Argomenti di Teoria di Lie

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Definition 1.1. *We say that a graded Lie algebra $\mathfrak{g}$ is* ***non-degenerate*** *if $\forall n > 0$ the pairing*

$$\begin{array}{ccc} \mathfrak{g}_{-n} \times \mathfrak{g}_n & \to & \mathbb{C} \\ (a, b) & \mapsto & \lambda([a, b]) \end{array}$$

is nondegenerate ***for generic*** *λ, that is, a Zariski-open of $\mathfrak{g}_0^\times$.*

2 20.02.2018 - Universal enveloping algebra

2.1 The case of Lie algebras

Recall that a **Lie algebra representation** is a linear map

$$\phi : \mathfrak{g} \to \mathrm{End}(V)$$

such that $\phi([a, b]) = \phi(a)\phi(b) - \phi(b)\phi(a)$.

The subrepresentation generated by $v \in V$ is

$$\langle v \rangle = \mathrm{Span}\{v; av | a \in \mathfrak{g}; abv | a, b \in \mathfrak{g}\} = \mathrm{Span}\left\{a_1 \cdots a_s v \;\middle|\; a_1, \dots, a_s \in \mathfrak{g}\right\}$$

2.2 The case of associative algebras

Recall that for an associative (unitary) algebra A, a representation is an associative algebra homomorphism $\phi : A \to \mathrm{End}(V)$ such that $\phi(ab) = \phi(a)\phi(b)$. If $v \in V$ is a vector, the subrepresentation generated by v is simply $\langle v \rangle = Av$.

2.3 The universal enveloping algebra

Since the situation with associative algebras is much easier, we'd like to reduce to this case. In particular, we want an associative algebra $U(\mathfrak{g})$ such that

$$\{\text{representations of } \mathfrak{g}\} \longleftrightarrow \{\text{representations of } U(\mathfrak{g})\}.$$

Theorem 2.1. *There exists a unique pair $(U(\mathfrak{g}), i)$ such that:*

1. *$U(\mathfrak{g})$ is a unitary associative algebra;*
2. *$i : \mathfrak{g} \to U(\mathfrak{g})$ is a Lie algebra homomorphism (where $U(\mathfrak{g})$ has its natural Lie structure given by $[a, b] = ab - ba$);*
3. *the following universal property holds: for every Lie algebra homomorphism $\phi : \mathfrak{g} \to A$, there exists a unique $\hat{\phi}$ such that the following diagram commutes*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi"] \arrow[d, "i"'] \arrow[dr, phantom, "\exists!"] & A \\ U(\mathfrak{g}) \arrow[ur, dotted, "\hat{\phi}"'] & \end{tikzcd}$$

Such a pair is called the universal enveloping algebra of $\mathfrak{g}$.

Remark 2.2. The universal property of $U(\mathfrak{g})$ implies that any representation $\phi : \mathfrak{g} \to \mathrm{End}(V)$ factors via $U(\mathfrak{g})$.

Proof. 1. Existence: we explicitly construct a pair $(U(\mathfrak{g}), i)$ as in the statement.

- Start with the tensor algebra $\mathcal{T}(\mathfrak{g})$.
- Consider the bilateral ideal $J = \langle a \otimes b - b \otimes a - [a, b] \mid a, b \in \mathfrak{g} \rangle$ and set $U(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/J$.
- Define i via the natural map $\mathfrak{g} \hookrightarrow \mathcal{T}(\mathfrak{g}) \to U(\mathfrak{g})$.
- Let us prove that i is a Lie algebra homomorphism. We have

$$i(a)i(b) - i(b)i(a) = \overline{a}\overline{b} - \overline{b}\overline{a} = \overline{a \otimes b - b \otimes a} = \overline{[a, b]} = i([a, b]).$$

- We now show that $(U(\mathfrak{g}), i)$ satisfies the universal property. Let $\phi : \mathfrak{g} \to A$ be a Lie algbera homomorphism from $\mathfrak{g}$ to a unitary associative algebra A. With reference to the following diagram,

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\phi"] \arrow[d, hook] & A \\
\mathcal{T}(\mathfrak{g}) \arrow[ur, dotted, "\tilde{\phi}"] \arrow[d, two heads] & \\
U(\mathfrak{g}) \arrow[uur, dotted, "\hat{\phi}"'] &
\end{tikzcd}$$

$\tilde{\phi}$ exists by the universal property of the tensor algebra, so (since $U(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/J$) it suffices to show that $\tilde{\phi}$ factors via $\mathcal{T}(\mathfrak{g})/J$, i.e. that $\tilde{\phi}(a \otimes b - b \otimes a - [a, b]) = 0$; and this is obvious, because by assumption $\phi([a, b]) = \phi(a)\phi(b) - \phi(b)\phi(a)$, so

$$\tilde{\phi}(a \otimes b - b \otimes a - [a, b]) = \phi(a)\phi(b) - \phi(b)\phi(a) - \phi([a, b]) = 0$$

as desired.

2. Uniqueness: general nonsense.

$\square$

3 Verma module

Definition 3.1. *A* ***highest weight representation*** V *of weight* $\lambda \in \mathfrak{g}_0^{\times}$ *is a* $\mathfrak{g}$*-module such that there exists* $v_\lambda \in V$ *such that:*

1. $av_\lambda = 0$ *for every* $a \in \mathfrak{g}_{>0}$*;*

2. $hv_\lambda = \lambda(h)v_\lambda \quad \forall h \in \mathfrak{g}_0$*;*

3. v_λ *generates* V*.*

The ***Verma module*** M_λ *is the universal representation of highest weight* λ*.*

Construction of the Verma module. One sets

$$M_\lambda = \frac{U(\mathfrak{g})}{U(\mathfrak{g})\left(\mathfrak{g}_{>0} + \mathrm{Span}\left\{h - \lambda(h) \mid h \in \mathfrak{g}_0\right\}\right)},$$

and takes the universal vector v_λ to be $1_\lambda = \overline{1}$. The universal property follows immediately from the universal property of the universal enveloping algebra.

Remark 3.2. Notice that $U(\mathfrak{g})$ a $\mathfrak{g}$-module via i (that is: given $B \in U(\mathfrak{g})$ and $a \in \mathfrak{g}$, the action is given by $a \cdot B = i(a)B$, where the product is taken in $U(\mathfrak{g})$).

Theorem 3.3 (Poincaré-Birkhoff-Witt)**.** *Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$. A basis of $U(\mathfrak{g})$ is given by $e_{i_1} \cdots e_{i_s}$, $i_1 \leq i_2 \leq \cdots \leq i_s$. In particular, $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective.*

Remark 3.4. Let $a_1, \ldots, a_m$ be a basis of $\mathfrak{g}_{<0}$, $h_1, \ldots, h_r$ be a basis of $\mathfrak{g}_0$, and $b_1, \ldots, b_n$ be a basis of $\mathfrak{g}_{>0}$. It follows from the PBW theorem that M_λ is generated by $\{\prod_s a_{i_s}\}$ (terms with b's vanish in the quotient, while the h's are scalars in the quotient); applying PBW again, this is nothing but $U(\mathfrak{g}_{<0})$.

Proof (of the PBW theorem). In order to establish that $M = \{\prod e_{i_j} \bigm| i_1 \leq i_2 \leq \ldots \leq i_n\}$ is a basis, we need to prove:

1. M is a generating set. Clearly *all* monomials form a generating set, so it suffices to prove: an arbitrary monomial $e_{j_1} \cdots e_{j_t}$ belongs to $\mathrm{Span}(M)$. We prove this by induction on (t, N), where t is the number of factors in $e_{j_1} \cdots e_{j_t}$ and N is $\#\{(p, q) : p < q, i_p > i_q\}$. If $N = 0$ or $t = 1$ we're done. Suppose $N \geq 1$: then there must be a p such that $i_p > i_{p+1}$. Then

$$e_{j_1} \cdots e_{j_p} e_{j_{p+1}} \cdots e_t = e_{j_1} \cdots e_{j_{p+1}} e_{j_p} \cdots e_t + e_{j_1} \cdots [e_{j_p}, e_{j_{p+1}}] \cdots e_t$$

   Both terms lie in $\mathrm{Span}(M)$: the first one because we've reduced N, the second one because we've reduced t[1].

2. The elements of M are linearly independent. Let V be a vector space with basis $\{v_{i_1,\ldots,i_s} \mid 1 \leq i_1 \leq \cdots \leq i_s \leq n\}$. We'd like to define a linear map $f : \mathcal{T}(\mathfrak{g}) \to V$ such that $f(e_{i_1} \otimes \cdots \otimes e_{i_s}) = v_{i_1,\ldots,i_s}$ whenever $i_1 \leq i_2 \leq \cdots \leq i_s$. This implies what we want. We require that f satisfies

$$\begin{aligned} &f(e_{j_1} \otimes \cdots \otimes e_{j_p} \otimes e_{j_{p+1}} \otimes \cdots \otimes e_{j_t}) \\ &\qquad = f(e_{j_1} \otimes \cdots \otimes e_{j_{p+1}} \otimes e_{j_p} \otimes \cdots \otimes e_{j_t}) + f(e_{j_1} \otimes \cdots \otimes [e_{j_p}, e_{j_{p+1}}] \otimes \cdots \otimes e_{j_t}) \end{aligned}$$

   It is quite clear that these conditions define f uniquely (if it exists), because (up to exchanging terms and replacing them with commutators) we can assume that the indices are increasing, and f is defined on products with increasing indices. To show that f exists, we prove by induction on (t, N) as above that the value of f on a monomial does not depend on the order of the exchange moves performed to reduce a monomial to one in increasing order. This is true, but not exciting: consider two pairs $j_p > j_{p+1}$ and $j_q > j_{q+1}$ (for now assume $p < p + 1 < q < q + 1$); omitting the $\otimes$ symbol for brevity, we find

$$\begin{aligned} f(e_{j_1} \cdots e_{j_p} e_{j_{p+1}} \cdots e_{j_q} e_{j_{q+1}} \cdots e_{j_t}) &= \\ = f(e_{j_1} \cdots e_{j_{p+1}} e_{j_p} \cdots e_{j_{q+1}} e_{j_q} \cdots e_t) &+ f(e_{j_1} \cdots e_{j_{p+1}} e_{j_p} \cdots [e_{j_q}, e_{j_{q+1}}] \cdots e_t) \\ = f(e_{j_1} \cdots e_{j_{p+1}} e_{j_p} \cdots e_{j_{q+1}} e_{j_q} \cdots e_t) &+ f(e_{j_1} \cdots [e_{j_p}, e_{j_{p+1}}] \cdots e_{j_{q+1}} e_{j_q} \cdots e_t) \\ + f(e_{j_1} \cdots e_{j_{p+1}} e_{j_p} \cdots [e_{j_q}, e_{j_{q+1}}] \cdots e_t) &+ f(e_{j_1} \cdots [e_{j_p}, e_{j_{p+1}}] \cdots [e_{j_q}, e_{j_{q+1}}] \cdots e_t) \end{aligned}$$

   All the terms are well-defined by induction, so making similar transformations in reverse we find that swapping the pair $(p, p + 1)$ or the pair $(q, q + 1)$ first leads to the same result. This leaves us with the case $p + 1 = q$, which is left as an exercise for the dedicated reader. Crucial remark: to prove *this case* one needs that $[\cdot, \cdot]$ satisfies the Jacobi identity.

□

[1] To be precise, I guess one should write $[e_p, e_{p+1}]$ as a linear combination of basis vectors...

4 21.02.2018 – Applications of the PBW theorem

4.1 Character of the Verma module

Remark 4.1. Suppose $\mathfrak{g}$ is equipped with a $\mathbb{Z}$-grading, that is $\mathfrak{g} \cong \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ with $[\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \mathfrak{g}_{m+n}$. Then $U(\mathfrak{g})$ admits a natural $\mathbb{Z}$-grading, given by declaring that a monomial $a_1 \cdots a_s$, where a_i has degree a_i in $\mathfrak{g}$, has degree $\sum n_i$ in $U(\mathfrak{g})$. This works because the tensor algebra $\mathcal{T}(\mathfrak{g})$ is naturally graded, and the ideal we quotient by is generated by relations that are homogeneous in the degree.

Definition 4.2. *Let V be a representation of highest weight $\lambda \in \mathfrak{g}_0^\times$, and write $V = U(\mathfrak{g})v_\lambda = U(\mathfrak{g}_{<0})v_\lambda$. We obtain*

$$V = \bigoplus_{n\in\mathbb{Z}} U(\mathfrak{g})[n]v_\lambda = \bigoplus_{n\geq 0} U(\mathfrak{g}_{<0})[-n]v_\lambda,$$

which we write as $V = \bigoplus_{n\geq 0} V[-n]$. We then define

$$\mathrm{ch}_q(V) = \sum_{n\geq 0} \dim V[-n] \cdot q^n.$$

Theorem 4.3. *The character of the Verma module M_λ is given by*

$$\prod_{j=0}^{\infty} \frac{1}{(1-q^j)^{\dim \mathfrak{g}_j}}.$$

Proof. Fix bases $e_1, \dots, e_N$ of $\mathfrak{g}_{>0}$, $h_1, \dots, h_r$ of $\mathfrak{g}_0$, $f_1, \dots, f_N$ of $\mathfrak{g}_{<0}$. We assume the grading to be nondegenerate (so that the positive and negative parts of the algebra are dual to each other, hence of the same dimension N), and we pick our bases to be homogeneous, that is $e_i, f_i \in \mathfrak{g}_{\pm\delta(i)}$ (e_i is homogeneous of degree $\delta(i)$, f_i is homogeneous of degree $-\delta(i)$). We obtain:

$$\begin{aligned}
\mathrm{ch}_q(M_\lambda) &= \sum_{n\geq 0} \dim M_\lambda[-n]\, q^n = \\
&= \sum_{n} \dim U(\mathfrak{g}_{<0})[-n]\, q^n \\
&= \sum_{n\geq 0} \# \left\{ n_1, \dots, n_N \geq 0 \mid \sum_{\ell} \delta(\ell) n_\ell = n \right\} q^n \\
&= \sum_{n_1,\dots,n_N\geq 0} q^{\delta(1)n_1 + \dots + \delta(N)n_N} \\
&= \prod_{i=1}^{N} \frac{1}{1-q^{\delta(i)}} \\
&= \prod_{j=0}^{\infty} \frac{1}{(1-q^j)^{\dim \mathfrak{g}_j}}
\end{aligned}$$

□

Remark 4.4. Recall that the PBW theorem implies that a basis of $U(\mathfrak{g})$ is given by the set of monomials $f_{i_1} \cdots f_{i_s} h_{k_1} \cdots h_{k_p} e_{j_i} \cdots e_{j_t}$, with $i_1 \leq \dots \leq i_s, k_1 \leq \dots \leq k_p, j_1 \leq \dots \leq j_t$. From this one obtains

$$\begin{aligned}
U(\mathfrak{g}) &= U(\mathfrak{g})\mathfrak{g}_{>0} \oplus U(\mathfrak{g}_{\leq 0}) \\
&= U(\mathfrak{g})\mathfrak{g}_{>0} \oplus \mathfrak{g}_{<0}U(\mathfrak{g}_{\leq 0}) \oplus U(\mathfrak{g}_0) \\
&= U(\mathfrak{g})\mathfrak{g}_{>0} \oplus \mathfrak{g}_{<0}U(\mathfrak{g}_{\leq 0}) \oplus S(\mathfrak{g}_0),
\end{aligned}$$

where $U(\mathfrak{g}_0)$, since $\mathfrak{g}_0$ is abelian, is nothing but the symmetric algebra $S(\mathfrak{g}_0)$. Let now $\lambda \in \mathfrak{g}_0^\times$. Extend λ to $S(\mathfrak{g}_0) \to \mathbb{C}$. The previous decomposition then allows us to extend λ to $U(\mathfrak{g})$, by declaring that it acts trivially on $U(\mathfrak{g})\mathfrak{g}_{>0} \oplus \mathfrak{g}_{<0}U(\mathfrak{g}_{\leq 0})$ and as itself on $S(\mathfrak{g}_0)$.

4.2 An invariant bilinear form

Remark 4.5. Let $n > 0$: then $U(\mathfrak{g})[n] \subset U(\mathfrak{g})\mathfrak{g}_{>0}$ (because I need at least one positive factor in order to have positive degree). Similarly, $U(\mathfrak{g})[-n] \subseteq \mathfrak{g}_{<0}U(\mathfrak{g})$, and therefore λ acts as zero on $U(\mathfrak{g})[n]$ and on $U(\mathfrak{g})[-n]$; in other words, λ is concentrated in degree 0.

Remark 4.6. We have defined

$$M_\lambda^+ = \frac{U(\mathfrak{g})}{U(\mathfrak{g})\left(\mathfrak{g}_{>0} \oplus \{h - \lambda(h) \mid h \in \mathfrak{g}_0\}\right)} \cong U(\mathfrak{g}_{<0})1_\lambda^+;$$

we called it M_λ, but we now add a superscript $+$ to remind ourselves that these are highest weight representations. We could equally well have defined the lowest weight Verma module,

$$M_\lambda^- = \frac{U(\mathfrak{g})}{U(\mathfrak{g})\left(\mathfrak{g}_{<0} \oplus \{h - \lambda(h) \mid h \in \mathfrak{g}_0\}\right)} \cong U(\mathfrak{g}_{>0})1_\lambda^-.$$

Definition 4.7. *For any $A = a_1 \dots a_k \in U(\mathfrak{g})$ we set*

$$s(A) = (-1)^k a_k \cdots a_1.$$

Proposition 4.8. *There exists a unique $\mathfrak{g}$-invariant bilinear form $(\cdot \mid \cdot) : M_{-\lambda}^- \times M_\lambda^+ \to \mathbb{C}$ such that $(1_{-\lambda}^-, 1_\lambda^+) = 1$. Here $\mathfrak{g}$-invariant means $(au|v) + (u|av) = 0$, for all $a \in \mathfrak{g}$ and for all $u \in M_{-\lambda}, v \in M_\lambda$. Furthermore, under this pairing, $M_{-\lambda}^-[n]$ is orthogonal to everything but $M_\lambda^+[-n]$.*

Proof.

$$\left(e_{i_1} \cdots e_{i_s} 1_{-\lambda}^- \mid f_{j_1} \cdots f_{j_t} 1_\lambda^+\right) = (-1)^s (1_{-\lambda}^- \mid e_{i_s} \cdots e_{i_1} f_{j_1} \cdots f_{j_t} 1_\lambda^+);$$

using our previous decomposition of $U(\mathfrak{g})$, write $e_{i_s} \cdots e_{i_1} f_{j_1} \cdots f_{j_t} \in U(\mathfrak{g})$ as $A_i e_i + f_j B_j + p(h_1, \dots, h_r)$. A term $A_i e_i$ kills 1_λ^+; a term $f_j B_j$ can be taken back to the left hand side of the scalar product, and it kills $1_{-\lambda}^-$. Finally, $p(h_1, \dots, h_r)1_\lambda^+ = p(\lambda(h_1), \dots, \lambda(h_r))1_\lambda^+$. The bilinear form is therefore given by $(A1_{-\lambda}^- | B1_\lambda^+) = \lambda(s(A)B)(1_{-\lambda}^- | 1_\lambda^+) = \lambda(s(A)B)$. □

Exercise 4.9. *Check that this bilinear pairing is well-defined, that is, independent of the representation of basis vectors as $A1_{-\lambda}^-$, $B1_\lambda^+$.*

Remark 4.10. $\lambda \mapsto (A1_{-\lambda}^-, B1_\lambda^+)$ is polynomial in λ.

