Area of intrinsic graphs in homogeneous groups

Valentino Magnani (University of Pisa)

in collaboration with Francesca Corni

(University of Bologna)

~

XXXIII Convegno Nazionale di Calcolo

delle Variazioni, Riccione, 16/2/2024

~

What is an "area forumla"?

The evasive answer is:

"It depends on the notion of Jacobian".

Metric space area formulas

Let X and Y be metric spaces and assume that

(covering relation) $V = \{(x,S) \mid x \in S \subset X\}$

V is a μ-Vitali relation if

$\forall\, C \subset V$ such C is fine at each point of $Z \subset X$, then $\exists\, \mathcal{F} \subset C(Z)$ countable and disjoint such that

$$\mu\left(Z \setminus \bigcup \mathcal{F}\right) = 0$$

Valentino Magnani. An area formula in metric spaces. *Colloq. Math.*, 124(2):275–283, 2011.

DEFINITION Let $f : E \to Y$ be continuous and let $x \in E$. Then we introduce two *metric Jacobians* of f at x as follows:

$$J_f(x) = (V) \limsup_{S \to x} \frac{\nu(f(S \cap E))}{\mu(S)}, \qquad Jf(x) = (V) \limsup_{S \to x} \frac{f^*\nu(S)}{\mu(S)}.$$

Assuming that V is a μ-Vitali relation $\Longrightarrow$

THEOREM (Area formula). *Let $f : E \to Y$ be continuous and assume that the pull-back $f^*\nu$ is finite on bounded sets and absolutely continuous with respect to μ. Then Jf is μ-a.e. finite and for all μ-measurable sets $A \subset E$, we have*

$$\int_A Jf(x)\, d\mu(x) = \int_Y N(f, A, y)\, d\nu(y).$$

When the metric spaces have more structure, a less tautological notion of Jacobian is expected, that makes it computable and possibly related to a suitable notion of differential.

equipped with Euclidean norm $|\cdot|$

The case of $f : \mathbb{R}^k = X \longrightarrow Y$

The metric differential of f at x is a seminorm $\|\cdot\|_x$ on $\mathbb{R}^n$ such that

$$\rho_Y(f(x), f(y)) - \|y-x\|_x = o(|y-x|) \quad y \to x$$

Kirchheim, B.: Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. Proc. Am. Math. Soc. **121**(1), 113–123 (1994)

Theorem *Let $f : \mathbb{R}^n \to (X, \|\cdot\|)$ be Lipschitzian, and let $A \subset \mathbb{R}^n$ be Lebesgue measurable. Then*

$$\int_A \mathscr{J}(MD(f, x))\, d\mathscr{L}^n(x) = \int_X N(f|A, x)\, d\mathscr{H}^n_{\|\cdot\|}(x),$$

where $N(f|A, x)$ denotes the cardinality of the set $A \cap f^{-1}(x)$.

metric Jacobian of f at x

The metric Jacobian is defined with respect to the metric differential, which is a seminorm s.

$$\mathcal{J}(s) = \frac{n\,\omega_n}{\int_{\mathbb{S}^{n-1}} s(x)^{-n}\, d\mathcal{H}^{n-1}(x)}$$

domain from $\mathbb{R}^n$ to stratified groups $\mathbb{G}$

Kirchheim's area formula extends to domains made by measurable subsets of a stratified Lie group $\mathbb{G}$ (real and finite dimensional).

$A \subset \mathbb{G}$ meas. $f : A \longrightarrow Y$

Y is a metric space

Magnani, V., Rajala, T.: Radon-Nikodym property and area formula for Banach homogeneous group targets. Int. Math. Res. Not. IMRN **23**, 6399–6430 (2014)

As usual, the point of an area formula is its notion of Jacobian. The *metric Jacobian* $J(s)$ of the homogeneous seminorm s is defined as follows:

$$J(s) = \begin{cases} \dfrac{\mathcal{H}_s^Q(B_1)}{\mathcal{H}_d^Q(B_1)} & \text{if } s \text{ is a homogeneous norm,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem Let $A \subset \mathbb{G}$ be measurable, let $f : A \longrightarrow Y$ be Lipschitz and almost everywhere metrically differentiable. It follows that

$$\int_A J(mdf(x))\, \mathrm{d}\mathcal{H}_d^Q(x) = \int_Y N(f, y)\, \mathrm{d}\mathcal{H}_\rho^Q(y),$$

What happens when the codomain Y is the Heisenberg group?

