proposition 3.5.7 we have that $\overline{\varsigma}(t)$ belongs to $\Omega^1_{k+1,k}$ for any $t \in (-\varepsilon, \varepsilon)$. Moreover, by the very definition of $\overline{\varsigma}$ and Proposition 3.5.8 we easily get

$$\begin{gathered}
(a_1 \dot{\circ} \overline{\varsigma})(0) = (b_1 \dot{\circ} \overline{\varsigma})(0) = (c_1 \dot{\circ} \overline{\varsigma})(0) = 0, \\
(a_1 \ddot{\circ} \overline{\varsigma})(0) = 16 \cos \overline{\alpha}_{k+1,k} \sin \overline{\alpha}_{k+1,k}, \\
(b_1 \ddot{\circ} \overline{\varsigma})(0) = (c_1 \ddot{\circ} \overline{\varsigma})(0) = -8 \cos \overline{\alpha}_{k+1,k} \sin \overline{\alpha}_{k+1,k},
\end{gathered} \tag{3.38}$$

and

$$(a_i \dot{\circ} \overline{\varsigma})(0) = (b_i \dot{\circ} \overline{\varsigma})(0) = (c_i \dot{\circ} \overline{\varsigma})(0) = (a_i \ddot{\circ} \overline{\varsigma})(0) = (b_i \ddot{\circ} \overline{\varsigma})(0) = (c_i \ddot{\circ} \overline{\varsigma})(0) = 0, \quad i = 2, \dots, k. \tag{3.39}$$

Let us set for any $t \in [0, \varepsilon^2)$

$$\widehat{\varsigma}^a_1(t) = \pi(\overline{\varsigma})(\sqrt{t}) \in X^{\text{geod}}_1(\pi_1(X_k)).$$

By Corollary 2.4.4 the map $\widehat{\varsigma}^a_1$ is of class C^1 at every point in $[0, \varepsilon)$. Moreover by equations (3.38), (3.39) we have

$$\begin{gathered}
(\overline{a}_1 \dot{\circ} \widehat{\varsigma}^a_1)(0) = 8 \cos \overline{\alpha}_{k+1,k} \sin \overline{\alpha}_{k+1,k}, \\
(\overline{b}_1 \dot{\circ} \widehat{\varsigma}^a_1)(0) = (\overline{c}_1 \dot{\circ} \widehat{\varsigma}^a_1)(0) = -4 \cos \overline{\alpha}_{k+1,k} \sin \overline{\alpha}_{k+1,k}, \\
(\overline{a}_i \dot{\circ} \widehat{\varsigma}^a_1)(0) = (\overline{b}_i \dot{\circ} \widehat{\varsigma}^a_1)(0) = (\overline{c}_i \dot{\circ} \widehat{\varsigma}^a_1)(0) = 0, \ i = 2, \dots, k.
\end{gathered}$$

For every $i = 1, \ldots, h$, the very same procedure also provides C^1-curves $\widehat{\varsigma}_i^a, \widehat{\varsigma}_i^b, \widehat{\varsigma}_i^c : [0, \varepsilon^2) \to X_h^{\mathrm{geod}}(\pi_1(X_k))$ with the following properties:

$$\begin{aligned}
&\dot{(\overline{a}_i \circ \widehat{\varsigma}_i^a)} = -2\dot{(\overline{b}_i \circ \widehat{\varsigma}_i^a)} = -2\dot{(\overline{c}_i \circ \widehat{\varsigma}_i^a)} \neq 0,\\
&\dot{(\overline{b}_i \circ \widehat{\varsigma}_i^b)} = -2\dot{(\overline{a}_i \circ \widehat{\varsigma}_i^b)} = -2\dot{(\overline{c}_i \circ \widehat{\varsigma}_i^b)} \neq 0,\\
&\dot{(\overline{c}_i \circ \widehat{\varsigma}_i^c)} = -2\dot{(\overline{a}_i \circ \widehat{\varsigma}_i^c)} = -2\dot{(\overline{b}_i \circ \widehat{\varsigma}_i^c)} \neq 0,\\
&\dot{(\overline{a}_j \circ \widehat{\varsigma}_i^a)} = \dot{(\overline{b}_j \circ \widehat{\varsigma}_i^a)} = \dot{(\overline{c}_j \circ \widehat{\varsigma}_i^a)} = \dot{(\overline{a}_j \circ \widehat{\varsigma}_i^b)} = \dot{(\overline{b}_j \circ \widehat{\varsigma}_i^b)} = \dot{(\overline{c}_j \circ \widehat{\varsigma}_i^b)}\\
&= \dot{(\overline{a}_j \circ \widehat{\varsigma}_i^c)} = \dot{(\overline{b}_j \circ \widehat{\varsigma}_i^c)} = \dot{(\overline{c}_j \circ \widehat{\varsigma}_i^c)} = 0, \quad j = 1, \ldots, k,\ j \neq i.
\end{aligned}$$