Theorem 4.11. *Suppose $\mathfrak{g}$ is nondegenerate. Then $\forall n \geq 0$ the pairing*

$$(\cdot \mid \cdot) : M_{-\lambda}^-[n] \times M_\lambda^+[-n] \to \mathbb{C}$$

is nondegenerate for λ in a Zariski-open (which may depend on n).

Remark 4.12. By assumption, since $\forall n > 0$ the pairing $\mathfrak{g}_n \times \mathfrak{g}_{-n} \to \mathbb{C}$ is nondegenerate (for generic λ), we also obtain that (for generic λ) the pairing

$$\mathfrak{g}_{0<\cdot\leq n} \times \mathfrak{g}_{-n\leq\cdot<0} \to \mathbb{C}$$

is nondegenerate. Hence we can fix dual bases $e_1, \dots, e_k$ and $f_1, \dots, f_k$, and the induced pairing on symmetric algebras $S^\bullet(\mathfrak{g}_{0<\cdot\leq n})$, $S^\bullet(\mathfrak{g}_{-n\leq\cdot<0}) \to \mathbb{C}$ is also nondegenerate. We normalize in such a way that the sets

$$\left\{e_1^{n_1} \dots e_k^{n_j}\right\}, \quad \left\{\frac{f_1^{n_1} \cdots f_k^{n_k}}{n_1! \cdots n_k!}\right\}$$

form dual bases; the pairing that achieves this duality is given by

$$(a_1 \cdots a_s \mid b_1 \cdots b_t) = \delta_{s,t} \sum_{\sigma \in S_s} \lambda([a_1, b_{\sigma(1)}]) \cdots \lambda([a_s, b_{\sigma(s)}]);$$

the presence of $\sum_{\sigma\in S_s}$ explains the need for the factorials in the denominator. The pairing

$$(\cdot \mid \cdot)^0_{\lambda,n} : S^s(\mathfrak{g}_{>0})[n] \times S^s(\mathfrak{g}_{<0})[-n] \to \mathbb{C}$$

is nondegenerate for the same λ as before.

Proof. Fix $n \geq 0$ and consider

$$(\cdot \mid \cdot) : U(\mathfrak{g}_{>0})[n] \times U(\mathfrak{g}_{<0})[-n] \to \mathbb{C},$$

where we have replaced $M^-_{-\lambda}[n]$ with $U(\mathfrak{g}_{>0})[n]$, and similarly for $M^+_\lambda[-n]$ (i.e. 'we omit the $1^\pm_{\pm\lambda}$'). We want to show that it is nondegenerate; we do it the good old way of writing down the matrix and computing the determinant. The generic entry of the matrix is $\left(e_{i_1}\cdots e_{i_s}1^-_{-\lambda} \mid f_{j_1}\cdots f_{j_t}1^+_\lambda\right)$, where $\sum\delta(i_m) = n$ and $\sum\delta(j_m) = n$. Now the determinant of this gigantic matrix is a polynomial in the $\lambda(h_j)$; to show that it's generically nonzero, it suffices to show that it's not identically zero. In order to do this, we compute the term of highest degree and show that it doesn't vanish. For a single term

$$\left(1^-_{-\lambda} \mid e_{i_s}\cdots e_{i_1}f_{j_1}\ldots f_{j_t}1^+_\lambda\right),$$

the number of λ's is the number of h's that are obtained by swapping the e's and the f's. In particular, there are at most $\min\{s,t\}$ factors λ, and for $s=t$ the term of degree s is

$$\sum_{\sigma\in S_t}\lambda([e_{i_1}, f_{j_{\sigma(1)}}])\cdots\lambda([e_{i_s}, f_{j_{\sigma(s)}}]).$$

Permuting bases if necessary, we can assume that our matrix is organized in blocks, where the block in position (s,t) is formed by those basis elements with given values of s and t. In block (s,t), all the entries are polynomials in λ of degree at most $\min\{s,t\}$. It follows that the determinant of the matrix is of the form

$$\prod_{s=1}^n \det\left(\cdot \mid \cdot\right)^0_{\lambda,n}\big|_{S^s(\mathfrak{g}_{>0})[n]\times S^s(\mathfrak{g}_{<0})[-n]\to\mathbb{C}} + \text{lower degree terms},$$

which is nonzero for generic λ for what we've seen before. Notice that the term of maximal degree essentially comes from the generalized diagonal, and has degree $1\dim(s=1)+2\dim(s=2)+\cdots$ ∎

Remark 4.13. Consider again $(\cdot\mid\cdot)_\lambda : M^-_{-\lambda}\times M^+_\lambda \to \mathbb{C}$. Consider the left- and right- kernels of this form,

$$J^+_\lambda \subset M^+_\lambda, \quad J^-_\lambda \subset M^-_{-\lambda}$$

- Generically, $J^\pm_\lambda$ are zero, but for special λ they could be nontrivial.
- As the form is invariant, $J^\pm_\lambda$ is a submodule of $M^\pm$, and in fact it is a *proper* submodule, because 1 pairs nontrivially with 1 on the other side.
- $J^+_\lambda = \bigoplus J^+_\lambda[n]$, that is, J^+_λ is graded (because the form is of degree 0)

Theorem 4.14. *1. $J^+_\lambda \subset M^+_\lambda$ is a proper* ***maximal*** *submodule, so $V^+_\lambda = M^+_\lambda/J^+_\lambda$ is an irreducible representation of highest weight λ*

2. J^+_λ is the unique maximal proper submodule that is graded. In general, there might well be other maximal proper submodules that are not graded.

3. If there exists $L\in\mathfrak{g}$ such that $[L,\cdot] =$ degree (that is $[L,a] = na$ for all $a\in\mathfrak{g}[n]$), then J^+_λ is the unique maximal proper submodule of M^+_λ. More precisely, all submodules are graded.

Remark 4.15. Part (3) of the theorem applies to the Virasoro algebra; we shall show that (for the Heisenberg algebra) there are in fact other maximal proper submodules.

5 27.02.2018 – The kernel of the natural pairing $(\cdot \mid \cdot)_\lambda$

5.1 Previously...

1. A Lie algebra $\mathfrak{g}$ is said to be **graded** if

$$\mathfrak{g} \cong \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$$

where $\mathfrak{g}_0$ is abelian and $\mathfrak{g}_n$ is finite-dimensional for every $n \in \mathbb{Z}$. It is **nondegenerate** if the pairing

$$\begin{array}{ccc} \mathfrak{g}_{-n} \times g_n & \to & \mathbb{C} \\ (a,b) & \mapsto & \lambda([a,b]) \end{array}$$

is nondegenerate for λ generic.

2. The **Verma module** of highest weight λ is

$$M_\lambda^+ = \frac{U(\mathfrak{g})}{U(g)\left(\mathfrak{g}_{>0} + \{h - \lambda(h) \mid \lambda \in g_0\}\right)};$$

similarly, we have $M_{-\lambda}^-$.

3. We have seen that there exists a unique pairing $(\cdot|\cdot) : M_{-\lambda}^- \times M_\lambda^+ \to \mathbb{C}$ that is $\mathfrak{g}$-invariant, nondegenerate and normalized in such a way that $(1_{-\lambda}^-|1_\lambda^+) = 1$. It is defined by

$$(A1_{-\lambda}^-|B1_\lambda^+) = \lambda(s(A)\cdot B),$$

where s is the antipode $s(a_1 \dots a_s) = (-a_s)(-a_{s-1})\cdots(-a_1)$.

4. λ can be extended to a map $\mathfrak{g} \to \mathbb{C}$ by setting $\lambda(\mathfrak{g}_{>0}) = 0$ and $\lambda(\mathfrak{g}_{<0}) = 0$; it can be further extended to $U(\mathfrak{g})$ by writing

$$U(\mathfrak{g}) = (\mathfrak{g}_{<0}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}_{>0}) \oplus U(\mathfrak{g}_0),$$

observing that $U(\mathfrak{g}_0) = S(\mathfrak{g}_0)$, and setting λ to be trivial on $(\mathfrak{g}_{<0}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}_{>0})$ and to be the obvious polynomial map on $S(\mathfrak{g}_0)$.

Remark 5.1. One can define a pairing on all of $U(\mathfrak{g})$ by the formula $(A|B)_\lambda = \lambda(s(A)B)$. This pairing factors through

$$J_\lambda^+ = U(\mathfrak{g})\left(\mathfrak{g}_{>0} + \{h - \lambda(h)|h \in \mathfrak{g}_0\}\right)$$

on the right, and similarly it factors through the analogous J_λ^- on the left.

5. We have also proven the following theorem:

Theorem 5.2. *If $\mathfrak{g}$ is nondegenerate, for every $n \geq 0$ the pairing $M_{-\lambda}^-[n] \times M_\lambda^+[n] \to \mathbb{C}$ is nondegenerate for a Zariski-open set of λs. This open set may depend on n.*

Corollary 5.3. *The pairing*

$$M_{-\lambda}^- \times M_\lambda^+ \to \mathbb{C}$$

is nondegenerate for λ sufficiently generic (i.e. lying in the intersection of all the previous Zariski opens).

5.2 Properties of $(\cdot \mid \cdot)_\lambda$

Define

$$J_\lambda^+ := \text{right kernel of } (\cdot \mid \cdot)_\lambda \subseteq M_\lambda^+.$$

5.2.1 Basic observations

1. J^+_λ is a submodule of M^+_λ (because $(\cdot \mid \cdot)_\lambda$ is $\mathfrak{g}$-invariant)

2. J^+_λ is graded: $J^+_\lambda = \bigoplus_{n\geq 0} J^+_\lambda[-n]$ (because $(\cdot \mid \cdot)_\lambda$ is of degree 0)

3. J^+_λ is a proper submodule, because $1^+_\lambda \notin J^+_\lambda$

4. As a consequence, $V^+_\lambda := M^+_\lambda / J^+_\lambda$ is a (nonzero) graded $\mathfrak{g}$-module. This is still a highest-weight representation; we shall show that it is irreducible.

5.2.2 Structure theorem

Theorem 5.4. *The following hold:*

1. *J^+_λ is a maximal proper submodule, i.e. V^+_λ is irreducible.*

2. *J^+_λ is the unique maximal proper graded submodule of M^+_λ. (In particular, since highest weight representations are graded, V^+_λ is the unique irreducible representation of highest weight λ).*

3. *In general, there can be other maximal proper submodules of M^+_λ.*

4. *If there exists $L \in \mathfrak{g}$ such that $[L, \cdot] = \text{degree}$, then all submodules $U \subset M^+_\lambda$ are graded, hence J^+_λ is the unique maximal proper submodule of M^+_λ.*

Corollary 5.5. *For generic λ, the Verma module $M^+_\lambda = V^+_\lambda$ is irreducible.*

Proof. We know that for generic λ the natural pairing $(\cdot \mid \cdot)_\lambda$ is nondegenerate, hence $J^+_\lambda = (0)$. □

Proof of Theorem 5.4. 1. Let W be a $\mathfrak{g}$-submodule of M^+_λ such that $J^+_\lambda \subset W \subsetneq M^+_\lambda$. We want to prove that $W = J^+_\lambda$. Let $w \in W$; we can write $w = w_0 + w_{-1} + \cdots + w_{-m}$, where each w_i belongs to $M^+_\lambda[-i]$. Suppose by contradiction that we can find a w in $W \setminus J^+_\lambda$; we choose such a w with m minimal. Notice that must have $m \geq 1$, for otherwise w would be a (nonzero) multiple of 1^+_λ, from which it would follow that $W = M^+_\lambda$, contradicting our assumption that W is a *proper* submodule.

For every $a \in \mathfrak{g}_j$, where $j > 0$, we know that W contains $aw = aw_0 + aw_{-1} + \cdots + aw_{-m}$. The term aw_{-i} has degree $j - i$, so it is zero whenever $j - i > 0$; in particular, aw has degree $-(m - j)$, so by minimality of m we have $aw \in J^+_\lambda$. As J^+_λ is graded, we have in particular $aw_{-n} \in J^+_\lambda$ for every $a \in \mathfrak{g}_{>0}$. We now show that $w_{-m} \in J^+_\lambda$, that is,

$$\left(e_{i_1} \cdots e_{i_s} 1^-_{-\lambda} \mid w_{-m}\right) = 0 \quad \forall i_1, \ldots, i_s.$$

Indeed, if $s = 0$ the claim follows from the fact that w_{-m} is in degree $-m$ while $1^-_{-\lambda}$ is in degree 0. But if there is at least one e_i (which we can assume is in $\mathfrak{g}_{>0}$), then – using the invariance and moving e_1 to the other side – we obtain

$$\left(e_{i_2} \cdots e_{i_s} 1^-_{-\lambda} \mid e_{i_1} w_{-m}\right) = 0$$

as desired (we have used $e_{i_1} w_{-m} \in J^+_\lambda$). Hence $w_{-m} \in J^+_\lambda$, and (since w_{-m} belongs to $J^+_\lambda \subseteq W$) by difference we obtain that $w_0 + w_{-1} + \ldots + w_{-(m-1)}$ belongs to W. By minimality of m again, this implies that $w_0 + w_{-1} + \ldots + w_{-(m-1)}$ belongs to J^*_λ, and therefore that $w = (w_0 + w_{-1} + \ldots + w_{-(m-1)}) + (w_{-m})$ is also an element of J^+_λ.

2. Let $W = \bigoplus_n W[-n] \subsetneq M^+_\lambda$ be a proper graded submodule. We shall show that $W \subseteq J^+_\lambda$. Notice that $J^+_\lambda \subseteq J^+_\lambda + W$; since J^+_λ and W are both graded and both trivial in degree 0, their sum $J^+_\lambda + W$ is again a *proper, graded* submodule of M^+_λ. By part 1, i.e. the maximality of J^+_λ among all *proper* submodules, this implies $J^+_\lambda + W = J^+_\lambda$, that is $W \subseteq J^+_\lambda$.

4. The usual argument with Vandermonde determinants shows that every submodule is graded. **There is a subtlety**: if $\operatorname{ad} L = [L, \cdot]$ is the degree, then (a) L is in degree 0, because $[L, L] = 0$, and (b) the action of L on a vector $u = u_0 + u_{-1} + \cdots + u_{-n}$ is

$$Lu = (0 + \lambda(L))u_0 + (-1 + \lambda(L))u_{-1} + \cdots + (-n + \lambda(L))u_{-n},$$

because $LA1_\lambda^+ = [L, A]1_\lambda^+ + AL\lambda^+ = \deg(A)A1_\lambda^+ + \lambda(L)A1_\lambda^+$.

3. Let us oonsider the Heisenberg algebra and the Verma module $M_{0,0}^+$. We have seen in an exercise[2] that

$$M_{0,0}^+ = \mathbb{C}[a_{-1}, a_{-2}, a_{-3}, \ldots],$$

where a_m acts as 0 for all $m \geq 0$, while a_{-m} acts naturally on the polynomial algebra ($\forall m > 0$); K also acts trivially. One similarly obtains that $M_{00}^- = \mathbb{C}[a_1, a_2, a_3, \ldots]$ with trivial action of a_{-n} for $n \geq 0$ and of K. The pairing between $P(a_1, a_2, \ldots)$ and $Q(a_{-1}, a_{-2}, \ldots)$ is given by $P(0)Q(0)$; it follows that $J_{0,0}^+ = \langle a_{-1}, a_{-2}, a_{-3}, \ldots \rangle$ (polynomials that vanish at 0); for any value of the variables, we have another maximal subdmoule given by $\langle a_{-1} - \alpha_{-1}, a_{-2} - \alpha_{-2}, \cdots \rangle$.

□

5.3 Category $\mathcal{O}$

Definition 5.6. *A $\mathfrak{g}$-module M is an object of category $\mathcal{O}$ if:*

1. *it is graded (over $\mathbb{C}$), that is:*
   - *(a) $M = \bigoplus_{d \in \mathbb{C}} M[d]$;*
   - *(b) for every n we have $\mathfrak{g}_n \cdot M[d] \subseteq M[d+n]$;*
2. *the degrees are bounded above, that is, there exists $C \in \mathbb{R}$ such that $M[d] \neq 0 \Rightarrow \Re(d) \leq C$, where $\Re(d)$ denotes the real part of the complex number d;*
3. *the degrees d for which $M[d] \neq 0$ belong to a finite number of $\mathbb{Z}_{<0}$-strings (where a $\mathbb{Z}_{<0}$-string is a set of the form $\{d + n \mid n \in \mathbb{Z}_{<0}\}$, where d is a fixed complex number)*

Example 5.7. The following are objects in category $\mathcal{O}$:

1. M_λ^+;
2. graded submodules of M_λ^+;
3. V_λ^+.

Proposition 5.8. *The irreducible modules in category $\mathcal{O}$ are precisely the V_λ^+ for λ ranging over $\mathfrak{g}_0^\times$. Moreover, these modules are pairwise non-isomorphic.*

Definition 5.9. *Given $M \in |\mathcal{O}|$, a **singular vector** of M of weight $\mu \in \mathfrak{g}_0^\times$ is an element $v \in V \setminus \{0\}$ such that*

1. *$a \cdot v = 0 \quad \forall a \in \mathfrak{g}_{>0}$;*
2. *$h \cdot v = \mu(h)v \quad \forall h \in \mathfrak{g}_0$.*

Notice that these are precisely the first two conditions in the definition of a highest weight representation. We shall write $M_\mu^{\mathfrak{g}_{>0}}$ for the set of singular vectors of weight μ in M.

[2] which I did not write down...

Remark 5.10. 1. If $v \in M$ is singular of weight μ, writing $v = \sum v_d$ with $v_d \in M[d]$ we have that every v_d is either zero or singular of weight μ. Equivalently,

$$M_\mu^{\mathfrak{g}_{>0}} \cong \bigoplus_{d \in \mathbb{C}} M_\mu^{\mathfrak{g}_{>0}}[d].$$

2. There is a correspondence

$$\begin{array}{ccc} M_\mu^{\mathfrak{g}_{>0}} & \cong & \mathrm{Hom}_{\mathfrak{g}}(M_\mu^+, M) \\ v & \mapsto & \left(\begin{array}{ccc} M_\mu^+ & \to & M \\ 1_\mu^+ & \mapsto & v \end{array} \right) \end{array}$$

3. In V_λ^+ the only singular vectors are the multiples of 1_λ^+:

$$\left(V_\lambda^+\right)_\mu^{\mathfrak{g}_{>0}} = \delta_{\lambda,\mu} \mathbb{C} 1_\lambda^+.$$

Indeed, take a singular vector v of weight μ in V_λ^+. By irreducibility, the submodule generated by v is V_λ^+, hence v (which we might assume to be homogeneous) must be of degree 0 (otherwise it could not generate the weight-0 part of V_λ^+). It follows that v is a multiple of 1_λ^+, which is singular of weight λ (and of no other weight).

Definition 5.11. *A* ***degree*** *of a module $M \in |\mathcal{O}|$ is a complex number d such that $M[d] \neq (0)$.*

Proof of Proposition 5.8. 1. First we show that V_λ^+ and V_μ^+ are not isomorphic for $\lambda \neq \mu$. A putative isomorphism $V_\lambda^+ \to V_\mu^+$ would carry singular vectors to singular vectors, hence 1_λ^+ to (a scalar multiple of) 1_μ^+. Clearly this can only happen when $\lambda = \mu$.