Ambrosio, L., Kirchheim, B.: Rectifiable sets in metric and Banach spaces. Math. Ann. **318**(3), 527–555 (2000)

Magnani, V. "Unrectifiability and rigidity in stratified groups." *Archiv der Mathematik* 83, no. 6 (2004): 568–76.

imply that $\mathcal{H}^2_d(f(A))=0$ $\quad \forall A \subset \mathbb{R}^2$

where $f : A \longrightarrow \mathbb{H}^1$ is Lipschitz continuous

$$\Longrightarrow \mathcal{H}^3_d(f(A))=0.$$

Notice that H-dim$(\mathbb{H}^1) = 4$ and obviously top-dim$(\mathbb{H}^1) = 3$

Theorem

If $\Sigma \subset \mathbb{H}^1$ is smooth, then $\mathcal{H}^3_d(\Sigma) > 0$ and then it is not 2-rectifiable, so Kirchheim's area formula does not apply!

Kirchheim's area forumla is then replaced by the "measure-theoretic area forumla"

The point is to differentiate a suitable measure on the surface.

Valentino Magnani. On a measure-theoretic area formula. *Proc. Roy. Soc. Edinburgh Sect. A*, 145:885–891, 2015.

THEOREM (differentiation with respect to the spherical Hausdorff measure). *Let X be a diametrically regular metric space, let $\alpha > 0$ and let μ be a Borel regular measure over X such that there exists a countable open covering of X whose elements have μ finite measure. If $B \subset A \subset X$ are Borel sets and $\mathcal{S}_{\mu,\zeta_{b,\alpha}}$ covers A finely, then $\theta^\alpha(\mu,\cdot)$ is Borel on A. In addition, if $\mathcal{S}^\alpha(A) < +\infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{S}^\alpha \llcorner A$, then we have*

$$\mu(B) = \int_B \theta^\alpha(\mu,x)\, d\mathcal{S}^\alpha(x).$$

Further extensions of the measure-theoretic area formula can be found in the paper

Giacomo Maria Leccese and Valentino Magnani. A study of measure-theoretic area formulas. *Ann. Mat. Pura Appl. (4)*, 201(3):1505–1524, 2022.

In the measure-theoretic area formula

$$\mu(B) = \int_B \theta^\alpha(\mu, x)\, \mathrm{d}\mathcal{S}^\alpha(x)$$

we have defined the following

spherical Federer density

$$\theta^\alpha(\mu, x) = \inf_{\varepsilon>0} \sup \left\{ \frac{\mu(\mathbb{B})}{r(\mathbb{B})^\alpha} : x \in \mathbb{B} \in \mathcal{F}_b,\ \mathrm{diam}(\mathbb{B}) < \varepsilon \right\} \in [0, +\infty]$$

$\mathbb{B}$ = metric unit closed ball

$\mathcal{F}_b$ = family of closed metric balls

Aim: compute the spherical Federer of a suitable measure on intrinsic graphs

What is an intrinsic graph?

We need first to consider what a homogeneous group G is. It is a Lie group bianalytic to $\mathbb{R}^n$ and it is equipped with a graded Lie algebra $\mathrm{Lie}\, G = V_1 \oplus \cdots \oplus V_r$, $[V_i, V_j] \subset V_{i+j}$, $i,j \geq 1$, $V_k = \{0\}$ for $k > r$.