This easily implies that a suitable linear combination of the starting tangent vectors of these curves provides an element $v \in T_{\chi_0} X_h^{\mathrm{geod}}(\pi_1(X_k))$ with

$$(d\overline{a}_i)_{\chi_0}(v) > (d\overline{b}_i)_{\chi_0}(v) > (d\overline{c}_i)_{\chi_0}(v), \quad i = 1, \ldots, h,$$

as required. □

We are now ready to prove the following:

Theorem 3.7.15. *Fix $1 \leqslant h \leqslant k$. Then there exists a sequence $\{W_h^n\}_{n \in \mathbb{N}}$ of pairwise non-homeomorphic complete finite-volume hyperbolic manifolds with geodesic boundary with the following properties:*

- *Each W_h^n is obtained by Dehn filling the first h cusps of X_k;*
- *For any $n \in \mathbb{N}$ there exist at least three (including W_h^n itself) pairwise non-commensurable hyperbolic Dehn fillings of X_k which are geometrically similar to W_h^n.*

Proof: Let us fix a neighbourhood $Z \subset \pi(V'')$ of χ_0 in $X^{\mathrm{geod}}(\pi_1(X_k))$ with a diffeomorphism $\varphi : Z \to \mathbb{C}^k \cong \mathbb{R}^{2k}$ such that $\varphi(\chi_0) = 0$ and $\varphi(Z \cap X_h^{\mathrm{geod}}(\pi_1(X_k))) = \mathbb{C}^{k-h} \times \{0\} \cong \mathbb{R}^{2(k-h)}$. Let $w \in \mathbb{R}^{2(k-h)} \cong T_0\mathbb{R}^{2(k-h)}$ be a unit vector with $d(\overline{a}_i \circ \varphi^{-1})_0(w) > d(\overline{b}_i \circ \varphi^{-1})_0(w) > d(\overline{c}_i \circ \varphi^{-1})_0(w)$ for $i = 1, \ldots, h$. We choose an infinite sequence $\{y_n\}_{n \in \mathbb{N}} \subset V'' \cap \Omega_{k+1,k}^h$ such that the following conditions hold: $lim_{n \to \infty} y_n = x_0$, for any $n \in \mathbb{N}$ the point y_n determines integral indivisible Dehn filling coefficients on every end of X_k, and $\lim_{n \to \infty} \varphi(\pi(y_n))/||\varphi(\pi(y_n))|| = w$ in $S^{2(k-h)-1} \subset \mathbb{R}^{2(k-h)}$. We then set $W_h^n = \hat{X}_k(y_n)$ for any $n \in \mathbb{N}$.

We first observe that W_h^n is obtained from X_k by Dehn filling the first h cusps of X_k: since y_n belongs to $\Omega_{k+1,k}^h$ the last $k-h$ cusps of $X_k(y_n)$ have to be complete; moreover, up to extracting a subsequence we can suppose $a_i(y_n) > b_i(y_n) > c_i(y_n)$ for all $i = 1, \ldots, h$, whence the angles along the compact edges of $\Delta_{2i-1}, \Delta_{2i}$ are not equal to each other, and the i-th cusp of $X_k(y_n)$ is not complete.