2. Let $M \in |\mathcal{O}|$ be any module. Let $\hat{d}$ be a degree such that $\Re(\hat{d})$ is maximal (among the degrees). The abelian Lie algebra $\mathfrak{g}_0$ acts on $M[\hat{d}]$, and (since we are over $\mathbb{C}$ and all the operators in $\mathfrak{g}_0$ commute) this action admits a common eigenvector v. Let $\mu : \mathfrak{g}_0 \to \mathbb{C}$ be the weight of this action, that is, $h \cdot v = \mu(h) v$ for all $h \in \mathfrak{g}_0$. We show that v is a singular vector of weight μ: condition (2) in the definition is satisfied by construction, and condition (1) follows from the fact that for every $a \in \mathfrak{g}_n$ with $n > 0$ we have $a \cdot v \in M[\hat{d} + a] = (0)$ (by maximality of $\Re(\hat{d})$). By our previous remarks, this gives a nonzero graded homomorphism $\phi : M_\mu^+ \to M$. If M is irreducible, this homomorphism is surjective (otherwise we would have found a nontrivial proper submodule), so ϕ factors through the quotient by the maximal proper graded submodule, which is J_λ^+. It follows that $M \cong M_\lambda^+ / J_\lambda^+ = V_\lambda^+$ as desired.

□

Example 5.12. Let us take again $\mathfrak{g} = \mathfrak{sl}_2 = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$, where $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$. The three direct summands are the graded pieces of degree $1, 0, -1$ respectively. Pick a weight $\lambda \in \mathfrak{g}_0^\times \cong \mathbb{C}$. What does M_λ^+ look like? We have

$$M_\lambda^+ = \frac{U(\mathfrak{g})}{U(\mathfrak{g})(e, h - \lambda)} = \mathbb{C}[f] \cdot 1_\lambda^+,$$

because if we eliminate e and h from all monomials all that is left are monomials in f. Likewise, one finds $M_{-\lambda}^- = \mathbb{C}[e] 1_{-\lambda}^-$. Let's now describe the pairing $(\cdot \mid \cdot)_\lambda : M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$:

$$\begin{aligned} (e^m 1_{-\lambda}^- \mid f^n 1_\lambda^+)_\lambda &= (-1)^m (1_{-\lambda}^- \mid e^m f^n 1_\lambda^+) \\ &= \delta_{m,n} (-1)^n (1_{-\lambda}^- \mid e^n f^n 1_\lambda^+) \end{aligned}$$

By induction one shows $e f^n 1_\lambda^+ = n(\lambda - n + 1) f^{n-1} 1_\lambda^+$ for all $n \geq 0$; from this, it follows easily that $e^n f^n 1_\lambda^+ = \left(\prod_{i=1}^n i(\lambda - i + 1)\right) 1_\lambda^+ = n!(\lambda)(\lambda - 1)\ldots(\lambda - n + 1)$, from which we finally obtain

$$(e^m 1_{-\lambda}^- \mid f^n 1_\lambda^+)_\lambda = \delta_{m,n} (-1)^n n! \cdot \lambda(\lambda - 1) \cdots (\lambda - n + 1)$$

Hence:

1. if $\lambda \notin \mathbb{Z}_{\geq 0}$, all these pairings are nonzero, and since the graded pieces $M_\lambda^+[n]$ are 1-dimensional this proves that the pairing is nondegenerate. In this case M_λ^+ is irreducible.

2. if $\lambda \in \mathbb{Z}_{\geq 0}$, then $J_\lambda^+ = \mathrm{Span}\left\{ f^n \mid n \geq \lambda + 1 \right\}$. It follows that $V_\lambda^+ = \frac{M_\lambda^+}{J_\lambda^+} = \langle 1, f, \dots, f^\lambda \rangle$, which is the $(\lambda+1)$-dimensional representation of $\mathfrak{sl}_2$ we had already considered in the first lecture.

6 28.02.2018 – Unitary structures

6.1 Unitary structure on $\mathfrak{g}$

Idea (from physics). The vector space V is supposed to be the (Hilbert) space of states, and therefore is equipped with a positive-definite Hermitian inner product. The elements $a \in \mathfrak{g}$ are the physical observables; a good observable should be *self-adjoint* with respect to this inner product, i.e. $A^\dagger = A$.

Example 6.1. $\mathfrak{g} = \mathfrak{sl}_2$ comes into play because of the isomorphism $\mathfrak{sl}_2 \cong \mathfrak{so}_3$ and because we're interested in the observable 'angular momentum'. One takes as fundamental observables/generators of the Lie algebra the three operators L_x, L_y, L_z (components of angular momentum along the three axes); in the unique 2-dimensional representation of $\mathfrak{sl}_2$ (1/2-spin representation), one typically takes these to be represented by the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Definition 6.2. *A* ***unitary structure*** *on $\mathfrak{g}$ is an operator $\dagger : \mathfrak{g} \to \mathfrak{g}$ (usually denoted $a \mapsto a^\dagger$) that satisfies:*

1. $(a^\dagger)^\dagger = a$
2. $[a, b]^\dagger = [b^\dagger, a^\dagger]$
3. $(\lambda a)^\dagger = \overline{\lambda} a^\dagger$
4. *$\dagger$ induces an isomorphism $\mathfrak{g}_n \to \mathfrak{g}_{-n}$*

Example 6.3. All of our standard examples of Lie algebras possess such an anti-involution:

1. $\mathfrak{sl}_2$: one sets $e^\dagger = f, f^\dagger = e, h^\dagger = h$;
2. $\mathfrak{gl}_n$: one can take $A^\dagger$ to be the conjugate transpose of A;
3. Heisenberg: $a_n^\dagger = a_{-n}$, $K^\dagger = K$;
4. Virasoro: $L_n^\dagger = L_{-n}$, $C^\dagger = C$;
5. Kac-Moody $\hat{\mathfrak{g}}$: $(at^n)^\dagger = a^\dagger t^{-n}$

Remark 6.4. The operator $\dagger$ satisfies $\dagger^2 = 1$, so $\mathfrak{g}$ decomposes as the direct sum of two (real) subspaces $\mathfrak{g}^+, \mathfrak{g}^-$ on which $\dagger$ acts as the identity (resp. as minus the identity). Multiplying by i exchanges $\mathfrak{g}^+, \mathfrak{g}^-$. It follows that $\mathfrak{g} = \mathfrak{g}^- \oplus i\mathfrak{g}^- = \mathfrak{g}^- \otimes_{\mathbb{R}} \mathbb{C}$. Notice that $\mathfrak{g}^-$ is a Lie subalgebra:

$$[a, b]^\dagger = [b^\dagger, a^\dagger] = [-b, -a] = [b, a] = -[a, b].$$

6.2 Unitary structure on a representation V

Let V be a $\mathfrak{g}$-module in category $\mathcal{O}$.

Definition 6.5. *A unitary structure on V is a Hermitian product*

$$(\cdot \mid \cdot) : V \times V \to \mathbb{C}$$

such that

1. $(au \mid v) = (u \mid a^\dagger v)$;
2. $(\cdot \mid \cdot)$ *is positive definite.*

Remark 6.6. The (almost) universal convention in physics is that Hermitian products should be linear in their *right* argument and anti-linear in their *left* one.

Problem 6.7. *Given $\lambda \in \mathfrak{g}_0^\times$, consider the $\mathfrak{g}$-module M_λ^+ (or any representation of highest weight λ, that is, a quotient of M_λ^+; in particular, we're interested in V_λ^+).*

1. *Does this $\mathfrak{g}$-module admit a Hermitian product for which $\dagger$ is the adjunction map*[3]*?*
2. *is this Hermitian product nondegenerate?*
3. *is it positive definite? That is, is it a unitary structure?*

Remark 6.8. Part (3) is much harder; we'll see a complete answer in the case of the Virasoro algebra.

Definition 6.9. *A weight $\lambda \in \mathfrak{g}_0^\times$ is said to be **real** if, writing $\mathfrak{g}_0 = \mathfrak{g}_0^+ \oplus \mathfrak{g}_0^-$, one has $\lambda(\mathfrak{g}_0^+) \subseteq \mathbb{R}$ (and therefore $\lambda(\mathfrak{g}_0^-) \subseteq i\mathbb{R}$).*

Proposition 6.10.

1. *A necessary condition for question (1) to have a positive answer is that λ be a real weight.*
2. *When λ is real, there exists a contravariant Hermitian product on V, and it is unique up to scalars. We normalize it by $(1_\lambda^+ \mid 1_\lambda^+) = 1$.*
3. *This Hermitian product is nondegenerate if and only if V is the unique irreducible representation of highest weight λ ($V = V_\lambda^+$).*

Remark 6.11. The $\mathfrak{g}^+$-eigenspace is the subspace of self-adjoint operators, which should correspond to physical observables. Therefore it is reassuring that their weight should be a real number!

Proof. 1. Consider

$$(v \mid 1_\lambda^+);$$

if $v \in V[-n]$, this vanishes because $v \in U(\mathfrak{g}_{<0})1_\lambda^+$, hence $(v \mid 1_\lambda^+) = (1_\lambda^+ \mid u^+ 1_\lambda^+)$ with $u^+ \in U(\mathfrak{g}_{>0})$, and this is zero because $u^+ 1_\lambda^+ = 0$. Moreover, if $(1_\lambda^+ \mid 1_\lambda^+) = 0$, then $(\cdot \mid \cdot) \equiv 0$ (proof: $(u \mid v) = (A1_\lambda^+ \mid v) = (1_\lambda^+ \mid A^\dagger v)$. Now 1_λ^+ is orthogonal to anything that does not live in degree 0, so $A^\dagger v$ can be replaced by some multiple of 1_λ^+). Hence we can assume that $(1_\lambda^+ \mid 1_\lambda^+) = 1$. Now pick any $a \in \mathfrak{g}_0^+$:

$$(a1_\lambda^+ \mid 1_\lambda^+) = (1_\lambda^+ \mid a^\dagger 1_\lambda^+) = (1_\lambda^+ \mid a1_\lambda^+) = (1_\lambda^+ \mid \lambda(a)1_\lambda^+) = \lambda(a);$$

on the other hand,

$$(a1_\lambda^+ \mid 1_\lambda^+) = (\lambda(a)1_\lambda^+ \mid 1_\lambda^+) = \overline{\lambda}(a),$$

so (for every $a \in \mathfrak{g}_0^+$) we must have $\lambda(a) = \overline{\lambda(a)}$, i.e. λ is real, as claimed.

[3] such a Hermitian product is said to be **contravariant**

2. We start by considering the form $(\cdot \mid \cdot)$ on all of $U(\mathfrak{g})$ given by $(A \mid B) = \lambda(A^\dagger B)$. We prove that this form is Hermitian. We have

$$(B \mid A) = \lambda(B^\dagger A) = \lambda((A^\dagger B)^\dagger) = \overline{\lambda(A^\dagger B)} = \overline{(A \mid B)},$$

which is what we need to show and where we have used the following lemma. Now we notice that $U(\mathfrak{g})(\mathfrak{g}_{>0} + \{h - \lambda(h) \mid h \in \mathfrak{g}_0\})$ is in the kernel of this Hermitian product, which therefore descends to a Hermitian product on M_λ^+. Uniqueness follows easily from the fact that $\dagger$ is coincides with the adjunction map.

3. If A is in the kernel of this Hermitian product, then $\lambda(B^\dagger A) = 0 \quad \forall B$, which is equivalent to $\lambda(s(B)A) = 0 \quad \forall B$, so $A1_\lambda^+$ is also in the kernel J_λ^+ of $(\cdot \mid \cdot)_\lambda$. It follows that the kernel of $(\cdot \mid \cdot)$ on M_λ^+ is J_λ^+, and the proof is complete.

□

Lemma 6.12. $\lambda(X^\dagger) = \overline{\lambda(X)}$.

Proof. Write $U(\mathfrak{g}) = (\mathfrak{g}_{<0}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}_{>0}) \oplus S(\mathfrak{g}_0)$. Since $\dagger$ exchanges $\mathfrak{g}_{<0}U(\mathfrak{g}), U(\mathfrak{g})\mathfrak{g}_{>0}$ and λ is zero on both of these, we only need to consider $X \in S(\mathfrak{g}_0)$. Write $X = \sum_{I \text{ multi-index}} a_I h^I$; since we can choose the h_i to be a basis of $\mathfrak{g}_0^+$ over $\mathbb{R}$ (recall that $\mathfrak{g}_0^+$ generates $\mathfrak{g}_0$ over $\mathbb{C}$), the claim now follows easily:

$$\lambda(X) = \sum_{I \text{ multi-index}} a_I \lambda(h)^I$$

and

$$\lambda(X^\dagger) = \sum_{I \text{ multi-index}} \overline{a_I}\,\overline{\lambda(h)}^I = \overline{\lambda(X)}.$$

In all these computations, $h^I := \prod_{i=1}^n h_i^{I_i}$ and $\lambda(h)^I := \prod_i \lambda(h_i)^{I_i}$. □

Consider the Virasoro algebra. We have $\mathfrak{g}_0 = \langle L_0, C\rangle$, $L_0^\dagger = L_0$, $C^\dagger = C$, and a real weight λ is determined by its values (h, c) on L_0 and on C. The questions are:

1. for which (c, h) is $M_{c,h} \neq V_{c,h}$? That is, for which weights is the Verma module degenerate?

2. For which (c, h) is the Hermitian form induced on $V_{c,h}$ positive definite?

We show that there is at least one pair (c, h) for which $V_{c,h}$ is positive definite; we start with an analogous observation for the Heisenberg algebra. This is the Sugawara construction:

Example 6.13. The Heisenberg Lie algebra $\mathcal{A}$ admits an irreducible representation on the Fock space $B_\mu = \mathbb{C}[x_1, x_2, \ldots]$, where:

1. the action of a_n is $\frac{\partial}{\partial x_n}$;

2. the action of a_{-n} is nx_n;

3. the action of a_0 is through μ;

4. the action of K is trivial.

The claim is that (for μ real) $\mathcal{B}$ is a unitary representation. Let's compute the Hermitian product explicitly:

$$\begin{aligned}
\left(x_1^{m_1} \cdots x_s^{m_s} \mid x_1^{n_1} \cdots x_t^{n_t}\right) &= \left((a_{-1}/1)^{m_1} \cdots (a_{-s}/s)^{m_s} \mid x_1^{n_1} \cdots x_t^{n_t}\right) \\
&= \frac{1}{\prod i^{m_i}} \left(1 \mid \left(\frac{\partial}{\partial x_1}\right)^{m_1} \cdots \left(\frac{\partial}{\partial x_s}\right)^{m_s} x_1^{n_1} \cdots x_t^{n_t}\right) \\
&= \frac{\delta_{s,t}\delta_{m_1,n_1} \cdots \delta_{m_s,n_s}}{\prod i^{m_i}} \left(1 \mid m_1! m_2! \cdots m_s! 1\right) \\
&= \frac{\delta_{s,t} \prod_i \delta_{m_i,n_i} n_i!}{\prod i^{m_i}}
\end{aligned}$$

This means that, up to rescaling, the monomial basis is orthonormal, and therefore B_μ is unitary. Notice that the conditions $s = t, m_1 = n_1$, etc, all come from the fact that if they are *not* satisfies, then either one variable or one derivation is left, and since the adjoint of a variable is a derivation in both cases we end up with an expression that involves the derivative of a constant, that is, zero.

We now try to find a unitary representation of the Virasoro algebra:

Example 6.14. $\mathcal{A} = \mathbb{C}[t^{\pm 1} \oplus \mathbb{C}K$, where $a_n = t^n$. Then $W = \mathbb{C}[t^{\pm 1}]\frac{\partial}{\partial t}$, which acts on $\mathcal{A}$ by derivations. We then constructed $W \rtimes \mathcal{A}$ (the semi-direct product), with bracket

$$[f(t)\frac{\partial}{\partial t}, g(t)] = f(t)g'(t).$$

Recall that we have a basis L_n of W given by $L_n = -t^{n+1}\frac{\partial}{\partial t}$, with commutator $[L_m, L_n] = (m-n)L_{m+n}$. The commutators $[L_n, a_m]$ are given by$-ma_{m+n}$. K is central.

We have also defined the Virasoro algebra $\mathrm{Vir} = \mathbb{C}[t^{\pm 1}]\frac{\partial}{\partial t} \oplus \mathbb{C}C$. Finally, one can construct $\mathrm{Vir} \ltimes \mathcal{A}$, with commutation rules

1. $[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C$;
2. $[a_m, a_n] = m\delta_{m+n,0}K$;
3. $[L_n, a_m] = -ma_{m+n}$;
4. C, K central.

Question: can we extend the action of $\mathcal{A}$ on B_μ to an action of either $W \ltimes A$ or $\mathrm{Vir} \ltimes \mathcal{A}$? That is: we want to construct operators $L_n : B_\mu \to B_\mu$, for $n \in \mathbb{Z}$, such that:

1. $[L_n, a_m] = -ma_{m+n}$;
2. $[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C$ for some C (for $C = 0$ we get an action of $W \ltimes \mathcal{A}$, for $C \neq 0$ one of $\mathrm{Vir} \ltimes \mathcal{A}$)
3. $L_n^\dagger = L_{-n}$, where $\dagger$ is the adjoint with respect to the Hermitian product on B_μ we constructed above.

The first remark is that if L_n exists, then it is uniquely determined by condition (1) up to an additive constant (that is, a constant times the identity). This follows from Schur's lemma: if L'_n, L''_n both satisfy condition (1), then (for all m)

$$[L'_n - L''_n, a_m] = 0,$$

so (since B_μ is an irreducible representation of the Heisenberg algebra), by Schur's lemma we obtain that $L'_n - L''_n$ is a scalar. We now construct the operators L_n as

$$L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{j+n}.$$

We have

$$\begin{aligned}
[a_m, L_n] &= \frac{1}{2}\sum_j [a_m, a_{-j}a_{j+n}] \\
&= \frac{1}{2}\sum_j [a_m, a_{-j}]a_{j+n} + \frac{1}{2}\sum_j a_{-j}[a_m, a_{j+n}] \\
&= \frac{1}{2}\sum_j m\delta_{m,j}a_{j+n} + \frac{1}{2}\sum_j a_{-j}m\delta_{m,-j-n} \\
&= \frac{1}{2}ma_{m+n} + \frac{1}{2}a_{m+n}m = ma_{m+n},
\end{aligned}$$

which is what we want. Similarly, one obtains $[L_m, L_n] = (m-n)L_{m+n}$. Notice that these would be the commutation rules of the Witt algebra. In fact, we should have obtained the Virasoro algebra:

Problem. These operators (at least, L_0) make no sense! We should specify what the infinite sums mean. If $n \neq 0$, a_{-j} and a_{j+n} commute, so we can assume that the terms with positive index are on the right; these act as derivatives, and therefore only a finite number of them will act on any given polynomial. Thus the operators L_n with $n \neq 0$ can be given a meaning, but

$$L_0 = \frac{\mu^2}{2} + \frac{1}{2}\sum_{j>0} j x_j \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{j>0} j \frac{\partial}{\partial x_j} x_j,$$

and the rightmost sum is divergent.