In local coordinates $G \simeq \mathbb{R}^n$ and has a noncommutative polynomial group operation $x \cdot y = p(x,y) \in \mathbb{R}^n$

Factorization

Let $\mathbb{W}, \mathbb{V} \subset \mathbb{G}$ be homogeneous subgroups of $\mathbb{G}$ such that

$$\mathbb{W} \cdot \mathbb{V} = \mathbb{G} \quad \text{and} \quad \mathbb{W} \cap \mathbb{V} = \{0\}$$

Then we say that $(\mathbb{W}, \mathbb{V})$ is a couple of complementary subgroups of $\mathbb{G}$.

→ this notion emphasizes the order $wv = x$ (since $v'w' = x$ and $v \neq v'$, $w \neq w'$)

Projections and intrinsic graphs

$(\mathbb{W}, \mathbb{V})$ is a couple of complementary subgroups

$$\pi_{\mathbb{W}} = \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W}} : \mathbb{G} \longrightarrow \mathbb{W}, \qquad x = wv \longmapsto w$$

$$\pi_{\mathbb{V}} = \pi^{\mathbb{V},\mathbb{W}}_{\mathbb{V}} : \mathbb{G} \longrightarrow \mathbb{V} \qquad x = wv \longmapsto v$$

are the group projection associated with $(\mathbb{W}, \mathbb{V})$.

~

Definition of intrinsic graph

$(\mathbb{W}, \mathbb{V})$ complementary subgroups, $A \subset \mathbb{W}$

$$\phi : A \longrightarrow \mathbb{V}, \qquad \text{graph}\,\phi = \{ w\phi(w) : w \in A \}$$

Many of the subsequent notions were first introduced by Franchi, Serapioni and Serra Cassano in stratified groups:

Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. Intrinsic Lipschitz graphs in Heisenberg groups. *J. Nonlinear Convex Anal.*, 7(3):423–441, 2006.

Bruno Franchi, Marco Marchi, and Raul Paolo Serapioni. Differentiability and approximate differentiability for intrinsic Lipschitz functions in Carnot groups and a Rademacher theorem. *Anal. Geom. Metr. Spaces*, 2:258–281, 2014.

Bruno Franchi and Raul Paolo Serapioni. Intrinsic Lipschitz graphs within Carnot groups. *J. Geom. Anal.*, 26(3):1946–1994, 2016.

Francesco Serra Cassano. Some topics of geometric measure theory in Carnot groups. In *Geometry, Analysis and Dynamics on sub-Riemannian manifolds. Vol. 1*, EMS Ser. Lect. Math., pages 1–121. Eur. Math. Soc., Zürich, 2016.

some of them were first introduced in Heisenberg groups:

Gabriella Arena and Raul Serapioni. Intrinsic regular submanifolds in Heisenberg groups are differentiable graphs. *Calculus of Variations and Partial Differential Equations*, 35(4):517–536, 2009.

Intrinsic translations

$x \in \mathbb{G}$, $\sigma_x : \mathbb{W} \longrightarrow \mathbb{W}$, $\boxed{\sigma_x(w) = \pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}}(xw)}$

and satisfy $(\sigma_x)^{-1} = \sigma_{x^{-1}}$;

the **translation of $\phi : A \subset \mathbb{W} \longrightarrow \mathbb{V}$ by x**

is a function $\phi_x : \sigma_x(A) \longrightarrow \mathbb{V}$ which

must satisfy $\boxed{x \operatorname{graph} \phi = \operatorname{graph} \phi_x}$.

imposing the equality

$$x\,\underbrace{w\,\phi(w)}_{\text{graph}(\phi)} = \pi_W(xw)\,\pi_V(xw)\,\phi(x^{-1}xw)$$

$$= \sigma_x(w)\,\pi_V(xw)\,\phi(x^{-1}\sigma_x(w)\,\pi_V(xw))$$

$$= \eta\,\pi_V(x\,\sigma_{x^{-1}}(\eta))\,\phi(x^{-1}\eta\,\pi_V(x\,\sigma_{x^{-1}}(\eta)))$$

$$= \eta\,\pi_V(x\,\pi_W(x^{-1}\eta))\,\phi(x^{-1}\eta\,\pi_V(x\,\pi_W(x^{-1}\eta)))$$