Let now $r \in \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ be the element acting as a positive (resp. negative) rotation by an angle of $\pi/3$ on T_i for $i \in \{1, \ldots, k\}$, i odd (resp. even), and let $\Theta : \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k) \to \mathrm{Sym}(\Omega_{k+1,k})$ be the isomorphism described in Proposition 3.6.1. We set $y'_n = \Theta(r)(y_n)$ and $y''_n = \Theta(r^2)(y_n)$. It is readily seen that $\Theta(r), \Theta(r^2)$ belong to $\mathrm{Sym}^+(\Omega_{k+1,k})$, whence $W_h^n = \hat{X}_k(y_n), \hat{X}_k(y'_n)$ and $\hat{X}_k(y''_n)$ are pairwise geometrically similar. Moreover an easy computation shows that for every $x \in \Omega_{k+1,k}$ we have

$$\begin{aligned} a_i(\Theta(r^2)(x)) = c_i(\Theta(r)(x)) = b_i(x), \\ b_i(\Theta(r^2)(x)) = a_i(\Theta(r)(x)) = c_i(x), \\ c_i(\Theta(r^2)(x)) = b_i(\Theta(r)(x)) = a_i(x), \end{aligned}$$

whence $a_i(y_n) > a_i(y_n'') > a_i(y_n')$, and $W_h^n = \hat{X}_k(y_n), \hat{X}_k(y_n'), \hat{X}_k(y_n'')$ are pairwise non-commensurable by Proposition 3.7.13. □

Remark 3.7.16. Let M be an element of $\mathcal{M}_{g,k}$ with canonical decomposition $\mathcal{T}$. Suppose that the arrangement of compact and non-compact tetrahedra around the compact edge of $\mathcal{T}$ is sufficiently irregular and let $\mathcal{S} = \{s_{i_1}, \ldots, s_{i_h}\}$ be a set of slopes for M such that s_{i_l} is not equivalent to s_{i_m} for $l \neq m$. The same argument used to prove Theorem 3.7.15 shows that typically the Dehn fillings of M which are geometrically similar to $M(\mathcal{S})$ are expected to be non-commensurable with each other.

We conclude the section with some examples of non-homeomorphic geometrically similar commensurable Dehn fillings of elements in $\mathcal{M}_{g,k}$.

Theorem 3.7.17. *Let $k \geqslant 4$. Then there exists an infinite sequence of pairs $\{Y_1^n, Y_2^n\}_{n\in\mathbb{N}}$ of complete finite-volume hyperbolic manifolds with geodesic boundary such that for every $n \in \mathbb{N}$ the following conditions hold:*

- *Y_1^n is obtained by Dehn filling the first cusp of X_k;*
- *Y_2^n is obtained by Dehn filling the third cusp of X_k;*
- *Y_1^n is geometrically similar to Y_2^n;*
- *Y_1^n is commensurable with Y_2^n;*
- *Y_1^n is not homeomorphic to Y_2^n.*

Proof: Let us fix a neighbourhood $Z \subset \pi(V'')$ of χ_0 in $X^{\text{geod}}(\pi_1(X_k))$ with a diffeomorphism $\varphi : Z \to \mathbb{C}^k \cong \mathbb{R}^{2k}$ such that $\varphi(\chi_0) = 0$ and $\varphi(Z \cap X_1^{\text{geod}}(\pi_1(X_k))) = \mathbb{C}^{k-1} \times \{0\} \cong \mathbb{R}^{2(k-1)}$. Let $w \in \mathbb{R}^{2(k-1)} \cong T_0\mathbb{R}^{2(k-1)}$ be a unit vector with $d(\overline{a}_1 \circ \varphi^{-1})_0(w) > d(\overline{b}_1 \circ \varphi^{-1})_0(w) > d(\overline{c}_1 \circ \varphi^{-1})_0(w)$. We choose an infinite sequence $\{y_n\}_{n\in\mathbb{N}} \subset V'' \cap \Omega^1_{k+1,k}$ such that the following conditions hold: $lim_{n\to\infty} y_n = x_0$, for any $n \in \mathbb{N}$ the point y_n determines integral indivisible Dehn filling coefficients on every end of X_k, and $\lim_{n\to\infty} \varphi(\pi(y_n))/||\varphi(\pi(y_n))|| = w$ in $S^{2(k-1)-1} \subset \mathbb{R}^{2(k-1)}$.