Solution. One introduces the **normal ordered product**:

$$: a_i a_j := \begin{cases} a_i a_j, & \textit{if } i \le j \\ a_j a_i, & \textit{if } j \le i \end{cases}$$

and re-defines L_n to be

$$L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} : a_{-j} a_{j+n} :$$

Now these operators are well-defined (in the physics lingo, we've put all the destruction operators on the right). But we need to recompute the commutators! The commutator $[a_m, L_n]$ is the same as before, because $: a_{-j}a_{j+n} :$ differs from $a_{-j}a_{j+n}$ by a commutator, which is a multiple of K. As K is central, this poses no problem. Now, however, we need to recompute $[L_m, L_n]$, which is trickier. We do a **regularization** procedure: we truncate sums to $j \in [-N, N]$ for some large N, and take the limit $N \to \infty$ only at the end. Let's do this[4]:

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{j=-N}^{N} [L_m, : a_{-j}a_{j+n} :] \\
&= \frac{1}{2}\sum_{j=-N}^{N} [L_m, a_{-j}a_{j+n}] \quad (: a_{-j}a_{j+n} : \text{ and } a_{-j}a_{j+n} \text{ differ at most by a central element}) \\
&= \frac{1}{2}\sum_{j=-N}^{N} ([L_m, a_{-j}]a_{j+n} + a_{-j}[L_m, a_{j+n}]) \quad \text{(by the Leibniz rule for the commutator)} \\
&= \frac{1}{2}\sum_{j=-N}^{N} (j a_{m-j}a_{j+n} + (-j-n)a_{-j}a_{j+n+m}) \quad \text{(by the commutation rule } [L_r, a_s] = -s a_{r+s}) \\
&= \frac{1}{2}\sum_{i=-N-m}^{N-m} (m+i)a_{-i}a_{m+n+i} + \frac{1}{2}\sum_{j=-N}^{N} (-j-n)a_{-j}a_{j+n+m} \quad \text{(setting } j = m+i).
\end{aligned}$$

We now make some remarks. If a_r and a_s commute, $: a_r a_s := a_r a_s$ independently of what r, s are. Furthermore, if $r < s$ then $: a_r a_s := a_r a_s$, so if we want to replace normal products with ordered products we only need to worry about expressions of the form $a_{-r}a_r$ with r negative (these being the only operators that do not commute and for which the first index is larger than the second). For these,

$$a_{-r}a_r =: a_{-r}a_r : + [a_{-r}, a_r] =: a_{-r}a_r : -rK,$$

and since K acts as 1 on B_μ we can simply write $a_{-r}a_r =: a_{-r}a_r : -r$. Consider for example the

[4] from here on I redid the computations myself; any mistakes are entirely due to me

first sum:

$$\begin{aligned}
&\frac{1}{2}\sum_{i=-N-m}^{N-m}(m+i)a_{-i}a_{m+n+i}\\
&\quad=\frac{1}{2}\sum_{i=-N-m}^{-1}(m+i)(:a_{-i}a_{m+n+i}:+\delta_{m+n,0}[a_{-i},a_i])+\frac{1}{2}\sum_{i=0}^{N-m}(m+i)(:a_{-i}a_{m+n+i}:)\\
&\quad=\frac{1}{2}\delta_{m+n,0}\sum_{i=-N-m}^{-1}(m+i)[a_{-i},a_i]+\frac{1}{2}\sum_{i=-N-m}^{N-m}(m+i)(:a_{-i}a_{m+n+i}:)
\end{aligned}$$

Now, however, we know that the sum of *normally ordered* operators is well-defined and independent of N for N large enough (indeed, the operators with positive indices act as derivations, so only finitely many of them survive when applied to a given polynomial), so (in the limit $N\to\infty$) the previous sum can be further rewritten as

$$\frac{1}{2}\delta_{m+n,0}\sum_{i=-N-m}^{-1}(m+i)[a_{-i},a_i]+\frac{1}{2}\sum_{i=-\infty}^{\infty}(m+i):a_{-i}a_{m+n+i}:$$

Similarly, the second sum we need to compute becomes[5]

$$\frac{1}{2}\delta_{m+n,0}\sum_{i=-N}^{-1}(-j-n)[a_{-j},a_j]+\frac{1}{2}\sum_{j=-\infty}^{\infty}(-j-n):a_{-j}a_{m+n+j}:$$

The two infinite sums combine to give

$$\frac{1}{2}\sum_{j=-\infty}^{\infty}(m+j-j-n):a_{-j}a_{m+n+j}:=(m-n)L_{m+n},$$

while the finite ones give

$$\begin{aligned}
&\frac{1}{2}\delta_{m+n,0}\sum_{i=-N-m}^{-1}(m+i)[a_{-i},a_i]+\frac{1}{2}\delta_{m+n,0}\sum_{j=-N}^{-1}(-j-n)[a_{-j},a_j]\\
&\quad=\frac{1}{2}\delta_{m+n,0}\left(\sum_{i=-N-m}^{-1}(m+i)(-i)+\sum_{j=-N}^{-1}(-j-n)(-j)\right)\\
&\quad=\frac{1}{2}\delta_{m+n,0}\left(-\sum_{i=-N-m}^{-1}(m+i)(i)+\sum_{j=-N}^{-1}(j-m)(j)\right),
\end{aligned}$$

where in the last line we have used that $n=-m$ because of the $\delta_{m+n,0}$ factor. Now this finally involves only numbers (as opposed to operators), and a straightforward computation leads to

$$\frac{1}{2}\delta_{m+n,0}\sum_{i=-N-m}^{-1}(m+i)[a_{-i},a_i]+\frac{1}{2}\delta_{m+n,0}\sum_{j=-N}^{-1}(-j-n)[a_{-j},a_j]=\delta_{m+n,0}\frac{m^3-m}{12}.$$

Putting everything together we have shown

$$[L_m,L_n]=(m-n)L_{m+n}+\delta_{m+n,0}\frac{m^3-m}{12};$$

thus, as desired, this is a representation of the Virasoro algebra, with central charge C equal to 1.

[5]this is not quite precise, of course: there is equality only when we apply the operators to any given polynomial, and N is large enough

Remark 6.15. The representation B_μ we constructed is not a quotient of a Verma module. However, 1 is a singular vector: $L_{n>0}1 = 0$, $L_0(1) = \frac{\mu^2}{2} \geq 0$, $C(1) = 1$. Hence the subrepresentation generated by 1 is of highest weight and unitary: for all μ and for $c = 1$ we have constructed the unitary representation we wanted.

7 19.03.2018 – The Fermionic space and $\mathfrak{gl}_\infty$

Problem. Given $h, c \in \mathbb{R}$, is the irreducible representation of highest weight $V_{h,c}$ (of the Virasoro algebra) unitary?

We've seen the Sugawara construction, to get a unitary representation of Vir on B_μ. $1 \in B_\mu$ is a *singular vector* of weight $(h = \mu^2/2, c = 1)$.

Remark 7.1.
1. V unitary implies V completely reducible (given a subrepresentation, one can take the orthogonal complement).
2. V unitary and of highest weight is irreducible. Indeed, there is a surjective map $V \to V_\lambda$, with kernel say K. Then V_λ is isomorphic to $K^\perp$ (because $V_\lambda \cong V/K \cong K^\perp$), hence $V = K \oplus V_\lambda$; since 1 (the highest weight vector) must lie in V_λ, this implies $V = V_\lambda$.
3. the tensor product of two unitary representations is unitary
4. $\lambda_1, \lambda_2 \in \mathfrak{h}^*$ unitary weights (i.e. $V_{\lambda_1}, V_{\lambda_2}$ unitary); then $\lambda_1 + \lambda_2$ is unitary, because one can just consider $V_{\lambda_1} \otimes V_{\lambda_2}$. More precisely, one can take the subrepresentation of $V_{\lambda_1} \otimes V_{\lambda_2}$ generated by $v_{\lambda_1} \otimes v_{\lambda_2}$: this is $V_{\lambda_1+\lambda_2}$, which is then unitary since it is a subrepresentation of a unitary representation. In fact, it is also of highest weight $\lambda_1 + \lambda_2$.
5. As a consequence of the previous remark and of the Sugawara construction, all weights with integral c (and positive h) are unitary.

Exercise 7.2. *(variant of Sugawara's construction) Fix $\lambda \in \mathbb{R}$ and define*

$$L_n^{(\lambda)} = \frac{1}{2} \sum_{j \in \mathbb{Z}} a_{-j} a_{j+n} + i\lambda n a_n, \quad n \neq 0$$

$$L_0^{(\lambda)} = \sum_{j>0} a_{-j} a_{j+n} + \frac{\lambda^2 + \mu^2}{2}$$

as operators on the Fock space B_μ. Prove that the following hold:

- $[L_n^\lambda, a_m] = -m a_{m+n} + i\lambda m^2 \delta_{m+n,0}$
- $[L_m^{(\lambda)}, L_n^{(\lambda)}] = (m-n)L_{m+n} + \delta_{m+n,0} \frac{m^3-m}{12}(1 + 12\lambda^2)$

The conclusion is that this is a unitary representation with central charge $1+12\lambda^2$, and the singular vector 1 has weight $(h, c) = \left(\dfrac{\lambda^2+\mu^2}{2}, 1 + 12\lambda^2\right)$.

Remark 7.3. As a consequence of the previous exercise, we find that all pairs (h, c) such that $h \geq \frac{c-1}{24}, c \geq 1$ correspond to unitary representations. Taking tensor products we can then prove that pairs with $h \geq \frac{\{c\}}{24}$ are unitary.

Remark 7.4. All unitary pairs (h, c) lie in the quadrant $h \geq 0, c \geq 0$. Indeed, let $V_{h,c}$ be unitary. Then $\forall n > 0$ we have

$$\begin{aligned} 0 &\leq \langle L_{-n} v_{hc} \mid L_{-n} v_{hc} \rangle = \langle v_{hc} \mid L_n L_{-n} v_{hc} \rangle = \langle v_{hc} \mid ([L_n, L_{-n}] + L_{-n} L_n) v_{hc} \rangle \\ &= \langle v_{hc} \mid [L_n, L_{-n}] v_{hc} \rangle = (2nh + \frac{n^3 - n}{12} c) \langle v_{hc} \mid v_{hc} \rangle, \end{aligned}$$

which taking $n = 1$ implies $h \geq 0$, and taking $n \gg 1$ implies $c \geq 0$.

Remark 7.5. Combining the various remarks and constructions we've seen so far, we know which weights are unitary with the exception of those pairs (c, h) with $0 \leq c < 1$ and of those with $c > 1$ that do not satisfy the inequality $h \geq \frac{\{c\}}{24}$.

7.1 Virasoro action on the free Fermion

Definition 7.6. *The* ***free Fermion algebra*** *(or* ***Clifford algebra****) is C_δ (where $\delta \in \{0, 1/2\}$) defined as follows:*

- *C_δ is the associative algebra generated by symbols ψ_n for $n \in \delta + \mathbb{Z}$.*
- *the (anti)commutation rules are given by*

$$\{\psi_m, \psi_n\} := \psi_m \psi_n + \psi_n \psi_m = \delta_{m+n,0}$$

When $\delta = 0$ we say that we are in the ***Ramond sector****; when $\delta = 1/2$ we say that we are in the* ***Neveau-Schwarz sector****. Notice that for $\delta = 0$ we have an element ψ_0 that satisfies $\psi_0^2 = \dfrac{1}{2}$*

Definition 7.7. *The* ***fermionic Fock space*** *is*

$$F = F_\delta = \bigwedge \left(\xi_n, n \in \delta + \mathbb{Z}_{\geq 0}\right),$$

that is, the algebra of polynomials on infinitely many ***anticommuting*** *variables.*

There is an action of the Clifford algebra on F given as follows: ψ_n acts as $\frac{\partial}{\partial \xi_n}$ for $n > 0$, ψ_{-n} acts as multiplication by ξ_n for $n > 0$, and $\psi_0 = \frac{1}{\sqrt{2}}\left(\xi_0 + \frac{\partial}{\partial \xi_0}\right)$

Remark 7.8. The derivatives are defined in such a way that $\frac{\partial}{\partial \xi_2}(\xi_2\xi_3) = \xi_3$, but $\frac{\partial}{\partial \xi_2}(\xi_1\xi_2) = -\xi_1$. With this definition,

$$(\psi_n\psi_{-n} + \psi_{-n}\psi_n)(p(\xi_i)) = \frac{\partial}{\partial \xi_n}(\xi_n p) + \xi_n \frac{\partial p}{\partial \xi_n} = p - \xi_n \frac{\partial p}{\partial \xi_n} + \xi_n \frac{\partial p}{\partial \xi_n} = p,$$

so that indeed $\{\psi_n, \psi_{-n}\} = 1$. Moreover,

$$\psi_0^2 = \frac{1}{2}\left(\xi_0 + \frac{\partial}{\partial \xi_0}\right)\left(\xi_0 + \frac{\partial}{\partial \xi_0}\right) = \frac{1}{2}\left(\xi_0 \frac{\partial}{\partial \xi_0} + \frac{\partial}{\partial \xi_0}\xi_0\right) = \frac{1}{2}\left\{\xi_0, \frac{\partial}{\partial \xi_0}\right\} = \frac{1}{2},$$

where we have used $\xi_0^2 = 0$ (which is certainly true in the exterior algebra) and $\left(\frac{\partial}{\partial \xi_0}\right)^2 = 0$.

Remark 7.9. F is a unitary representation with respect to the unique scalar product for which $\{\xi_{i_1} \cdots \xi_{i_s} \mid i_1 < \ldots < i_s\}$ is an orthonormal basis. With this choice of scalar product, adjunction is given by $\psi_n^\dagger = \psi_{-n}$.

Theorem 7.10. *There is a representation of* Vir *on F_δ given by*

$$L_n = \frac{1}{2} \sum_{j \in \delta + \mathbb{Z}} j : \psi_{-j}\psi_{j+n} : + \delta_{n,0} \frac{1 - 2\delta}{16}$$

where the ordered product is given by

$$: \psi_m \psi_n := \begin{cases} \psi_m \psi_n, m \leq n \\ -\psi_n \psi_m, m > n \end{cases}$$

The following commutation rules hold:

- $[\psi_m, L_n] = \left(m + \frac{n}{2}\right)\psi_{m+n}$;
- $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{24}\delta_{m+n,0}$.

Proof. Let's check the first commutation rule.

$$\begin{aligned}
[\psi_m, L_n] &= \left[\psi_m, \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j : \psi_{-j}\psi_{j+n} : + \frac{1-2\delta}{16}\delta_{n,0}\right] \\
&= \left[\psi_m, \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j : \psi_{-j}\psi_{j+n} :\right] \quad \text{because numbers commute with everything} \\
&= \left[\psi_m, \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j \psi_{-j}\psi_{j+n}\right] \quad \begin{array}{c}\text{because an ordered product}\\ \text{and a product differ at most by a number}\end{array} \\
&= \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j(\psi_m\psi_{-j}\psi_{j+n} - \psi_{-j}\psi_{j+n}\psi_m) \\
&= \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j(\{\psi_m, \psi_{-j}\}\psi_{j+n} - \psi_{-j}\{\psi_{j+n}, \psi_m\}) \\
&= \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j(\delta_{j,m}\psi_{j+n} - \psi_{-j}\delta_{j+n,-m}) \\
&= \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} (m\psi_{m+n} + (m+n)\psi_{m+n}) \\
&= \left(m + \frac{n}{2}\right)\psi_{m+n}.
\end{aligned}$$

The second commutation rule is left as an exercise (and is supposed to be harder, because it involves the central charge – so one needs to regularize...). □

Remark 7.11. Notice that $\frac{1-2\delta}{16} = 0$ when $\delta = 1/2$.

7.1.1 Consequences of the theorem

One checks that the adjoint of L_n is

$$\begin{aligned}
L_n^\dagger &= \frac{1}{2}\sum_j j : \psi_{-j-n}\psi_j : + \delta_{n,0}\frac{1-2\delta}{16} \\
&= \frac{1}{2}\sum_j (-j) : \psi_j\psi_{-j-n} : + \delta_{n,0}\frac{1-2\delta}{16} \\
&= \frac{1}{2}\sum_i i : \psi_{-i}\psi_{i-n} : + \delta_{n,0}\frac{1-2\delta}{16} = L_{-n},
\end{aligned}$$

so the action is unitary (with respect to the scalar product defined in remark 7.9).

Remark 7.12. There is a well-defined notion of parity for the elements in F: one has $F_\delta = F_\delta^+ \oplus F_\delta^-$, where F_δ^+ (resp. F_δ^-) is the subspace of polynomials of even (resp. odd) degree. Notice that the Virasoro operators are of even degree (they're essentially quadratic, up to a constant term), so $F_\delta^\pm$ are subrepresentations. Each of them contains a singular vector: 1 is a singular vector in F_δ^+ and ξ_δ is a singular vector in F_δ^-.

Remark 7.13. What is the weight of these representations? The central charge is certainly $\frac{1}{2}$, because that's the coefficient of $\frac{m^3-m}{12}\delta_{m+n,0}$. As for h, we find it as the eigenvalue of L_0 acting on 1 (resp. ξ_0, resp. $\xi_{1/2}$): in the three cases, it is given by

$$L_0 \cdot 1 = \frac{1-2\delta}{16},\ L_0 \cdot \xi_0 = \frac{1}{16}\xi_0,\ L_0 \cdot \xi_{1/2} = 2 \cdot \frac{1}{2} \cdot \frac{1}{2}\psi_{-1/2}\psi_{1/2}\xi_{1/2} = \frac{1}{2}\xi_{1/2},$$

that is, $h = \frac{1-2\delta}{16}, h = \frac{1}{16}, h = \frac{1}{2}$. So:

- $F_0^+ \supset V_{\frac{1}{2},\frac{1}{16}}$, $F_0^- \supset V_{\frac{1}{2},\frac{1}{16}}$
- $F_{1/2}^+ \supset V_{\frac{1}{2},0}$, $F_{1/2}^- \supset V_{\frac{1}{2},\frac{1}{2}}$.

7.2 $\mathfrak{gl}_\infty$

Definition 7.14.

$$\mathfrak{gl}_\infty = \{M \textit{ matrix on } \mathbb{Z}\times\mathbb{Z} : M \textit{ has a finite number of nonzero coefficients}\} = \bigoplus_{i,j\in\mathbb{Z}} \mathbb{C}E_{i,j}.$$

$\mathfrak{gl}_\infty$ has a defining representation on $V = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}v_n$, where the action is $E_{ij}v_n = \delta_{jn}v_i$. From V, one obtains many representations of $\mathfrak{gl}_\infty$ by considering $V^{\otimes n}, S^n(V), \bigwedge^n(V)$, etc. However, we don't like these representations very much, because they tend not to be highest weight representations. On the other hand, $\mathfrak{gl}_\infty$ is an algebra of the kind we've considered in this course: for example, $\mathfrak{gl}_\infty$ is graded, $\mathfrak{gl}_\infty = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}[j]$, where

$$\mathfrak{g}[j] = \{\text{matrices with nonzero entries only on the } j\text{-th diagonal}\}$$

There is a corresponding triangular decomposition $\mathfrak{gl}_\infty = \mathfrak{g}_{>0} \oplus \mathfrak{g}_{<0} \oplus \mathfrak{h}$, where $\mathfrak{g}_{>0}$ (resp. $\mathfrak{g}_{<0}$) is the subspace of upper- (resp lower-) triangular matrices, and $\mathfrak{h}$ is the subspace of diagonal matrices.