$$= \eta\,\pi_V(x^{-1}\eta)^{-1}\,\phi(x^{-1}\eta\,\pi_V(x^{-1}\eta)^{-1})$$

$$= \eta\,\pi_V(x^{-1}\eta)^{-1}\,\phi(\pi_W(x^{-1}\eta))$$

$$= \eta\,\boxed{\pi_V(x^{-1}\eta)^{-1}\,\phi(\sigma_{x^{-1}}(\eta))}$$

$$= \eta\,\boxed{\phi_x(\eta)}, \qquad \eta \in \sigma_x(A)$$

Intrinsically linear map

An intrinsically linear map $\ell : \mathbb{W} \longrightarrow \mathbb{V}$

is characterized by the request that

$$\text{graph}\,\ell = H$$

is a homogeneous subgroup of $\mathbb{G}$,

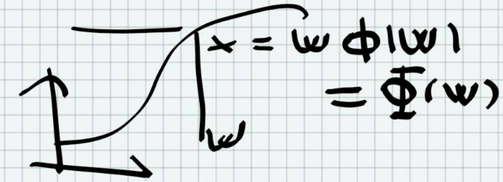
namely $\delta_r H \subset H \quad \forall r>0$ and $HH \subset H$

Remark

Intrinsically linear map are not linear!

(They have a polynomial form)

Intrinsic differentiability

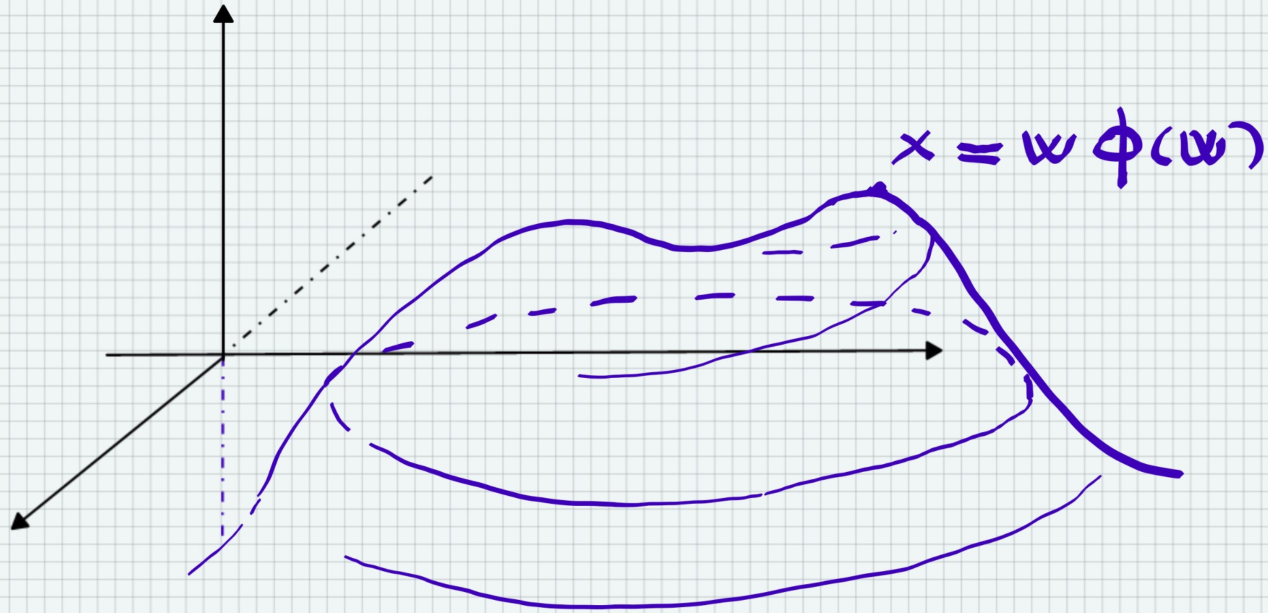


We say that $\phi : A \subset \mathbb{W} \longrightarrow \mathbb{V}$ is intrinsically differentiable at $w \in A$ if there exists an intrinsically linear map $\ell : \mathbb{W} \longrightarrow \mathbb{V}$ such that