Let now $\tau_{13} \in \mathrm{Sym}^+(\Omega_{k+1,k})$ be the element which exchanges the first cusp of X_k with the third one and let $y_n' = \tau_{13}(y_n)$. We set $Y_1^n = \hat{X}_k(y_n)$ and $Y_2^n = \hat{X}_k(y_n')$. It is easily seen that Y_1^n is obtained by Dehn filling the first cusp of X_k, while Y_2^n is obtained by Dehn filling the third one. Of course Y_1^n is geometrically similar to Y_2^n. Moreover an easy computation shows that for every $x \in \Omega_{k+1,k}$ we have

$$\begin{array}{llll} a_1(\tau_{13}(x)) = a_3(x), & b_1(\tau_{13}(x)) = b_3(x), & c_1(\tau_{13}(x)) = c_3(x), & \\ a_3(\tau_{13}(x)) = a_1(x), & b_3(\tau_{13}(x)) = b_1(x), & c_3(\tau_{13}(x)) = c_1(x), & \\ a_j(\tau_{13}(x)) = a_j(x), & b_j(\tau_{13}(x)) = b_j(x), & c_j(\tau_{13}(x)) = c_j(x), & j = 2, 4, 5, 6, \ldots, k. \end{array}$$

This easily implies $a(y_n) = a(y'_n)$, $b(y_n) = b(y'_n)$, $c(y_n) = c(y'_n)$, so Y_1^n is commensurable with Y_2^n by Proposition 3.7.13.

Let us prove that Y_1^n is not homeomorphic to Y_2^n. Up to passing to a subsequence we can suppose that the added geodesic $Y_1^n \setminus X_k$ (resp. $Y_2^n \setminus X_k$) is the shortest geodesic of Y_1^n (resp. Y_2^n). Let $f_n : Y_1^n \to Y_2^n$ be a homeomorphism. By Mostow-Prasad's rigidity Theorem we may assume that f_n is an isometry, which implies $f_n(Y_1^n \setminus X_k) = Y_2^n \setminus X_k$. Thus f_n restricts to a homeomorphism $f'_n : X_k \to X_k$ taking the first cusp onto the third one. By rigidity again we can homotope f'_n into an isometry. This isometry should take the first cusp of X_k onto the third one, against Proposition 3.7.7. $\square$

Appendix A

Cohomology theory of representations

In this appendix we briefly recall some general results relating the local properties of representation spaces to suitably defined cohomology groups, and we outline how these results, together with classical theorems by Weil and Garland [Wei64, Gar67], provide a nice description of neighbourhoods of holonomy representations for complete hyperbolic structures.

The tangent space to a representation Let G be a Lie group with associated Lie algebra $\mathfrak{g}$ and Γ be any group. We say that a path of representations $\{\rho_t \in \mathcal{R}(\Gamma, G) : t \in (-\varepsilon, \varepsilon)\}$ is *smooth* if $\rho_t(\gamma)$ is a smooth function of t for any $\gamma \in \Gamma$. If $\{\rho_t\}$ is a smooth path of representations, the tangent vector to the map $t \mapsto \rho_t(\gamma)$ at 0 gives an element in $T_{\rho_0(\gamma)}G$. Identifying this tangent space with $\mathfrak{g} = T_1 G$ by *right* translation we get an element $\dot{\rho}(\gamma)$ of the Lie algebra $\mathfrak{g}$:

$$\dot{\rho}(\gamma) = \frac{d}{dt}\Big|_{t=0} \left(\rho_t(\gamma)\rho_0(\gamma)^{-1}\right).$$

Differentiating the homomorphism relation $\rho_t(\gamma_1\gamma_2) = \rho_t(\gamma_1)\rho_t(\gamma_2)$ we see that $\dot{\rho} : \Gamma \to \mathfrak{g}$ satisfies the so-called *cocycle* relation

$$\dot{\rho}(\gamma_1\gamma_2) = \dot{\rho}(\gamma_1) + \mathrm{Ad}(\rho_0(\gamma_1))(\dot{\rho}(\gamma_2)),$$

where $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ is the usual adjoint representation.