Definition 7.15. *A representation of highest weight $\lambda \in \mathfrak{h}^*$ is a representation V such that there exists $v_\lambda \in V$ with the following properties:*

- $\mathfrak{g}_{>0}v_\lambda = 0$*;*
- $hv_\lambda = \lambda(h)v_\lambda \quad \forall h \in \mathfrak{h}$*;*
- $V = U(\mathfrak{g})v_\lambda$*.*

Remark 7.16. The representation $V = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}v_n$ of $\mathfrak{gl}_\infty$ introduced above is not a representation of highest weight (there is no singular vector).

To construct highest weight representations we introduce the following object:

Definition 7.17.

$$\bigwedge^{\infty/2} V = \operatorname{Span}\left\{v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\cdots \,\middle|\, i_0 > i_1 > i_2 > \cdots, i_{k+1} = i_k - 1 \textit{ for } k \gg 0\right\}$$

It is a representation of $\mathfrak{gl}_\infty$ with the action given by

$$A(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\cdots) = A(v_{i_0})\wedge v_{i_1}\wedge v_{i_2}\wedge\cdots + v_{i_0}\wedge A(v_{i_1})\wedge v_{i_2}\wedge\cdots + v_{i_0}\wedge v_{i_1}\wedge A(v_{i_2})\wedge\cdots$$

This action is well-defined, because A has only finitely many nonzero coefficients (hence it acts as 0 on v_n for $n \gg 0$).

Remark 7.18. There is a handy pictorial representation of the basis vectors: if we draw the line $\mathbb{Z}$ in *decreasing* order, a basis vector is a choice of finitely many points on this line, plus a (right) half-line. This looks very much like Dirac's sea: all negative spots are filled, except for finitely many exceptions, and all positive states are empty, except for finitely many. The action of $E_{i,j}$ moves a particle from state j to state i if possible (i.e. if state j is nonempty and i is empty), and otherwise is zero.

8 20.03.2018 – Bosonic-Fermionic correspondence

We decompose $\bigwedge^{\infty/2} V$ as $\bigoplus_{m\in\mathbb{Z}} \bigwedge^{\infty/2,m} V$, where

$$\bigwedge^{\infty/2,m} V = \{v_{i_0} \wedge \cdots \wedge v_{i_n} \wedge \cdots \mid i_0 > i_1 > \cdots, i_k = m - k \ \forall k \gg 0\}$$

We define in particular $\psi_m = v_m \wedge w_{m-1} \wedge w_{m-2} \wedge \cdots \in \bigwedge^{\infty/2,m} V$.

Finally, we define a hermitian scalar product on $\Lambda^{\infty/2,m}$ by declaring that the basis $v_{i_0} \wedge \cdots \wedge v_{i_n} \wedge \cdots$ is orthonormal.

Proposition 8.1. *The natural action of $\mathfrak{gl}_\infty$ on $\Lambda^{\infty/2,m}$ equips this space with the structure of a unitary highest weight representation with highest weight vector ψ_m and highest weight*

$$\omega_m(E_{ii}) = \begin{cases} 1, & \textit{if } i \le m \\ 0, & \textit{otherwise} \end{cases}$$

Proof. **Highest weight vector.** To show that ψ_m is a highest weight vector we need to show that $E_{ij}\psi_m = 0$ for all $i < j$ (in other words, that all upper-triangular matrices kill ψ_m). The action of E_{ij} tries to move particle in state j to state i; however, either there is no particle in state j (hence E_{ij} kills ψ_m), or there is already a particle in state i, hence again E_{ij} kills ψ_m.

Highest weight. A diagonal matrix E_{ii}, applied to ψ_m, sends it to

$$\begin{cases} \psi_m, & \text{if } i \le m \\ 0, & \text{otherwise} \end{cases}$$

Irreducibility. Given any nonzero state, acting with E_{ij} I can move particles to the lowest empty state and produce (a multiple of) ψ_m. Once I have ψ_m, acting with E_{ij} I can get any other state (moving particles to excited states).

Unitarity. Adjunction is given by $E_{ij}^\dagger = E_{ji}$; as already remarked, the scalar product is such that the basis $v_{i_0} \wedge \cdots \wedge v_{i_n} \wedge \cdots$ is orthonormal.

□

Corollary 8.2. *All weights of the form*

$$\lambda = (\cdots a, a, \cdots, a, a + n_1, a + n_1 + n_2, \cdots, a + n_1 + n_2 + \ldots + n_k, b, b, \cdots, b, \cdots),$$

where $a \in \mathbb{R}$, $n_i \in \mathbb{N}$ and $b = \sum_{i=1}^k n_i$, are unitary.

Proof. The weight in question can be written as

$$\begin{aligned} \lambda &= (\cdots, a, a, a, \cdots) + \sum_{i=1}^k (\cdots, 0, 0, \cdots, 0, \underbrace{n_i}_{m_i-\text{th spot}}, n_i, n_i, \cdots) \\ &= a(\cdots, 1, 1, 1, \cdots) + \sum_{i=1}^k n_i \omega_{m_i}, \end{aligned}$$

so it appears in the representation $\mathbb{C}_a \otimes V_{\omega_{m_1}}^{n_1} \otimes \cdots \otimes V_{\omega_{m_k}}^{n_k}$, where $\mathbb{C}_a$ is the 1-dimensional representation on which $A \in \mathfrak{gl}_\infty$ acts as $a\,\mathrm{Tr}(A)$.

To realize a unitary representation of weight λ one can then take

$$V_\lambda = U(\mathfrak{gl}_\infty) \cdot v_\lambda \subset \mathbb{C}_a \otimes V_{\omega_{m_1}}^{n_1} \otimes \cdots \otimes V_{\omega_{m_k}}^{n_k},$$

with highest weight vector $v_\lambda = 1 \otimes v_{\omega_{m_1}}^{\otimes n_1} \otimes \cdots \otimes v_{\omega_{m_k}}^{\otimes n_k}$

□

Exercise 8.3. *If V_λ is a unitary representation of $\mathfrak{gl}_\infty$, then $\lambda = (\cdots, a, a, a, \cdots, b, b, b, b, \cdots)$.*

Definition 8.4. *We let $\overline{\mathfrak{a}_\infty}$ be the algebra of matrices on $\mathbb{Z} \times \mathbb{Z}$ that have nonzero elements only on finitely many diagonals. Formally,*

$$\overline{\mathfrak{a}_\infty} = \left\{ A \;\middle|\; A = \sum_{\substack{n \in I \subset \mathbb{N} \\ I \text{ finite}}} \sum_{j \in \mathbb{Z}} c_j^n E_{j,j+n} \right\}$$

Remark 8.5.
- One of the advantages of working with $\overline{\mathfrak{a}_\infty}$ is that this new algebra contains the identity matrix.
- In $\overline{\mathfrak{a}_\infty}$ everything works well **except** that it doesn't make sense to take traces.

8.0.1 Construction of $\mathfrak{a}_\infty$

We now construct a central extension $\mathfrak{a}_\infty$ of $\overline{\mathfrak{a}_\infty}$ that acts on $\Lambda^{\infty/2}$.

One checks without difficulty that, for $n \neq 0$, when a matrix $A = \sum_{j \in \mathbb{Z}} c_j E_{j,j+n}$ acts on a state we find a **finite** sum. Indeed, say that $n < 0$: then we try to act by moving particles from state j to state $j+n$ (which, in our graphical representation, lies to the **left** of j). Then for $j \gg 0$ there is no particle to move, so the corresponding summand is zero, while for $j \ll 0$ there are particles both in j and in $j+n$, so the corresponding summand is again zero. A similar argument applies for $n > 0$.

Unfortunately, for $n = 0$, when A acts on a state v we find

$$A(v) = \left(\sum_{j \text{ occupied}} c_j \right) v,$$

which is divergent in general. We modify the action of E_{ii} as follows: we introduce a new representation $\tilde{\rho}$ given by

$$\tilde{\rho}(E_{ij}) = \begin{cases} E_{ij}, & \text{if } i \neq j \\ E_{ii}, & \text{if } i = j > 0 \\ E_{ii} - 1, & \text{if } i = j \leq 0 \end{cases}$$

With this definition, when A is diagonal we have

$$\tilde{\rho}(A)(v) = \left(\sum_{j>0 \text{ occupied}} c_j - \sum_{j \leq 0 \text{ empty}} c_j \right) v,$$

which is at least finite. Of course there is the problem of deciding whether $\tilde{\rho}$ is a Lie algebra action. Unfortunately (or fortunately...) it isn't! Let's check this. It will be useful to notice that

$$\tilde{\rho}(E_{ij}) = E_{ij} - \delta_{i,j} \mathbf{1}_{i \leq 0},$$

where $\mathbf{1}_{i \leq 0}$ is 1 if $i \leq 0$ and 0 otherwise. For simplicity we write $\delta_{i,j \leq 0}$ for the last term.

Since $\tilde{\rho}(E_{ij})$ differs from E_{ij} at most by a scalar (which is central), we obtain

$$[\tilde{\rho}(E_{ij}), \tilde{\rho}(E_{hk})] = [E_{ij}, E_{hk}] = \delta_{jh} E_{ik} - \delta_{ik} E_{hj};$$

we should check that this is the same as $\tilde{\rho}([E_{ij}, E_{hk}])$. We compute

$$\tilde{\rho}([E_{ij}, E_{hk}]) = \delta_{jh} E_{ik} - \delta_{ik} E_{hj} + \delta_{i,k} \delta_{j,h \leq 0} - \delta_{j,h} \delta_{i,k \leq 0}$$

which is in general **different** from $[\tilde{\rho}(E_{ij}), \tilde{\rho}(E_{hk})]$. The error term is

$$\alpha = \begin{cases} 1, \text{ if } i = k > 0, j = h \leq 0 \\ -1, \text{ if } i = k \leq 0, j = h > 0 \\ 0, \text{ otherwise} \end{cases}$$

However, this is not too bad: it differs from the correct expression by a central element, so this can be repaired by taking a central extension of $\overline{\mathfrak{a}_\infty}$. We define

$$\mathfrak{a}_\infty = \overline{\mathfrak{a}_\infty} \oplus \mathbb{C}$$

with commutator

$$[A, B]^\sim = AB - BA + \operatorname{tr}(A_{12}B_{21} - A_{21}B_{12}),$$

where $A_{11}, A_{12}, A_{21}, A_{22}$ are the sub-matrices on indices $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, $\mathbb{Z}_{\leq 0} \times \mathbb{Z}_{>0}$, $\mathbb{Z}_{\leq 0} \times \mathbb{Z}_{>0}$, $\mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\leq 0}$. There is a number of properties one should check: let me just list them.

- $A_{12}, A_{21}, B_{12}, B_{21}$ are in $\mathfrak{gl}_\infty$
- $\alpha(A, B) := \operatorname{tr}(A_{12}B_{21} - A_{21}B_{12})$ is a 2-cocycle (necessary to get a central extension)
- $\alpha(A, B)$, evaluated on the elementary matrices E_{ij}, gives back the expression $\delta_{i,k}\delta_{j,h\leq 0} - \delta_{j,h}\delta_{i,k\leq 0}$ *with the opposite sign* (because it should cancel with our previous α)

8.0.2 Remarks on $\mathfrak{a}_\infty$

Let $T^n = \sum_{j\in\mathbb{Z}} E_{j,j+n}$ be the matrix whose only nonzero coefficients are on the n-th diagonal, which is all filled with 1.

All matrices T^i belong to $\mathfrak{a}_\infty$; in $\overline{\mathfrak{a}_\infty}$, $T^iT^j = T^jT^i = T^{i+j}$, but it is not clear that T^i, T^j should commute in $\mathfrak{a}_\infty$. Let's check:

$$\begin{aligned} [T^m, T^n] &= \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} [E_{i,i+m}, E_{j,j+m}]^\sim \\ &= \sum_{i,h\in\mathbb{Z}} (-\delta_{i,h+n>0}\delta_{i+m,h\leq 0} + \delta_{i,h+n\leq 0}\delta_{i+m,h\geq 0}) \end{aligned}$$

Now $\delta_{i,h+n>0}\delta_{i+m,h\leq 0}$ can be nonzero only if $m + n = 0$ (since $h + n = i, h = i + m$), and the previous expression becomes

$$\delta_{m+n,0} \sum_{i\in\mathbb{Z}} (-\delta_{0<i\leq -m} + \delta_{-m<i\leq 0}) = m\delta_{m+n,0}$$

As a consequence, the Heisenberg algebra $\mathcal{A}$ embeds in $\mathfrak{a}_\infty$, with a_n corresponding to T^n (and K corresponding to 1). We know that $\mathcal{A}$ acts on the bosonic Fock space B_μ, in such a way that a_0 acts as μ. On the other hand, $T^0 = \mathrm{id}$ (which corresponds to a_0) acts on $\bigwedge^{\infty/2,m}$ as a scalar (because it's central, even in $\mathfrak{a}_\infty$). How does id act on ψ_m?

$$\mathrm{id} \cdot \psi_m = \left(\sum_{j>0 \text{ full}} 1 - \sum_{j\leq 0 \text{ empty}} 1 \right) \psi_m = m\psi_m$$

In particular, if $\mu = m$ is integral, $a_0 \leftrightarrow \mathrm{id}$ acts in the same way on B_m and on $\bigwedge^{\infty/2,m} V$.

Moreover, B_m is the **unique** irreducible representation of highest weight m (in fact, the Verma module is irreducible: $M_m = V_m$). On the other hand we know that $\psi_m \in \bigwedge^{\infty/2,m}$ is a singular vector of weight $(m, 1)$ (weights for a_0, K). Even better:

Proposition 8.6. $\bigwedge^{\infty/2,m}$ *is irreducible as a representation of $\mathcal{A} \subset \mathfrak{a}_\infty$; it has a singular vector, hence it is a representation of highest weight.*

Proof. There is a map $\sigma : B_m \to F_m$ induced by the universal property of the Verma module. Since B_m is irreducible, σ is injective; we now show that it's also surjective.

Define a **degree**:

- on $B_m = \mathbb{C}[x_1, x_2, \ldots] = \bigoplus_k B_m[k]$: the degree of $x_1^{n_1} \cdots x_s^{n_s}$ is $1 \cdot n_1 + 2 \cdot n_2 + \ldots + sn_s$. The operator $a_n = \frac{\partial}{\partial x_n}$ is of degree $-n$, while a_{-n} is of degree $+n$.
- on F_m: the basis vector $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots$ has degree $(i_0 - m) + (i_0 - (m-1)) + (i_2 - (m-2)) + \cdots$ (this is a finite sum since $i_k = m - k$ for $k \gg 0$). The operator a_n is likewise of degree $-n$.

To check that σ is surjective, it suffices to show that $B_m[n]$ and $F_m[n]$ have the same dimension:

- $\dim B_m[k] = \{n_1 \geq 0, \ldots, n_s \geq 0 \mid n_1 + 2n_2 + \cdots + sn_s = k\} = p(k)$, the number of partitions of k
- $(i_0 - m) + (i_1 - (m-1)) + \ldots + (i_2 - (m-2)) = k$: notice that $i_0 - m \geq i_1 - (m-1) \geq i_2 - (m-2) \geq \ldots$ (indeed, these are the 'distances of a particle from the lowest state it can occupy', and on a drawing it's clear that these are decreasing). Hence we're writing k as sum of non-increasing numbers, and these are again precisely the partitions of k.

□

Corollary 8.7. *There is an isomorphism $B_m \cong F_m := \bigwedge^{\infty/2,m}$, which is usually called the* ***boson-fermion correspondence****.*

8.0.3 Action of the Clifford algebra on the Fermionic space

Let

$$\mathcal{C}\ell = \{\hat{v}_i, \check{v}_i \mid i \in \mathbb{Z}\}$$

be the algebra defined by the relations

$$\begin{cases} \hat{v}_i \hat{v}_j + \hat{v}_j \hat{v}_i = 0 \\ \check{v}_i \check{v}_j + \check{v}_j \check{v}_i = 0 \\ \hat{v}_i \check{v}_j + \check{v}_j \hat{v}_i = \delta_{ij} \end{cases}$$

This is essentially the algebra $C_{1/2}$ from yesterday, up to renumbering.

There is a natural action of $\mathcal{C}\ell$ on $\bigwedge^{\infty/2}$, given by declaring $\hat{v}_i$ to be the creation operator for particles in state i and $\check{v}_i$ to be the corresponding destruction operator. In symbols:

$$\hat{v}_i(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots) = v_i \wedge v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots$$

(where in particular $\hat{v}_i(...) = 0$ if state i is already occupied), and

$$\check{v}_i(v_{i_0} \wedge v_{i_1} \wedge \cdots) = \begin{cases} (-1)^k v_{i_0} \wedge v_{i_1} \wedge \cdots \wedge \widehat{v_{i_k}} \wedge \cdots, & \text{if } i = i_k \\ 0, & \text{if } i \text{ is empty} \end{cases}$$

Exercise 8.8. *Check that this is a representation of $\mathcal{C}\ell$.*

Remark 8.9. $\mathcal{C}\ell$ does **not** act on $\bigwedge^{\infty/2,m}$, because it does not preserve the number of particles. It does, however, act on $F = \bigwedge^{\infty/2} V = \bigoplus_{m \in \mathbb{Z}} F_m$. Since we have isomorphisms $F_m \cong B_m$, we are tempted to introduce

$$B = \bigoplus_{m \in \mathbb{Z}} B_m,$$

which is isomorphic to F as a representation of the Heisenberg algebra $\mathcal{A}$.

One can easily represent the elementary matrices E_{ij} via the creation/destruction operators $\hat{v}_i / \check{v}_j$: one has

$$E_{ij} = \hat{v}_i \check{v}_j - \delta_{i,j \leq 0}.$$

The last sentence in the previous remark leads to defining

Definition 8.10. *The **ordered product** of two operators in $\mathcal{C}\ell$ is*

$$: \hat{v}_i \check{v}_j := \begin{cases} \hat{v}_i \check{v}_j, & \text{if } i \neq j \text{ or } i = j > 0 \\ -\check{v}_j \hat{v}_i, & \text{if } i = j \leq 0 \end{cases}$$

Notice that $\{\hat{v}_i, \check{v}_i\} = 1$, so $-\check{v}_j \hat{v}_i$ is precisely $\hat{v}_i \check{v}_i - 1$.

Definition 8.11. *In order to keep track of which B_m we are working with, since every B_m is $\mathbb{C}[x_1, \ldots, x_s, \ldots]$, we write B as*

$$B = \bigoplus_{m \in \mathbb{Z}} u^m \mathbb{C}[x_1, \ldots, x_s, \ldots] = \mathbb{C}[u^{\pm 1}, x_1, \ldots, x_s, \ldots].$$

Here u is a dummy variable which keeps track of the number of particles/occupied states.

Remark 8.12. The fermionic-bosonic correspondence is such that ψ_m on the fermionic side corresponds to u^m on the bosonic side. Given this, there is a unique map (isomorphism) of representations of the Heisenberg algebra from F to B.