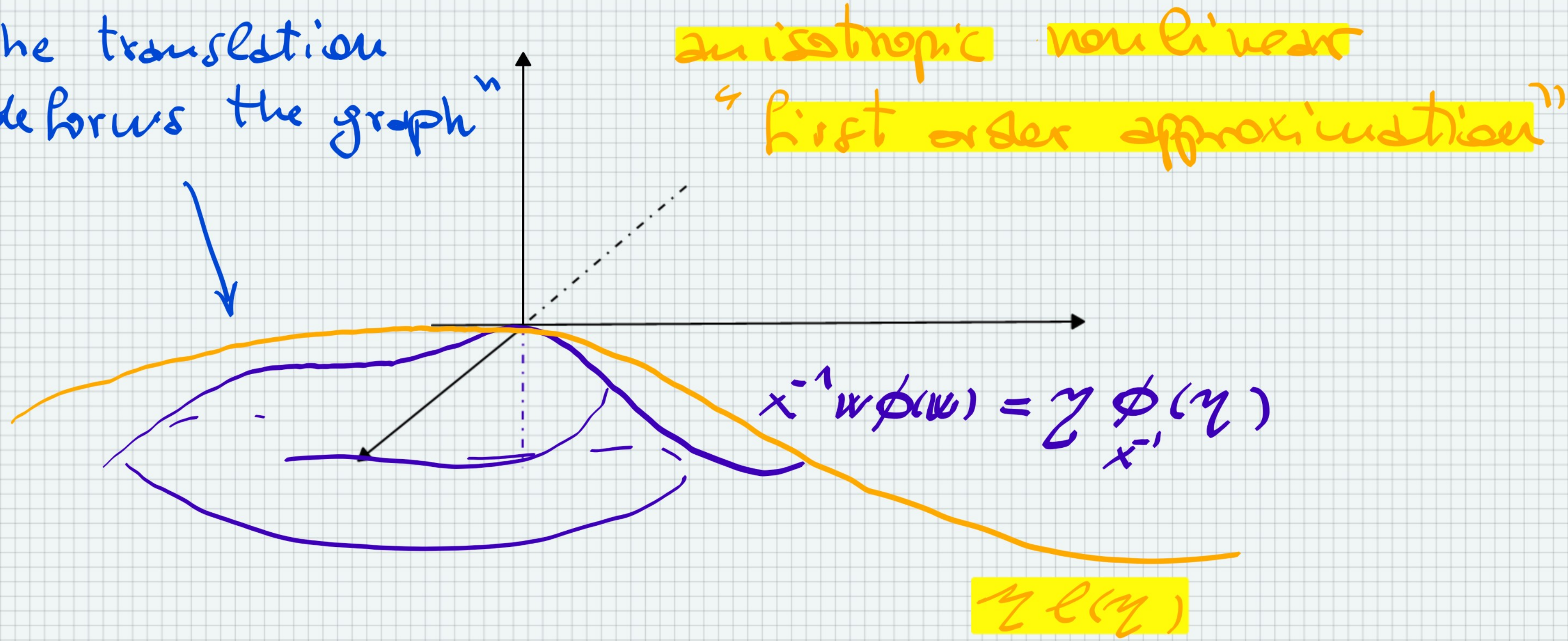
$$\frac{d(\phi_{x^{-1}}(u), \ell(u))}{\|u\|} \xrightarrow{u \to 0} 0$$

where we have set $x = w\phi(w)$.

Therefore $\pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}}(x) = w$.



$x = w\,\phi(w)$



the translation "deforms the graph"
anisotropic nonlinear
"first order approximation"
$x^{-1} w\,\phi(w) = \eta\,\phi_{x^{-1}}(\eta)$
$\eta\,\ell(\eta)$

Recent results on the area formula for intrinsic graphs

Antoine Julia, Sebastiano Nicolussi Golo, and Davide Vittone. Area of intrinsic graphs and coarea formula in Carnot groups. *Math. Z.*, 301(2):1369–1406, 2022.