Now consider a *trivial* deformation of ρ_0, *i.e.* let $t \mapsto g_t$ be a smooth path in G starting at the identity and set $\rho_t(\gamma) = g_t\rho_0(\gamma)g_t^{-1}$ for any $\gamma \in \Gamma$. Then differentiation shows that

$$\dot{\rho}(\gamma) = \dot{g} - \mathrm{Ad}(\rho_0(\gamma))(\dot{g})$$

for any $\gamma \in \Gamma$, where $\dot{g} \in \mathfrak{g}$ is the tangent vector to $t \mapsto g_t$ at $t = 0$. We are now ready to give the following:

Definition A.1. Let Γ be a group and M be an Abelian group. Let also $\varphi : \Gamma \to \mathrm{Aut}(M)$ be an action of Γ on M. Then we define 1-cocycles and 1-coboundaries with respect to φ to be respectively the groups:

$$\begin{aligned} Z^1(\Gamma; M; \varphi) &= \{c : \Gamma \to M : \ c(\gamma_1\gamma_2) = c(\gamma_1) + \varphi(\gamma_1)(c(\gamma_2))\}, \\ B^1(\Gamma; M; \varphi) &= \{b : \Gamma \to M : \ c(\gamma) = m - \varphi(\gamma)(m) \text{ for some } m \in M\}. \end{aligned}$$

Since $B^1(\Gamma; M; \varphi)$ is contained in $Z^1(\Gamma; M; \varphi)$ it makes sense to define the first cohomology group of Γ with respect to φ to be the quotient module:

$$H^1(\Gamma; M; \varphi) = Z^1(\Gamma; M; \varphi)/B^1(\Gamma; M; \varphi).$$

The above discussion shows that $Z^1(\Gamma; \mathfrak{g}; \mathrm{Ad} \circ \rho_0)$ should correspond to the tangent space of $\mathcal{R}(\Gamma, G)$ at ρ_0 (however this holds true only in a quite sophisticated sense). Under this identification the group $B^1(\Gamma; \mathfrak{g}; \mathrm{Ad} \circ \rho_0)$ should represent the tangent space to trivial deformations of ρ_0. Thus it makes sense to expect $H^1(\Gamma; \mathfrak{g}; \mathrm{Ad} \circ \rho_0)$ to be something similar to the tangent space of $\mathcal{R}(\Gamma, G)/G$ at $[\rho_0]$. However in general there could be cocycles in $Z^1(\Gamma; \mathfrak{g}; \mathrm{Ad} \circ \rho_0)$ which do not arise as tangent vectors to smooth deformations of ρ_0. Moreover, both $\mathcal{R}(\pi_1(N), G)$ and $\mathcal{R}(\pi_1(N), G)/G$ need not be smooth manifolds in a neighbourhood of ρ_0 (the latter space could in general be even non-Haussdorff). When G is algebraic, these difficulties can be overcome by considering the algebraic *scheme* $\overline{\mathcal{R}}(\pi_1(N), G)$ rather than the algebraic set $\mathcal{R}(\pi_1(N), G)$, and the algebraic quotient $\overline{\mathcal{R}}(\pi_1(N), G)//G$ rather than the set-theoretic quotient $\mathcal{R}(\pi_1(N), G)/G$. We do not go into details here, addressing the reader to [LM85, Por97] for a thorough description of this technique.

Two easy lemmas Let Γ be a group, M be an Abelian group and φ be an action of Γ on M. Fix also a cocycle $c \in Z^1(\Gamma; M; \varphi)$, and set $\Gamma_0 = \{\gamma \in \Gamma : \ \varphi(\gamma) = \mathrm{Id}_M, \ c(\gamma) = 0\}$.