Question 8.13. *The description of the previous remark is not completely satisfactory:*

- *can we write σ in more explicit terms?*
- *on F we have an action of a number of algebras ($\mathfrak{a}_\infty$, $\mathcal{C}\ell$...): how does $\mathcal{C}\ell$ act (via σ) on B? In formulas, we have*

  $$\sigma \hat{v}_i \sigma^{-1} : B \to B$$

  and

  $$\sigma \check{v}_j \sigma^{-1} : B \to B.$$

  *These are linear operators on polynomials: there **should** be a concrete way of describing them!*

9 21.03.2018 – Vertex operators

9.1 From yesterday

We have constructed the total Bosonic space

$$B = \mathbb{C}[z^{\pm 1}, x_1, x_2, \ldots],$$

which admits an action of the Heisenberg algebra $\mathcal{A}$ defined by the rules

- $a_n = \frac{\partial}{\partial x_n}$, $n > 0$
- $a_{-n} = n x_n$
- $a_0 = z \frac{\partial}{\partial z}$ (this acts as multiplication by the degree in z)
- $K = 1$

On the other hand we also have the fermionic space

$$F = \bigwedge^{\infty/2} V$$

on which $\mathfrak{a}_\infty, \mathcal{C}\ell, \mathcal{A}$ all act. The action of $\mathcal{C}\ell$ is given by

$$\hat{v}_i = v_i \wedge, \quad \check{v}_i = \frac{\partial}{\partial v_i},$$

and it induces an action of a_n via T^n (n-th shift), or – explicitly in terms of $\hat{v}_i, \check{v}_j$ –

$$a_n = \sum_{j\in\mathbb{Z}} : \hat{v}_j \check{v_{j+n}}.$$

Finally, we have constructed an isomorphism $\sigma : F \to B$, and we finished the lecture wondering how to carry the action of $\mathcal{C}\ell$ over to B. Concretely, we asked:

1. Given $\psi = v_{i_0} \wedge v_{i_1} \wedge ... \in F$, what is $\sigma(\psi) \in B$?
2. What do $\sigma\hat{v}_i\sigma^{-1} \in \mathrm{End}(B)$ look like? Same question for $\sigma\hat{v}_j\sigma^{-1} \in \mathrm{End}(B)$

9.2 Vertex operators

We consider the generating function

$$X(u) = \sum_{i\in\mathbb{Z}} \hat{v}_i u^i \in \mathrm{End}(F)[[u, u^{-1}]]$$

Definition 9.1. *A* ***quantum field*** *is $X(u) \in \mathrm{End}(F)[[u, u^{-1}]]$ such that the evaluation $X(u)(f)$ lies in $F((u))$ (Laurent series in u) for every $f \in F$.*

We likewise introduce the generating function for the destruction operators,

$$X^*(u) = \sum_{i\in\mathbb{Z}} \check{v}_i u^{-i},$$

which is also a quantum field.

Theorem 9.2. *We have*

$$\sigma X(u)\sigma^{-1} =: e^{\int a(u)du} : z$$

and

$$\sigma X^*(u)\sigma^{-1} = z^{-1} : e^{-\int a(u)du} :$$

These two expressions are denoted by $\Gamma(u)$ (resp. $\Gamma^*(u)$) and are known as ***vertex operators****.*

Remark 9.3. In the previous theorem, $a(u)$ is the generating function of the operators a_n:

$$a(u) = \sum_{n\in\mathbb{Z}} a_n u^{-n-1} : B \to B((u)),$$

and the integral is the formal one,

$$\int a(u)du = -\sum_{n\in\mathbb{Z}\setminus\{0\}} a_n \frac{u^{-n}}{n} + a_0 \log(u).$$

Now we don't quite know what $\log(u)$ means, but for sure $\exp(\log(u)) = u$. Finally, we need to discuss the meaning of the ordered exponential: write

$$\int a(u)du = -\sum_{n>0} \frac{a_n}{n} u^{-n} + \sum_{n>0} \frac{a_{-n}}{n} u^n + a_0 \log u = -\sum_{n>0} \frac{1}{n}\frac{\partial}{\partial x_n} u^{-n} + \sum_{n>0} x_n u^n + a_0 \log u.$$

The ordered exponential is then defined by putting the exponential of the destruction operators on the right and the exponential of the creation operators on the left (and a_0 is central, so we can put it wherever):

$$: e^{\int a(u)du} := u^{a_0} e^{\sum_{n>0} x_n u^n} e^{-\sum_{n>0} \frac{1}{n}\frac{\partial}{\partial x_n} u^{-n}}$$

Notice that when I hit a vector in B_m with the operator $\Gamma(u)$ the factor z shifts the vector to B_{m+1}, so that u^{a_0} acts as u^{m+1}. The generating series of the destruction operators is likewise

$$u^{-a_0} e^{-\sum_{n>0} x_n u^n} e^{\sum_{n>0} \frac{1}{n}\frac{\partial}{\partial x_n} u^{-n}}$$

Since there are only finitely many derivatives that can act nontrivially on a given polynomial, Γ and Γ^* are themselves quantum fields.

Lemma 9.4. *We have the following commutation relations:*

1. (a) $[T^n, X(u)] = u^n X(u)$

   (b) $[T^n, X^*(u)] = -u^n X^*(u)$

2. (a) $[a_n, \Gamma(u)] = u^n \Gamma(u)$

   (b) $[a_n, \Gamma^*(u)] = -u^n \Gamma^*(u)$

Proof. Parts (1) and (2) are conceptually different, in that the former says something about the Fermionic space while the latter concerns the Bosonic space. On the other hand, parts (a) and (b) of (1) and of (2) are completely analogous.

- $$\begin{aligned}[T^n, X(u)] &= \left[\sum_{i\in\mathbb{Z}} :\hat{v}_i \check{v}_{i+n}: , \sum_{j\in\mathbb{Z}} \hat{v}_j u^j\right] \\ &= \left[\sum_{i\in\mathbb{Z}} \hat{v}_i \check{v}_{i+n} , \sum_{j\in\mathbb{Z}} \hat{v}_j u^j\right] \\ &= \sum_{i,j\in\mathbb{Z}} (\hat{v}_i \check{v}_{i+n} \hat{v}_j - \hat{v}_j \hat{v}_i \check{v}_{i+n}) u^j \\ &= \sum_{i,j\in\mathbb{Z}} (\hat{v}_i \check{v}_{i+n} \hat{v}_j + \hat{v}_i \hat{v}_j \check{v}_{i+n}) u^j \\ &= \sum_{i,j\in\mathbb{Z}} \hat{v}_i \{\check{v}_{i+n}, \hat{v}_j\} u^j \\ &= \sum_{i,j\in\mathbb{Z}} \hat{v}_i \delta_{i+n,j} u^j \\ &= \sum_{i\in\mathbb{Z}} \hat{v}_i u^{i+n} = X(u) u^n\end{aligned}$$

  where the first equality follows as usual from the fact that the ordered product differs from the usual product by a central element.

- $$\begin{aligned}[a_n, \Gamma(u)] &= \left[a_n, : e^{\int a(u)du} : z\right] \\ &=: e^{\int a(u)du} : z[a_n, \int a(u)du]\end{aligned}$$

  Here we use two facts: one, $[a_n, \cdot]$ satisfies Leibniz's rule, hence $[a_n, \exp(B)] = B[A_n, B']$. Two, this is not (in general) true for non-commuting operators, but the advantage here is that $[a_n, \cdot]$ is a number, hence commutes with everything and can be pulled out from the series and put on the right. Now, at least for $n \neq 0$,

  $$[a_n, \int a(u)du] = \sum_{i\neq 0} [a_n, a_i] \frac{u^{-i}}{i} = \sum_{i\neq 0} n\delta_{n+i,0} \frac{u^{-i}}{-i} = u^n.$$

  The only problem is with a_0, which commutes with all a_i *but not with* z. Hence

  $$[a_0, \Gamma(u)] = [a_0, : e^{\int a(u)du} : z] =: e^{\int a(u)du} : [a_0, z] =: e^{\int a(u)du} : (a_0 z - z a_0) =: e^{\int a(u)du} : z$$

  because z sends B_m to B_{m+1}, so (on an element $v \in B_m$) $a_0 z$ acts as $(m+1)z$, while $-za_0$ acts as $-zm$.

□

Corollary 9.5. *Conjugating with σ we obtain*

$$[a_n, \sigma X(u)\sigma^{-1}] = u^n \sigma X(u)\sigma^{-1};$$

since we also have

$$[a_n, \Gamma(u)] = u^n \Gamma(u)$$

we deduce that

$$[a_n, \sigma X(u)\sigma^{-1}\Gamma(u)^{-1}] = 0 \quad \forall n \in \mathbb{Z},$$

where the inverse of $\Gamma(u)$ is not really defined, but we don't care too much. In any case, what we want to deduce is that $\sigma X(u)\sigma^{-1}\Gamma(u)^{-1}$ a scalar, and in fact equal to one; we do this below.

Exercise 9.6. *Decide whether $\Gamma(u)$ is invertible as a formal power series.*

More formally, $\sigma X(u)\sigma^{-1}$ and $\Gamma(u)$ have the same commutation rules, so

$$\sigma X(u)\sigma^{-1} = C(u)\Gamma(u)$$

with C that commutes with the Heisenberg algebra. In particular, $C(u)$ acts as a scalar, possibly depending on u, on every B_m (because each B_m is an irreducible $\mathcal{A}$-module of countable dimension, so that Schur's lemma applies). The claim is that $C(u)$ acts as 1 on every B_m, independently of u. In order to show this, we compare

$$\langle z^{m+1} \mid \sigma X(u)\sigma^{-1} z^m \rangle \quad \text{with} \quad \langle z^{m+1} \mid \Gamma(u) z^m \rangle.$$

We have:

$$\begin{aligned}\langle z^{m+1} \mid \sigma X(u)\sigma^{-1} z^m \rangle &= \langle \psi_{m+1} \mid X(u)\psi_m \rangle \\ &= \langle v_{m+1} \wedge v_m \wedge w_{m-1} \wedge w_{m-2} \wedge ... \mid \sum_i \hat{v}_i u^i v_m \wedge w_{m-1} \wedge w_{m-2} \wedge ... \rangle\end{aligned}$$

Since the standard basis is orthonormal, the only summand on the right that contributes to the scalar product is the one corresponding to $\hat{v}_{m+1}$; thus this scalar product evaluates to u^{m+1}.

On the other hand,

$$\begin{aligned}\langle z^{m+1} \mid \Gamma(u) z^m \rangle &= \langle z^{m+1} \mid : e^{\int a(u)du} : z z^m \rangle \\ &= \langle z^{m+1} \mid u^{a_0} e^{\sum_{n>0} x_n u^n} e^{-\sum_{n>0} \frac{1}{n}\frac{\partial}{\partial x_n} u^{-n}} z z^m \rangle\end{aligned}$$

Since $\dfrac{\partial z^{m+1}}{\partial x_n} = 0$, so $e^{-\sum_{n>0} \frac{1}{n}\frac{\partial}{\partial x_n} u^{-n}} z z^m = z^{m+1}$. On the other hand,

$$e^{\sum_{n>0} x_n u^n} = 1 + \text{ polynomials in the } x_i,$$

and all polynomials in x_i are orthogonal to z^{m+1}, so they do not count[6]. Thus this scalar product evaluates to

$$\langle z^{m+1} \mid u^{a_0} z^{m+1} \rangle = \langle z^{m+1} \mid u^{m+1} z^{m+1} \rangle = u^{m+1}.$$

From the equality of these scalar products we deduce $C(u) \equiv 1$, which is what we needed to show.

Exercise 9.7. *Let $X(u), Y(u)$ be quantum fields such that $[a_n, X(u)] = u^n X(u)$ and $[a_n, Y(u)] = u^n Y(u)$ for all $n \in \mathbb{Z}$. Prove or disprove: for all $n \in \mathbb{Z}$, there exists $c_m(u)$ such that $Y(u)|_{B_m} = c_m(u) X(u)|_{B_m}$, or $X(u)|_{B_m} = c_m(u) Y(u)|_{B_m}$*

[6] equivalently: replace this exponential with its adjoint on the left. The adjoint is a certain $\exp \sum_{n>0} \frac{1}{n}\frac{\partial}{\partial x_n}$, which acts as 1 on z^{m+1}

9.3 Schur polynomials

Definition 9.8. *The **elementary Schur polynomials** are the polynomials $S_k(x_1, x_2, \ldots) \in \mathbb{Q}[x_1, x_2, \ldots]$ defined by the generating series*

$$e^{\sum_{j=1}^{\infty} x_j z^j} = \sum_{k=0}^{\infty} S_k(x) z^k.$$

In particular, $S_k(x) = 0$ for $k < 0$.

Example 9.9. It is useful to write the exponential as

$$1 + (x_1 z + x_2 z^2 + x_3 z^3 + \ldots) + \frac{1}{2}(x_1 z + x_2 z^2 + x_3 z^3 + \ldots)^2 + \frac{1}{6}(x_1 z + x_2 z^2 + x_3 z^3 + \ldots)^3 + \cdots$$

From this expression one easily obtains

- $S_0(x) = 1$ (evaluate the series at $z = 0$)
- $S_1(x)$ is the coefficient of z in the exponential, hence it is x_1.
- $S_2(x) = x_2 + \frac{1}{2}x_1^2$
- $S_3(x) = x_3 + x_1 x_2 + \frac{1}{6}x_1^3$

Definition 9.10. *A **partition** $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is a collection of non-increasing natural numbers such that $\lambda_i = 0$ for all sufficiently large i. Given a partition Λ, the Schur polynomial $S_\Lambda(x)$ is the determinant of*

$$\begin{pmatrix} S_{\lambda_1}(x) & S_{\lambda_1+1}(x) & S_{\lambda_1+2}(x) & S_{\lambda_1+3}(x) & \cdots & \cdots \\ S_{\lambda_2-1}(x) & S_{\lambda_2}(x) & S_{\lambda_2+1}(x) & S_{\lambda_2+2}(x) & \cdots & \cdots \\ S_{\lambda_3-2}(x) & S_{\lambda_3-1}(x) & S_{\lambda_3}(x) & S_{\lambda_3+1}(x) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & 1 & \ddots \end{pmatrix},$$

which makes sense because all but finitely many rows of this matrix have a 1 on the main diagonal and 0 to the left of it. In other words, the determinant of this infinite matrix can be defined as being the determinant of the topmost $t \times t$ block, where $t = \min\{j : \lambda_j = 0\}$.

Lemma 9.11.

$$S_k(x) = h_k(y_1, \ldots, y_N),$$

where $x_j = \frac{y_1^j + \cdots + y_N^j}{j}$ and $h_k(y)$ is the sum of all monomials of degree k on N variables.

Proof. We compare the generating functions:

$$\begin{aligned} \sum_{k \geq 0} h_k(y) z^k &= \sum_{k \geq 0} \sum_{\substack{k_i \geq 0 \\ k_1 + \cdots + k_N = k}} y_1^{k_1} \cdots y_N^{k_N} z^k \\ &= \sum_{k \geq 0} \sum_{\substack{k_i \geq 0 \\ k_1 + \cdots + k_N = k}} y_1^{k_1} \cdots y_N^{k_N} z^{k_1 + \cdots + k_N} \\ &= \sum_{k_i \geq 0} (z y_1)^{k_1} \cdots (z y_N)^{k_N} \\ &= \prod_{i=\ell}^{N} \frac{1}{1 - y_\ell z}. \end{aligned}$$

On the other hand, we have
$$\sum_{k\geq 0} S_k(x)z^k = e^{\sum_{j\geq 1} x_j z^j};$$
evaluating at $x_j = \dfrac{y_1^j + \cdots + y_N^j}{j}$ we find
$$\exp\left(\sum_{j\geq 1}\sum_{\ell=1}^{N} \frac{(y_\ell z)^j}{j}\right) = \exp\left(\sum_{\ell=1}^{N}\sum_{j\geq 1} \frac{(y_\ell z)^j}{j}\right) = \exp\left(-\sum_{\ell=1}^{N} \log(1-y_\ell z)\right) = \prod_{\ell=1}^{N} \frac{1}{1-y_\ell z}.$$
□

Theorem 9.12. *We study the bosonic-fermionic correspondence for F_0. Given a basis vector $\psi = v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$, consider the partition $\Lambda = (i_0, i_1+1, i_2+2, \ldots)$: we have*
$$\boxed{\sigma(\psi) = S_\Lambda(x)}$$

For the proof we need the following fact:

Exercise 9.13. *For every $n \times n$ matrix A we have*
$$(\Lambda^n A)(e_1 \wedge e_2 \wedge \cdots \wedge e_n) = \det(A) e_1 \wedge e_2 \wedge \cdots \wedge e_n$$
and
$$\langle e_{j_1} \wedge \cdots \wedge e_{j_k} \bigm| (\Lambda^k A)(e_{i_1} \wedge \cdots \wedge e_{i_k})\rangle = \det A_{JI},$$
where A_{JI} is the $k \times k$ minor with rows $j_1, \ldots, j_k$ and columns $i_1, \ldots, i_k$.

Proof. We compute the generating series (in the new variables y_i)
$$\left\langle 1 \bigm| \exp\left(\sum_{j\geq 1} a_j y_j\right) S_\Lambda(x) \right\rangle;$$
notice the fundantamental fact that $\exp(y\partial/\partial x)p(x) = p(x+y)$ for every polynomial $p(x)$ (Taylor series expansion). Since $a_j = \frac{\partial}{\partial x_j}$, the scalar product is therefore given by
$$\bigl\langle 1 \bigm| S_\Lambda(x_1+y_1, \ldots, x_n+y_n, \ldots)\bigr\rangle = S_\Lambda(y_1, \ldots).$$
It now suffices to compute
$$\left\langle 1 \bigm| \exp\left(\sum_{j\geq 1} a_j y_j\right) \sigma(\psi) \right\rangle,$$
which – in the fermionic space – reads
$$\left\langle v_0 \wedge v_{-1} \wedge v_{-2} \wedge \cdots \bigm| \exp\left(\sum_{j\geq 1} T^j y_j\right) (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots) \right\rangle.$$
Now the exponential acts as a group element, so the previous scalar product is
$$\left\langle v_0 \wedge v_{-1} \wedge v_{-2} \wedge \cdots \bigm| \exp\left(\sum_{j\geq 1} T^j y_j\right) v_{i_0} \wedge \exp\left(\sum_{j\geq 1} T^j y_j\right) v_{i_1} \wedge \exp\left(\sum_{j\geq 1} T^j y_j\right) v_{i_2} \wedge \cdots \right\rangle$$
and
$$\exp\left(\sum_{j\geq 1} T^j y_j\right) = \sum_{k=0}^{\infty} S_k(y) T^k = \begin{pmatrix} S_0(y) & S_1(y) & S_2(y) & \cdots \\ 0 & S_0(y) & S_1(y) & \cdots \\ 0 & 0 & S_0(y) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$
The claim now follows from the linear algebra exercise: the scalar product is the determinant of the submatrix given by the first rows and by the columns indexed by $i_0, i_1, \ldots$, which is by definition $S_\Lambda(y)$. □

10 26.03.2018 – Unitary representations of the Virasoro algebra

10.1 Two questions

Given $c, h \in \mathbb{R}_{\geq 0}$,

- is $M_{c,h} \cong V_{c,h}$? That is, is the Verma module irreducible? In this case, we shall say that $M_{c,h}$ is **nondegenerate**
- is $V_{c,h}$ unitary?