Theorem (Area formula) *Let $\mathbb{G}$ be a Carnot group and let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be a splitting. Let $A \subset \mathbb{W}$ be an open set, $\phi \in C^1_{\mathbb{W},\mathbb{V}}(A)$ and let $\Sigma := \{w\phi(w) : w \in A\}$ be the intrinsic graph of ϕ; let d be the homogeneous dimension of $\mathbb{W}$. Then, for all Borel functions $h : \Sigma \to [0, +\infty)$,*

$$\int_\Sigma h \mathrm{d}\psi^d = \int_A h(w\phi(w))\mathcal{A}(T^H_{w\phi(w)}\Sigma)\mathrm{d}\psi^d(w). \tag{1}$$

ψ^d = Hausdorff or spherical measure

we recall the graph mapping $\Phi(w) = w\phi(w)$

$$(\mathrm{I}) \qquad \mathcal{A}(T^H_{\Phi(w)}\Sigma) = \frac{\psi^d \mathcal{L}\, T^H_{\Phi(w)}\Sigma}{(\Phi)_\#(\psi^d \mathcal{L} \mathbb{W})}$$

$T^H_{\Phi(w)}\Sigma = \ker Df(\Phi(w))$ = tangent subgroup to the intrinsic graph

$L(w) = w\, l(w)$, $L : \mathbb{W} \longrightarrow \mathbb{G}$ graph map

$$L(\mathbb{W}) = T^H_{\Phi(w)}\Sigma = \ker Df$$

Comments

1) Their arguments only work on stratified groups, where the implicit function theorem holds.

2) The intrinsic differentiability of the implicit map does not appear.

3) The "area factor" (I) cannot be computed using the underlying Euclidean structure.

Gioacchino Antonelli and Andrea Merlo. On rectifiable measures in Carnot groups: representation. *Calc. Var. Partial Differential Equations*, 61(1):Paper No. 7, 52, 2022.

Theorem *Let $\mathbb{V}, \mathbb{L}$ be homogeneous complementary subgroups of a Carnot group $\mathbb{G}$ such that $h := \dim_H \mathbb{V}$. Let Γ be the graph of an intrinsic Lipschitz map $\varphi : A \subseteq \mathbb{V} \to \mathbb{L}$ with A Borel, such that $\mathcal{S}^h \llcorner \Gamma$ is $\mathscr{P}_h^c$-rectifiable with tangent measures $\mathcal{S}^h \llcorner \Gamma$-almost everywhere supported on homogeneous subgroups* **complemented by** *$\mathbb{L}$. Then, for every Borel function $\psi : \Gamma \to [0, +\infty)$ the following area formula holds*

$$\int_\Gamma \psi \, d\mathcal{C}^h \llcorner \Gamma = \int_A \psi(a \cdot \varphi(a)) \mathcal{A}(\mathbb{V}(a \cdot \varphi(a))) d\mathcal{C}^h \llcorner \mathbb{V},$$

where $\mathcal{C}^h$ is the centered Hausdorff measure, $\mathbb{V}(a \cdot \varphi(a))$ is the tangent on which it is supported the tangent measure of $\mathcal{S}^h \llcorner \Gamma$ at the point $a \cdot \varphi(a) \in \Gamma$, and $\mathcal{A}(\cdot)$ is the centered area factor defined with respect to the splitting $\mathbb{G} = \mathbb{V} \cdot \mathbb{L}$.

$$(\text{II}) \quad \mathcal{A}(\mathbb{V}(a\varphi(a))) = \frac{\mathcal{C}^h \llcorner \mathbb{V}(a\varphi(a))}{(\Phi)_\#(\mathcal{C}^h \llcorner \mathbb{W})}$$

$\mathcal{C}^h$ = h-dimensional centered Hausdorff measure

Φ = graph of intrinsic linear function with $\Phi(\mathbb{W}) = \mathbb{V}(a\varphi(a))$.