Lemma A.2. *Γ_0 is a normal subgroup of Γ.*

Lemma A.3. *The map*

$$\overline{c} : \Gamma/\Gamma_0 \to M, \qquad \overline{c}([\gamma]) = c(\gamma)$$

is well-defined and gives a cocycle $\overline{c} \in Z^1(\Gamma/\Gamma_0; M; \overline{\varphi})$, where $\overline{\varphi} : \Gamma/\Gamma_0 \to \mathrm{Aut}(M)$ is the action naturally induced by φ. Moreover, we have $\overline{c} \in B^1(\Gamma/\Gamma_0; M; \overline{\varphi})$ if and only if $c \in B^1(\Gamma; M; \varphi)$.

Classical rigidity results Let now N be a smooth 3-manifold without boundary and suppose $\rho_0 : \pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$ is the holonomy representation for an $(\mathbb{H}^3, \mathrm{Isom}^+(\mathbb{H}^3))$-structure on N. The following result is due to Weil [Wei64]:

Theorem A.4. *Suppose N is compact. Then $H^1(\pi_1(N); \mathfrak{sl}(2, \mathbb{C}); \mathrm{Ad} \circ \rho_0) = 0$.*

An easy analysis of the local structure of $\mathcal{R}(\pi_1(N), \mathrm{PSL}(2, \mathbb{C}))$ (essentially based on the implicit function theorem) now leads to the following result [Wei64, Hod86], which can be considered as a local version of Mostow's rigidity Theorem:

Theorem A.5. *If N is compact, then there exists a neighbourhood U of ρ_0 in $\mathcal{R}(\pi_1(N), G)$ such that any $\rho \in U$ is conjugated to ρ_0.*

Let now N be the interior of a compact 3-manifold $\overline{N}$ whose boundary $\partial\overline{N} = T_1 \sqcup \ldots \sqcup T_k$ consists of tori. Suppose N supports a complete finite-volume hyperbolic structure and let $\rho_0 : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ be a holonomy representation for this structure. We know that ρ_0 admits non-trivial deformations (see Theorem 1.3.3), so we cannot expect $H^1(\pi_1(N); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ \rho_0)$ to be trivial in this case.

For any $j = 1, \ldots, k$ and any cyclic subgroup W of $\pi_1(T_j)$ the natural injection $i_W : W \to \pi_1(N)$ induces a map on cohomology

$$i_W^* : H^1(\pi_1(N); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ \rho_0) \to H^1(W; \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ \rho_0 \circ i_W).$$

Let now

$$\mathcal{P} = \{W \subset \pi_1(N) :\ W \text{ is a cyclic subgroup of } \pi_1(T_j) \text{ for some } j = 1, \ldots, k\}$$

and set

$$H^1_{\mathrm{par}}(\pi_1(N); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ \rho_0) = \bigcap_{W \in \mathcal{P}} \mathrm{Ker}\ i_W^*.$$

The naïve correspondence between $H^1(\pi_1(N); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ \rho_0)$ and the tangent space of $\mathcal{R}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$ at ρ_0 induces an identification of $H^1_{\mathrm{par}}(\pi_1(N); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ \rho_0)$ with the space of infinitesimal deformations of ρ_0 through holonomies for *complete* structures on N. Thus the following result [Gar67, Rag72] can be considered an infinitesimal version of Mostow-Prasad's rigidity Theorem for complete finite-volume hyperbolic 3-manifolds:

Theorem A.6. *We have $H^1_{\mathrm{par}}(\pi_1(N); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ \rho_0) = 0$.*

It is maybe worth mentioning that *a priori* there is not any obvious reason for which $H^1_{\mathrm{par}}(\pi_1(N); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ \rho_0)$ should be equal to the intersection $\bigcap_{j=1}^k \mathrm{Ker}\, i^*_{\pi_1(T_j)}$ (by Theorem A.6, however, these vector spaces are both trivial). Under the identification between $H^1(\pi_1(N); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ \rho_0)$ and the tangent space of $\mathcal{R}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$ at ρ_0, the intersection $\bigcap_{j=1}^k \mathrm{Ker}\, i^*_{\pi_1(T_j)}$ should in fact correspond to the subspace of $H^1_{\mathrm{par}}(\pi_1(N); \mathfrak{sl}(2,\mathbb{C}); \mathrm{Ad} \circ \rho_0)$ representing those infinitesimal deformations of ρ_0 which do not affect the similarity class of the Euclidean structure on the cusps of N.

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