So far we have seen the following:

- there exists a unique Hermitian form
$$(\cdot \mid \cdot)_{c,h} : M_{c,h} \times M_{c,h} \to \mathbb{C}$$
such that $(L_{-n}u \mid v) = (u \mid L_n v)$ and $(1_{c,h} \mid 1_{c,h}) = 1$
- the kernel of $(\cdot \mid \cdot)_{c,h}$ is $J_{c,h} \subsetneq M_{c,h}$, the unique maximal submodule of $M_{c,h}$
- $M_{c,h}[m]$ is orthogonal to $M_{c,h}[n]$ for $m \neq n$

In particular:

- $M_{c,h}$ is nondegenerate iff for all n the determinant $d_n(c,h) = \det\left(\cdot \mid \cdot\right)|_{M_{c,h}[n] \times M_{c,h}[n]}$ is nonzero;
- if $V_{c,h}$ is unitary, then $d_n(c,h) \geq 0 \quad \forall n \geq 0$

10.2 Aims for today

1. Prove the **Kac determinant formula**, which computes $d_n(c,h)$
2. Construct all the unitary representations

10.3 An example: formulas for d_n for small n

- $n = 0$. There is a unique vector, $1_{c,h}$, which by definition has norm 1. Hence $d_0(c,h) = (1 \mid 1) = 1$.
- $n = 1$. $M[1] = \langle L_{-1}1 \rangle$, so
$$d_1(c,h) = \left(L_{-1}1 \mid L_{-1}1\right) = (\mid L_1 L_{-1} 1) = (1 \mid (2L_0 + L_{-1}L_1)1) = (1 \mid 2L_0 1) = 2h$$
In particular, one can only have unitary representations for $h > 0$ (which is something we already knew), and the determinant is zero iff $h = 0$.
- $n = 2$. $M_{c,h}[2] = \langle L_{-2}1, (L_{-1})^2 1 \rangle$. We need to compute the determinant of a 2×2 matrix whose entries are
$$\left((L_{-1})^2 1 \mid (L_{-1})^2 1\right) = 2(1 \mid L_1 L_0 L_{-1} 1) + (1 \mid L_1 L_{-1} L_1 L_{-1}) = 4h(h+1) + (2h)^2 = 4h(2h+1),$$
$$(L_{-2}1 \mid L_{-2}1) = (1 \mid L_2 L_{-2} 1) = (1 \mid (4L_0 + c/2)1) = 4h + \frac{c}{2}$$
and
$$(L_{-2} \mid (L_{-1})^2 1) = (1 \mid L_2 (L_{-1})^2 1) = 6h.$$
Thus $d_2(c,h) = \det\begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 4h(2h+1) \end{pmatrix} = 4h\left(8h^2 + hc - 5h + \frac{c}{2}\right)$. Since we already know $h \geq 0$, this gives a nontrivial condition on (c,h) which corresponds to lying 'outside' a certain hyperbola, which passes through $(1/2, 1/16)$, $(1/2, 1/2)$ and $(0, 5/8)$.

Remark 10.1. $d_n = 0 \Rightarrow d_{n+1} = 0$. Indeed, the Verma module is irreducible iff it contains a nontrivial singular vector v, and the determinant d_n is zero iff $M[n]$ contains a vector which lies in the representation generated by v. Acting with L_{-1} on such vectors, we obtain vectors that lie both in $M[n+1]$ and in the representation generated by v, therefore forcing d_{n+1} to be zero.

10.4 Kac's formula

Theorem 10.2.

$$d_n(c,h) = K_n \prod_{\substack{r,s\geq 1\\ rs\leq n}} (h - h_{r,s}(c))^{p(n-rs)},$$

where

$$K_n = \prod_{r,s=1}^{n} (s!(2r)^s)^{m(r,s)},$$

$$h_{r,s}(c) = \frac{1}{48}\left[(13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}(r^2-s^2) - 24rs - 2 + 2c\right]$$

$$p(n-rs) = \#\{\textit{partitions of } n-rs\}$$

$$m(r,s) = \#\{\textit{partitions of } n \textit{ in which } r \textit{ appears exactly } s \textit{ times}\}$$

10.4.1 Some consequences of Kac's theorem

The first remark one might make is that the formula *seems* to involve some square root of a polynomial in c, but they will have to somehow cancel out since $d_n(c,h)$ is (essentially by definition) polynomial in c and h.

Define

$$\varphi_{r,s}(h,c) = \begin{cases} h - h_{r,r}, & \text{if } r = s \\ (h-h_{r,s})(h-h_{s,r}), & \text{if } r \neq s \end{cases}$$

and observe that

$$d_n(c,h) = K_n \prod_{r\leq s} \varphi_{r,s}(h,c)^{p(n-rs)}$$

since $p(n-rs)$ is symmetric in r,s. One has

$$\varphi_{r,r}(c,h) = h + \frac{1}{24}(c-1)(r^2-1)$$

and

$$\varphi_{r,s}(h,c) = \left(h - \frac{(r-s)^2}{4}\right)^2 + \frac{h}{24}(r^2+s^2-2)(c-1) + \frac{1}{576}(r^2-1)(s^2-1)(c-1)^2 + \frac{1}{48}(r-s)^2(rs+1)(c-1)$$

This expression is horrible, but clearly positive for $h > 0$ and $c > 1$.

This implies, in particular, that for any such pair (c,h) the determinant does not vanish. As a consequence, all the $M_{c,h}$ are nondegenerate for $h > 0, c > 1$, which in turn implies that all these modules are unitary (all the eigenvalues are positive in at least one point, because we have constructed at least one unitary representation. By continuity of the determinant, this implies that all eigenvalues are positive for every $M_{c,h}$). In particular, this 'fills in' the small triangles for which we did not know whether $M_{c,h}$ was unitary or not.

Furthermore, since the set of unitarity is closed, this implies that all the representations with $h = 0, c \geq 1$ and all those with $c = 1, h \geq 0$ are unitary.

The discrete series In the strip $h \geq 0, 0 \leq c \leq 1$ the unitary representations are all degenerate, and are given by

$$c = c_m = 1 - \frac{6}{(m+2)(m+3)} \quad m \geq 0$$

$$h = h_{rs}(c) = \frac{((m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)} \quad \text{for some } 1 \leq s \leq r \leq m+1.$$

Example: representations in the discrete series

- $m = 1, c = 0$. If $m = 0$, the only possible choice for r, s is $(0,0)$, which gives $h = 0$
- $m = 1, c = 1/2$. With $m = 1$, we must choose $\{r, s\}$ in $\{(1,1),(1,2),(2,2)\}$, and the corresponding values of h are 0, 1/2, 1/16.

10.4.2 Proof of Kac's formula

Step 1. Asymptotic expansion of $d_n(c,h)$. We study $d_n(c,h)$ as a polynomial in h; in particular, we are interested in its leading term. We have

$$d_n(c,h) = \det\left(\langle L_{-i_1} \cdots L_{-i_s} 1 \mid L_{-j_1} \cdots L_{-j_t} 1 \rangle\right)_{\underline{i},\underline{j}},$$

where the matrix is indexed by $\underline{i} = (i_1 \geq i_2 \geq \cdots \geq i_s)$ and $\underline{j} = (j_1 \geq j_2 \geq \cdots \geq j_r)$, both partitions of n. Now a single scalar product is

$$\langle L_{-i_1} \cdots L_{-i_s} 1 \mid L_{-j_1} \cdots L_{-j_t} 1 \rangle = \langle 1 \mid L_{i_s} L_{i_1} L_{-j_1} \cdots L_{-j_t} 1 \rangle;$$

now we need to commute all the various L past each other, and every time we commute two adjacent L the number of L's go down by 1. In particular, the maximum number of h's is produced if every L_i is commuted only once with its L_{-i}[7]. The conclusion is that there is a maximal degree M, that appears on the diagonal and (potentially) in some coefficients *above* the diagonal, but certainly in no coefficient *below* it[8]. From this, we obtain that

$$d_n(c,h) = \prod_{\substack{i_1 \leq \cdots \leq i_s \\ i_1 + \cdots + i_s = n}} \langle L_{-i_1} \cdots L_{-i_s} 1 \mid L_{-i_1} \cdots L_{-i_s} 1 \rangle + \text{ terms of lower degree in } h$$

Reparametrizing partitions as $(n_1, \ldots, n_k)$ with $n_1 + 2n_2 + \ldots + kn_k = n$, we have to compute

$$\prod_{\substack{n_1 \geq 0, \ldots, n_k \geq 0 \\ n_1 + 2n_2 + \ldots + kn_k = n}} \langle L_{-1}^{n_1} \cdots L_{-k}^{n_k} 1 \mid L_{-1}^{n_1} \cdots L_{-k}^{n_k} 1 \rangle$$

up to terms of non-maximal order in h. As already observed, the only way for the maximal number of h to appear is for every L_i to commute with L_{-i} (because $[L_m, L_n]$ produces L_{m+n}, which acts as h if $m+n = 0$, but which needs to be commuted past the other L otherwise). Hence the leading term is

$$\prod_{\substack{n_1 \geq 0, \ldots, n_k \geq 0 \\ n_1 + 2n_2 + \ldots + kn_k = n}} \prod_i \langle L_{-i}^{n_i} 1 \mid L_{-i}^{n_i} 1 \rangle,$$

so we want to study

$$\langle L_{-r}^s 1 \mid L_{-r}^s 1 \rangle = \langle 1 \mid L_r \cdots L_r L_{-r} \cdots L_{-r} 1 \rangle.$$

[7] 'ci sono marito e moglie, e non vogliamo rapporti extraconiugali' – 'ma se ci sono tante L_i con lo stesso indice posso farlo in tanti modi...' – 'certo, certo, si può: quella si chiama poligamia, in certi paesi va bene...'

[8] one needs to think about the number of L that survive the commutation relation; an example where M is realized outside the diagonal is $(L_{-2} 1 \mid L_{-1} L_{-1} 1)$

Now $[L_r, L_{-r}^s] = sL_{-r}^{s-1}[L_r, L_{-r}] +$ lower degree terms (because the commutator is a multiple of h, which is central, plus something of lower degree in h; here we use the fact that $[L_r, \cdot]$ is a derivation). Hence $\langle L_{-r}^s 1 \mid L_{-r}^s 1\rangle = s!(2rh)^s +$ lower degree terms.

Combining everything we've proven so far,

$$\begin{aligned}
d_n(c,h) &= \prod_{n=n_1+\cdots+kn_k} \prod_{r=1}^{k} \langle L_{-r}^{n_r} 1 \mid L_r^{n_r} 1\rangle + \text{terms of lower degree} \\
&= \prod_{r=1}^{n} \prod_{\substack{s=1 \\ rs\le n}}^{n} \langle L_{-r}^{n_r} 1 \mid L_r^{n_r} 1\rangle^{\#\text{ partitions of } n \text{ with exactly } s \text{ parts equal to } r} + \text{terms of lower degree} \\
&= \prod_{r=1}^{n} \prod_{rs\le n} (s!(2r)^s)^{m(r,s)} h^{\sum_{rs\le m} sm(r,s)} + \text{ terms of lower degree}
\end{aligned}$$

Now let's check that the *degree* of $d_n(c,h)$ is the one claimed in Kac's formula. We have found that the leading term in h has degree $\sum_{rs\le m} sm(r,s)$, and

$$\begin{aligned}
m(r,s) &= \#\{\text{partitions of } n \text{ in which } r \text{ appears } \textit{at least } s \text{ times}\} \\
&\quad - \#\{\text{partitions of } n \text{ in which } r \text{ appears } \textit{at least } s+1 \text{ times}\} \\
&= p(n-rs) - p(n-r(s+1)),
\end{aligned}$$

so

$$\sum_{rs\le n} sm(r,s) = \sum_{rs\le n} [sp(n-rs) - sp(n-r(s+1))] = \sum_{rs\le n} p(n-rs)$$

as claimed in Kac's theorem. For the remainder of the proof (tomorrow) we'll show that the $h_{r,s}(c)$ are roots of $d_n(c,h)$ and that $p(n-rs)$ is a *lower* bound for their multiplicity, which will be enough to obtain the desired conclusion.

11 27.03.2018

12 28.03.2018 – Geddard-Kent-Olive, and the discrete series of the Virasoro algebra

12.1 Unitary representations of $\widehat{\mathfrak{sl}_2}$ of highest weight

V_ω with

$$\omega = mD + n\frac{h_1}{2} + rK,$$

where $m \ge n \ge 0$.

12.2 Sugawara's construction

Theorem 12.1. *Let $\mathfrak{g}$ be a reductive Lie algebra. Fix*

1. $(\cdot \mid \cdot)$ *a non-degenerate invariant symmetric bilinear form*
2. *a basis* $\{u_i\}_{i=1}^{\dim\mathfrak{g}}$ *and its dual basis* $\{u^i\}$
3. *the Casimir operator* Ω *(see below)*
4. $2h^\vee = \Omega|_{\mathfrak{g}}$, *where we consider the adjoint action of* Ω *(when* $\mathfrak{g}$ *is simple,* $h^\vee$ *is a number, called the* ***dual Coxeter number****)*
5. $k \ne -h^\vee$

6. *V representation of level k*

Define

$$L_n = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} \sum_{p\in\mathbb{Z}} :(u_i t^{-p})(u^i t^{p+n}):$$

where the normal ordered product is

$$:(at^p)(bt^q) := \begin{cases} (at^p)(bt^q), & \text{if } q \geq p \\ (bt^q)(at^p), & \text{if } q < p \end{cases}$$

Note that $L_n^\dagger = L_{-n}$. Then:

1. $[at^m, L_n] = mat^{m+n}$
2. $$[L_m, L_n] = (m-n)L_{m+n} + c\frac{m^3-m}{12}\delta_{m+n,0},$$

   where $c = \frac{k}{k+h^\vee}\dim\mathfrak{g}$.

Remark 12.2. Definition of the ordered product. What we are doing in the case $p > q$ is writing

$$(at^p)(bt^q) = (bt^q)(at^p) + [at^p, bt^q] = (bt^q)(at^p) + [a,b]t^{p+q} + \text{central element}$$

and eliminating the central element. More precisely,

$$[at^m, bt^n] = [a,b]t^{m+n} + km(a|b)\delta_{m+n,0}$$

Definition 12.3. *The **Casimir element** Ω of $\mathfrak{g}$ is $\sum_i u_i u^i$, considered as an element of $U(\mathfrak{g})$.*

Exercise 12.4. *Ω is independent from the choice of basis and is central in $U(\mathfrak{g})$.*

Example 12.5. One can take $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, with $(A \mid B) = \mathrm{tr}(AB)$ or $(A \mid B) = \mathrm{tr} \oplus \mathrm{tr}$. In the case of $\mathfrak{sl}_2$, we can take as basis $\{e, f, h\}$ and as dual basis $\{f, e, \frac{h}{2}\}$, which leads to $\Omega = ef + fe + \frac{1}{2}h^2$. Let us compute the adjoint action of the Casimir operator by looking at the equation

$$\mathrm{ad}(\Omega)(e) = (\text{number})e;$$

it gives

$$[e,[f,e]] + [f,[e,e]] + [h,[h/2,e]] = [e,-h] + 0 + [h,[h/2,e]] = 2e + 2e = 4e,$$

which implies $h^\vee = 2$.

Exercise 12.6. *$h^\vee(\mathfrak{sl}_n) = n$.*

Lemma 12.7. *Let u_i, u^i be dual bases. The element $A := \sum[u_i, u^i]$ is independent of the choice of basis, and is equal to zero.*

Proof. One checks independence on the choice of basis. Observing that (u^i, u_i) are also dual bases we get

$$A = \sum[u_i, u^i] = \sum[u^i, u_i],$$

which implies $A = 0$. □

Proof. We only show part (1). We proceed by regularization, truncating the sum on p to some large (but finite) interval $[-N, N]$. We compute

$$\left[at^n, \sum_{i=1}^{\dim \mathfrak{g}} \sum_{p=-N}^{N} :(u_i t^{-p})(u^i t^{p+n}):\right] = \sum_i \sum_p [at^n, u_i t^{-p}] u^i t^{p+n} + (u_i t^{-p})[at^m, u^i t^{p+n}]$$
$$= \sum_i \sum_p \big(([a, u_i] t^{m-p}) u^i t^{p+n} + (u_i t^{-p})[a, u^i] t^{p+m+n} + km(a|u_i)\delta_{m-p,0} u^i t^{p+n} + km(a|u^i)\delta_{m+n+p} u_i t^{-p}\big)$$

where in order to remove the ordered product we have used the fact that the ordered product and the normal product differ by a central element (which is irrelevant, since it appears inside the commutator) and by (a multiple of) $[u_i, u^i]$, which summed over i gives zero by the previous lemma.

Now we

- use the δ's to simplify the sums;
- apply the obvious identity $\sum_i (a|u_i)u^i = a$
- rename $m - p \to p$ to make the first two summands equal to each other

and obtain

$$2kmat^{m+n} + \sum_i \sum_{p=-N-m}^{N-m} [a, u_i] t^{-p} (u^i t^{p+m+n}) + \sum_i \sum_{p=-N}^{N} (u_i t^{-p}) [a, u^i] t^{p+m+n}.$$

We would now like to take the limit as $N \to \infty$; it is easier to do so if we work with the ordered products. Therefore we rewrite the previous expression as

$$2kmat^{m+n} + \sum_i \sum_{p=-N-m}^{N-m} :[a, u_i] t^{-p} (u^i t^{p+m+n}): + \sum_i \sum_{p=-N}^{N} :(u_i t^{-p}) [a, u^i] t^{p+m+n}:$$
$$+ \sum_i \sum_{p=-N-m}^{-(m+n)/2} ([[a, u_i], u^i] t^{m+n} + kp([a, u_i]|u^i)\delta_{m+n,0}$$
$$+ \sum_i \sum_{p=-N}^{-(m+n)/2} ([u_i, [a, u^i]] t^{m+n} + kp(u_i|[a, u^i])\delta_{m+n,0}$$

Exercise 12.8.

$$\sum_i [a, u_i] \otimes u^i + \sum_i u_i \otimes [a, u^i] = 0,$$

that is, the Casimir element is central in the tensor algebra (and not just in the universal enveloping algebra).