Comments

1) The existence of the blow-up comes from the assumption that we have a rectifiable measure (namely it has good tangent measures)

2) The argument does not rely on an implicit function theorem and it seems does not really rely on the stratified structure

3) Their area formula extends to intrinsic Lipschitz graphs that are assumed to be a.e. intrinsically differentiable

4) The issue of missing an explicit formula for the area factor (II) remains.

Let $\mathbb{G}$ be an arbitrary homogeneous group.

F. Corni & V. Magnani, ArXiv 2023

Theorem 1.2 (Area formula). *We consider a couple* $(\mathbb{W}, \mathbb{V})$ *of complementary subgroups of* $\mathbb{G}$. *Let* m *and* M *be the topological and the Hausdorff dimensions of* $\mathbb{W}$, *respectively. We consider an open set* $A \subset \mathbb{W}$ *and a mapping* $\phi : A \to \mathbb{V}$. *We also assume that* ϕ *is intrinsically differentiable at any point of* A *and that* $d\phi : A \to \mathcal{IL}(\mathbb{W}, \mathbb{V})$ *is continuous. Setting* $\Sigma = \Phi(A)$, *where* Φ *is the graph map of* ϕ, *then for every Borel set* $B \subset \Sigma$, *we have the formula*

$$\int_{\Phi^{-1}(B)} J\Phi(n)\, d\mathcal{H}^m_{|\cdot|}(n) = \int_B \beta_d(\mathbb{T}_x)\, d\mathcal{S}^M(x), \tag{5}$$

where $\mathbb{T}_x$ *is the tangent subgroup of* Σ *at* x.

Approximating by simple functions $\Longrightarrow$

$$\int_A h \circ \Phi(x)\, J\Phi(x)\, d\mathcal{H}^m_{|\cdot|}(x) = \int_\Sigma h(y)\, \beta_d(\mathbb{T}_y)\, d\mathcal{S}^M(y)$$

it has the same form of the Euclidean area formula

Low regularity of intrinsic graphs

Bernd Kirchheim and Francesco Serra Cassano. Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 3(4):871–896, 2004.

THEOREM 3.1. *There exists an $\mathbb{H}$-regular surface $S \subset \mathbb{H}^1$ such that*

$$\mathcal{H}^{(5-\varepsilon)/2}(S) > 0 \text{ for all } \varepsilon \in (0,1)\,. \tag{18}$$

In particular, S is not 2- Euclidean rectifiable.

Combining the results by F. Corni (PhD 2021) or D. Di Donato (P.A. 2021) and Corni-Magnani (ArXiv 2023)

$\Rightarrow$ S is the intrinsic graph of a continuously intrinsically differentiable function ϕ with $S = \{w\,\phi(w)\}$

Remark

Setting $h(y) = \dfrac{\psi(y)}{\beta_d(\Pi_y)} \implies$

Jacobian with explicit formula

$$A(\Pi_{\Phi(x)}) = \frac{J\Phi(x)}{\beta_d(\Pi_{\Phi(x)})} \cdot c_d$$

area factor of Julia Nicolussi-Golo Vittone

spherical factor

where $\mathcal{H}^m_{|\cdot|} = c_d \, \mathcal{S}^m_d$

New notion of Jacobian

$(\mathbb{W}, \mathbb{V})$ couple of complementary subgroups

$A \subset \mathbb{W}$ open set, $\phi : A \longrightarrow \mathbb{V}$, $w \in A$

ϕ intrinsically differentiable at w

with intrinsic differential $d\phi_w : \mathbb{W} \longrightarrow \mathbb{V}$

$$\text{(III)} \qquad J\phi(w) = \frac{\mathcal{H}^m_{1,1}\left(G(d\phi_w)(\mathbb{B}) \right)}{\mathcal{H}^m_{1,1}(\mathbb{B})}$$