Now we can take the limit in N for the first two sums (when acting on a given vector, the finite sum and the infinite one act in the same way when $N \gg 0$) and find 0 by the exercise. Moreover,

$$\sum_i ([a, u_i]|u^i) = \sum_i (a|[u_i, u^i]) = \left(a \;\Big|\; \sum_i [u_i, u^i]\right) = 0,$$

so the only terms that survive are $[[a, u_i], u^i]$ and $[u_i, [a, u^i]]$. We compute these by observing that

$$\sum_i [u_i, [a, u^i]] = \sum_i [u^i, [u_i, a]] = \operatorname{ad}\Omega(a) = 2h^\vee a$$

and likewise

$$\sum_i [u_i, [a, u^i]] = -2h^\vee a.$$

Putting everything together, the commutator we're trying to compute is

$$2kmat^{m+n} + \sum_{p=-N-m}^{-(m+n)/2} 2h^\vee at^{m+n} - \sum_{p=-N}^{-(m+n)/2} 2h^\vee at^{m+n} = 2(k+h^\vee)mat^{m+n}$$

□

Corollary 12.9. *Let V be a unitary representation of $\widehat{\mathfrak{sl}_2}$ of highest weight $\omega = mD + n\frac{h_1}{2} + rK$. Then V is a unitary representation of* Vir *(with the same Hermitian form and the same adjunction map) with central charge $c = \frac{m}{m+2} \cdot 3$ (which is either 0 or ≥ 1, and therefore does not give anything new with respect to our investigation of the discrete series).*

12.3 Geddard-Kent-Olive

There is a general construction which we will only describe in the special case of $\widehat{\mathfrak{sl}_2}$. We consider two particular unitary representations of $\widehat{\mathfrak{sl}_2}$, namely V_D and V_ω, with $\omega = mD + n\frac{h_1}{2}$ (i.e. $r = 0$). We look at the tensor product $V_D \otimes V_\omega$, which is a unitary representation of $\widehat{sl_2}$. This is a space with three different $\widehat{sl_2}$-actions:

1. $a \cdot (u \otimes v) = (a \cdot u) \otimes v + u \otimes (a \cdot v)$ (the diagonal action)
2. $a \cdot (u \otimes v) = (a \cdot u) \otimes v$ (the action on the first factor)
3. $a \cdot (u \otimes v) = u \otimes (a \cdot v)$ (the action on the second factor)

The levels (:= action of K) of these three representations are $K = m+1, K = 1, K = m$ respectively. The Sugawara construction gives us three actions of Vir on this same vector space:

1. $L_n^{(1)} = \frac{1}{2(1+2)}(\sum \ldots) \otimes 1$, central charge $c_1 = \frac{1}{1+2} \cdot 3 = 1$
2. $L_n^{(2)} = 1 \otimes \frac{1}{2(1+2)}(\sum \ldots)$, central charge $c_2 = \frac{m}{m+2} \cdot 3 = \frac{3m}{m+2}$
3. $L_n^{(\Delta)} = \frac{1}{2(m+1+2)}(\sum \ldots)$, central charge $c_\Delta = \frac{3(m+1)}{m+3}$

Theorem 12.10 (Geddard-Kent-Olive)**.** *Defining*

$$L_n = L_n^{(1)} + L_n^{(2)} - L_n^{(\Delta)}$$

we obtain a Virasoro action with the following properties:

1. *$[at^m, L_n] = 0$, that is, Virasoro commutes with $\widehat{\mathfrak{sl}_2}'$ (the diagonal action)*
2. *$[L_m, L_n] =$ Virasoro relation, with central charge $c = c_1 + c_2 - c_\Delta = 1 - \dfrac{6}{(m+2)(m+3)}$, which are precisely the admissible values of the central charge for the discrete series of Virasoro.*

Proof. Let's start by computing $[at^m, L_n]$, where by a we mean the diagonal action of $\widehat{\mathfrak{sl}_2}$, that is, $at^n \otimes 1 + 1 \otimes at^m$. We have

$$\begin{aligned}
[at^m, L_n] &= \left[at^m \otimes 1 + 1 \otimes at^m, L_n^{(1)} \otimes 1 + 1 \otimes L_n^{(2)} - L_n^{(\Delta)}\right] \\
&= [at^m, L_n^{(1)}] \otimes 1 + 1 \otimes [at^m, L_n^{(2)}] + [(at^m)^\Delta, L_n^{(\Delta)}] \\
&= mat^{m+n} \otimes 1 + 1 \otimes mat^{m+n} - m(at^{m+n})^\Delta = 0
\end{aligned}$$

Now for part (2): we want to compute

$$\left[L_m^{(1)} \otimes 1 + 1 \otimes L_m^{(2)} - L_m^{(\Delta)}, L_n\right],$$

where we observe that L_n (which is a sum of ordered products of things in $\widehat{\mathfrak{sl}_2}'$) commutes with $L_m^{(\Delta)}$. Hence we just need to compute

$$\begin{aligned}\Big[L_m^{(1)} \otimes 1 + 1 \otimes L_m^{(2)},\ L_n^{(1)} \otimes 1 + 1 \otimes L_n^{(2)} - L_n^{\Delta}\Big] &= \\ [L_m^{(1)}, L_n^{(1)}] \otimes 1 + 1 \otimes [L_m^{(2)}, L_n^{(2)}] &- [L_m^{(1)} \otimes 1 + 1 \otimes L_m^{(2)} - L_m^{(\Delta)} + L_m^{(\Delta)}, L_n^{\Delta}] \\ &= (m-n)L_{m+n} + (c_1 + c_2 - c_\Delta)\frac{m^3 - m}{12}\delta_{m+n,0},\end{aligned}$$

where we have used that L_m, L_n^{Δ} commute because again L_n^{Δ} is given by a combination of operators that commute with L_m by part (1). □

12.4 Decomposition of $V_D \otimes V_\omega$

We shall use the following known result, which gives the decomposition of $V_D \otimes V_\omega$ as a $\widehat{\mathfrak{sl}_2}$-representation (with respect to the diagonal action):

$$V_D \otimes V_\omega = \bigoplus_{\substack{k \in \mathbb{Z} \\ -(m+1-n)/2 \leq k \leq n/2}} \bigoplus_{j \geq k^2} V_{(m+1)D + (n-2k)\frac{h_1}{2} - jK}^{\oplus \Delta_{m,n,k}^j}$$

The multiplicities $\Delta_{m,n,k}^j$ are all strictly positive.

12.5 The discrete series of the Virasoro algebra

Remark 12.11. [9] Since the Virasoro action commutes with the action of $\widehat{\mathfrak{sl}_2}'$, then, if v is a singular vector (ie $(at^m)v = 0$ for all $m > 0$ and $ev = 0$), $L_n v$ is also a singular vector. It follows that Virasoro acts on the subspace of singular vectors, which is given by the sum of the 1-dimensional lines of singular vectors in each $V_{(m+1)D + (n-2k)\frac{h_1}{2} - jK}$. The subspace of singular vectors in $V_D \otimes V_\omega$ is

$$U = \bigoplus_k \bigoplus_{j \geq k^2} U_{m,n;k}^j,$$

where $\dim U_{m,n;k}^j = \Delta_{m,n,k}^j$ is the space of singular vectors in a single summand. Moreover, the action on $U_{m,n;k}$ is given by $K = m+1, h_1 = n - 2k$ and $D = -j$.

Now Virasoro commutes with K and with h_1, but not with D, so it preserves $\bigoplus_{j \geq k^2} U_{m,n,k}^j$ (but not necessarily every $U_{m,n,k}^j$ taken separately). In any case, for every k we get a Virasoro action on

$$U_{m,n,k} = \bigoplus_{j \geq k^2} U_{m,n,k}^j.$$

These are unitary representations of the Virasoro algebra, with central charge $c = 1 - \frac{6}{(m+2)(m+3)}$. We still need to compute the action of h on $U_{m,n,k}$, where by definition h is the value of L_0 on the highest weight vector, or equivalently the smallest eigenvalue of L_0. Recall that $L_0 = L_0^{(1)} \otimes 1 + 1 \otimes L_0^{(1)} - L_0^{(\Delta)}$. To simplify the computation we use the following trick: we rewrite L_0 as

$$(L_0^{(1)} + D) \otimes 1 + 1 \otimes (L_0^{(2)} + D) - (L_0^{(\Delta}) + D^{(\Delta)}$$

[9]'scema ma cruciale'

and notice that $L_0^{\text{Sugawara}} + D$ is central[10]. To see this, notice that $[K, L_0^{\text{Sugawara}} + D] = 0$ because K is central. Moreover,

$$\begin{aligned} &= [D, L_0^{\text{Sugawara}}] \\ &= [D, \sum : (u_i t^{-p})(u^i t^p) :]; \end{aligned}$$

observe that $[D, \cdot]$ is a derivation *and* it gives the degree in t, so the previous sum becomes a sum with coefficients $(-p) + (p) = 0$. Finally,

$$[at^m, L_0^{\text{Sugawara}} + D] = mat^{m+0} - \text{degree}(t^m)at^m = 0.$$

It follows that $L_0^{\text{Sugawara}} + D$ acts as a scalar on irreducible (or even just highest weight) representations. On the other hand, we can compute this scalar on a representation $V_{\tilde{\omega}}$ by looking at the action on a highest weight vector $v_{\tilde{\omega}}$. Write $\tilde{\omega} = \tilde{m}D + \tilde{n}\frac{h_1}{2} + \tilde{r}K$. Then the action of D on $v_{\tilde{\omega}}$ is $\tilde{r}$ by definition; we then get

$$L_0^{\text{Sugawara}} + D)v_{\tilde{\omega}} = \tilde{r}v_{\tilde{\omega}} + \frac{1}{2(\tilde{m}+2)} \sum_{i=1}^{3} \sum_{p} : (u_i t^{-p})(u^i t^p) : v_{\tilde{\omega}},$$

and we may observe that for $p \neq 0$ the *ordered* product hits $v_{\tilde{\omega}}$ with an operator that kills it, so

$$(L_0^{\text{Sugawara}} + D)v_{\tilde{\omega}} = \tilde{r}v_{\tilde{\omega}} + \frac{1}{2(\tilde{m}+2)}(ef + fe + \frac{1}{2}h^2)v_{\tilde{\omega}} = \tilde{r}v_{\tilde{\omega}} + \frac{\tilde{n}(\tilde{n}+2)}{4(\tilde{m}+2)}v_{\tilde{\omega}}$$

Combined with the previous remark that L_0 can be rewritten as a combination of $L_0^{(\cdot)} + D$, we obtain that the action of L_0 on a vector in $U^j_{m,n;k}$ is

$$0 + \frac{n(n+2)}{4(n+2)} + j - \frac{(n-2k)(n-2k+2)}{4(m+3)}$$

where the three terms come from the action on a representation of weight D, ω, $(m+1)D + (n - 2k)\frac{h_1}{2} - jK$ respectively. Now we want the *lowest* eigenvalue of L_0, so we have to take $j = k^2$, and we get

$$h = \frac{n(n+2)}{4(m+2)} - \frac{(n-2k)(n-2k+2)}{4(m+3)} + k^2.$$

Now the change of variables

$$\begin{cases} r = n+1 \\ s = n+1-2k \end{cases} \quad \text{if } k \geq 0, \quad \begin{cases} r = m-n+1 \\ s = m-n+2+2k \end{cases} \quad \text{otherwise}$$

shows that $h = h_{r,s}$, and completes the classification of the unitary representations of the Virasoro algebra.

[10]that is, it commutes with $\widehat{\mathfrak{sl}_2}$

13 Exercises

13.1 Week 2

1. Let $(\cdot \mid \cdot) : M_\lambda^+ \times M_\mu^- \to \mathbb{C}$ be a nonzero $\mathfrak{g}$-invariant bilinear form. Prove that $\mu = -\lambda$.

7. Let $U_{\leq n}$ be the image of $\sum_{r \leq n} \mathfrak{g}^{\otimes r}$ in $U(\mathfrak{g})$. Prove that $U_{\leq m} \cdot U_{\leq n} \subseteq U_{\leq m+n}$, and that $\mathrm{gr}(U) \cong S(\mathfrak{g})$.

   Solution. The first part is trivial: given $\alpha \in U_{\leq m}, \beta \in U_{\leq n}$ we can find $\tilde{\alpha} \in \sum_{r \leq m} \mathfrak{g}^{\otimes r}$ and $\tilde{\beta} \in \sum_{r \leq n} \mathfrak{g}^{\otimes r}$ such that α (resp. β) is the image of $\tilde{\alpha}$ (resp. $\tilde{\beta}$) in $U(\mathfrak{g})$. Now $\alpha\beta$ is the class of $\tilde{\alpha}\tilde{\beta}$, which clearly belongs to $\sum_{r \leq m+n} \mathfrak{g}^{\otimes r}$.

   To prove that $\mathrm{gr}(U(\mathfrak{g})) \cong S(\mathfrak{g})$ we apply the PBW theorem. The i-th graded piece of $\mathrm{gr}(U(\mathfrak{g}))$ has a basis given by *ordered* monomials of exact degree i in a basis of $\mathfrak{g}$; to show the desired isomorphism, therefore, it suffices to show that $\mathrm{gr}(U)$ is commutative. Since the i-th graded piece is generated by monomials in the basis of the 1st graded piece, it suffices to show the following: given $x \in \mathfrak{g}, y \in \mathfrak{g}$ which are part of a basis for $\mathfrak{g}$, their images in $\mathrm{gr}(U)$ commute. And this is true because $xy - yx \equiv xy - yx - [x, y] \bmod \mathrm{gr}^1(U)$, hence $xy = yx$ in $\mathrm{gr}^2 / \mathrm{gr}^1$, and therefore $xy = yx$ in $\mathrm{gr}(U)$.

8. Let $\mathfrak{g}$ be a Lie algebra. Prove that $U(\mathfrak{g})$ is a domain (i.e. $ab = 0 \Rightarrow a = 0$ or $b = 0$)

   Solution. This follows from the fact that (for any graded associative algebra U) if $\mathrm{gr}(U)$ is a domain, then so is U. Let me write a proof of this lemma:

   Proof. An element x of U is said to have *degree* d if it belongs to $U_d \setminus U_{d-1}$; the *leading term* of x is $\bar{x} = x + U_{d-1}$. By definition of the multiplication in $\mathrm{gr}(U)$, one has $\overline{xy} = \overline{x} \cdot \overline{y}$ provided that the latter is nonzero.

   Now suppose that $\mathrm{gr}(U)$ is a domain, and take $x, y \in U \setminus \{0\}$. Since $\overline{x}, \overline{y}$ are nonzero and $\mathrm{gr}(U)$ is a domain, $\overline{x} \cdot \overline{y}$ is nonzero, so it is equal to $\overline{xy}$, which is therefore also nonzero; it follows that xy cannot be zero. □

9. Let $\mathfrak{g} = \mathfrak{sl}_2$. Prove that $C = ef + fe + \frac{1}{2}h^2$ is central, and compute its action on M_λ.

   Solution. One has

$$
\begin{aligned}
&= e(ef + fe + \frac{1}{2}h^2) - (ef + fe + \frac{1}{2}h^2)e \\
&= e[e, f] + [e, f]e + \frac{1}{2}[e, h^2] \\
&= eh + he + \frac{1}{2}(eh^2 - heh + heh - h^2 e) \\
&= eh + he + \frac{1}{2}([e, h]h + h[e, h]) \\
&= eh + he - eh - he = 0,
\end{aligned}
$$

   and similarly for $[f, C]$. As for $[h, C]$, we get

$$
\begin{aligned}
= [h, ef + fe] &= hef - ehf + ehf - efh + hfe - fhe + fhe - feh \\
&= [h, e]f + e[h, f] + [h, f]e + f[h, e] \\
&= 2ef + e(-2f) + (-2f)e + f(2e) = 0.
\end{aligned}
$$

   Being central, C acts as a scalar. Thus it suffices to see how it acts on 1_λ:

$$
C1_\lambda = (ef + \frac{1}{2}h^2)1_\lambda = ([e, f] + fe + \frac{1}{2}h^2)1_\lambda = (h + \frac{1}{2}h^2)1_\lambda = \left(\lambda(h) + \frac{1}{2}\lambda(h)^2\right) 1_\lambda.
$$

13.2 Week 3

1. The following hold:

   (a) $f \cdot V[\lambda] \subset V[\lambda - 2]$

   (b) $e \cdot V[\lambda] \subset V[\lambda + 2]$

   Solution. This is obvious: if $v \in V[\lambda]$, then

   $$h \cdot (f \cdot v) = ([h, f] + fh) \cdot v = -2f \cdot v + f \cdot (h \cdot v) = -2(f \cdot v) + f \cdot (\lambda v) = (\lambda - 2)(f \cdot v),$$

   so $f \cdot v$ belongs to $V[\lambda - 2]$. The computation for $e \cdot v$ is completely analogous.

2. Let V be a module in our category. Then:

   (a) V is finitely generated as a $\mathfrak{g}$-module

   (b) $\dim V[\lambda] < +\infty \quad \forall \lambda \in \mathbb{C}$

   (c) $\exists \lambda_1, \ldots, \lambda_m \in \mathbb{C}$ such that

   $$P_V := \{\lambda \in \mathbb{C} : V[\lambda] \neq (0)\} \subset (\lambda_1 - 2\mathbb{N}) \cup \cdots \cup (\lambda_m - 2\mathbb{N})$$

   Solution.

   (a) True by assumption.

   (b) Let $v_1, \ldots, v_r$ be generators of V as a $\mathfrak{g}$-module. Since V is graded, we can (and will) assume that the v_i's are homogeneous, that is, that each of them belongs to a weight space. By the PBW theorem, a set of generators of V as vector space is given by

   $$\left\{ f^a e^b h^c v_i \;\middle|\; i = 1, \ldots, r; a, b, c \in \mathbb{N} \right\},$$

   and since h acts as a scalar on every v_i we can reduce this to the set

   $$\left\{ f^a e^b v_i \;\middle|\; i = 1, \ldots, r; a, b \in \mathbb{N} \right\}.$$

   Let λ_i be the weight of v_i; by the previous exercise, $f^a e^b v_i$ has weight $\lambda_i + 2(b-a)$. As V is graded and the $f^a e^b v_i$ are homogeneous, we can extract from

   $$\left\{ f^a e^b v_i \;\middle|\; i = 1, \ldots, r; a, b \in \mathbb{N} \right\}$$

   a generating set for $V[\lambda]$. However, an element $f^a e^b v_i$ belongs to $V[\lambda]$ if and only if $\lambda_i + 2(b-a) = \lambda$, which means that $b - a = \frac{\lambda - \lambda_i}{2}$. Thus for every i the difference $a - b$ is fixed; on the other hand, for b sufficiently large (depending on i), we know by assumption that $e^b v_i = 0$, so there are only finitely many vectors of the form $f^a e^b v_i$ (with $b - a = \frac{\lambda - \lambda_i}{2}$) that are nonzero. In particular, we have found a generating set of $V[\lambda]$ that contains only finitely many elements, so $\dim V[\lambda] < \infty$ as desired.

   (c) Consider generators $v_1, \ldots, v_r$ of V as a $\mathfrak{g}$-module. Since $V \cong \bigoplus V[\lambda]$, each v_i has components only along finitely many weight spaces. Let $\lambda_1, \ldots, \lambda_m$ be the complete list of weights such that at least one of the v_i has a nontrivial component in $V[\lambda_i]$. Then by the previous exercise it is clear that $\bigoplus_{i=1}^m \bigoplus_{n \in \mathbb{N}} V[\lambda_i - 2n]$ is a submodule of V, and since by construction it contains all the generators it is in fact equal to V, which proves the claim.

3. M_λ, L_λ are objects in $\mathcal{O}$ and $\{L_\lambda \mid \lambda \in \mathbb{C}\}$ is a complete list of the irreducible representations in $\mathcal{O}$.

   Solution. All the necessary properties are trivially true by construction. That the list is complete has been proven in class; in any case, it follows easily from the following argument. Take any eigenvector v for h. Then for a suitable $n \in \mathbb{N}$ the vector $e^n v$ is a (nonzero) eigenvector for h killed by e (this uses the hypothesis that for $N \gg 0$ we have $e^N v = 0$); by irreducibility, $v_+ := e^n v$ generates V. Hence V is a highest weight representation, so it is a quotient of the Verma module, and from here the conclusion follows immediately.

4. Every $V \in \mathcal{O}$ admits a Jordan-Hölder composition series of finite length.