$\mathbb{B} \subset \mathbb{W}$ unit metric ball

$$G(d\phi_w)(u) = u\, d\phi_w(u)$$

graph mapping

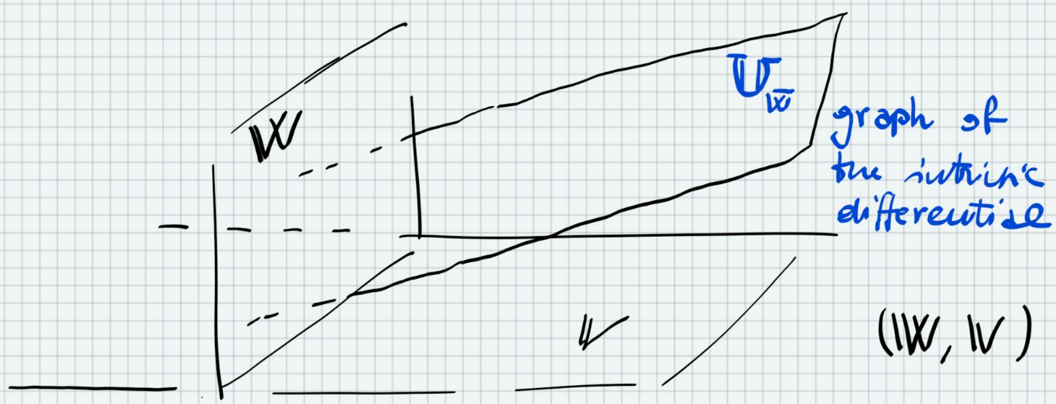
F. Corni & V. Magnani, ArXiv 2023

Proposition *Let $(\mathbb{W}, \mathbb{V})$ be a couple of complementary subgroups of $\mathbb{G}$. Let $A \subset \mathbb{W}$ be an open set, $\bar{w} \in A$ and let p be the topological dimension of $\mathbb{V}$. Let $\Phi : A \to \mathbb{G}$ be the graph map of $\phi : A \to \mathbb{V}$. Assume that ϕ is intrinsically differentiable at $\bar{w}$ and set*

$$\mathbb{U}_{\bar{w}} = \mathrm{graph}(d\phi_{\bar{w}}). \tag{52}$$

If $\mathbf{V}$ is an orienting unit p-vector of $\mathbb{V}$ and $\mathbf{W}$, $\mathbf{U}_{\bar{w}}$ are orienting unit $(q-p)$-vectors of $\mathbb{W}$ and $\mathbb{U}_{\bar{w}}$, respectively, then

$$J\Phi(\bar{w}) = \frac{|\mathbf{V} \wedge \mathbf{W}|}{|\mathbf{V} \wedge \mathbf{U}_{\bar{w}}|}.$$



$(\mathbb{W}, \mathbb{V})$ represent our system of coordinates

$$J\phi(w) = \frac{\mathcal{H}^m_{1,1}(G(d\phi_w)(A))}{\mathcal{H}^m_{1,1}(A)} = \frac{|V \wedge W|}{|V \wedge U_{\bar{w}}|}$$

does not depend on the set A

To see the independence from A we need the second blow-up, that is hidden in definition (III) of Jacobian and this is delicate since the graph map $G(d\phi_{\bar{w}})$ is nonlinear.

Behind the well-posedness of the definition of Jacobian there is an important tool of the paper.

The algebraic lemma

$$\mathcal{H}^m_{1\cdot 1}\left(\pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}}(B\cap\mathbb{T})\right) = \frac{|\mathbf{V}\wedge\mathbf{T}|}{|\mathbf{V}\wedge\mathbf{W}|}\,\mathcal{H}^m_{1\cdot 1}(B\cap\mathbb{T})$$

$\mathbf{V}$ is the unit p-vector orienting $\mathbb{V}$

$\mathbf{W}$ is the unit m-vector orienting $\mathbb{W}$

T is the unit m-vector orienting $\mathbb{T}$

$(\mathbb{W},\mathbb{V})$ and $(\mathbb{T},\mathbb{V})$ are complementary subgroups

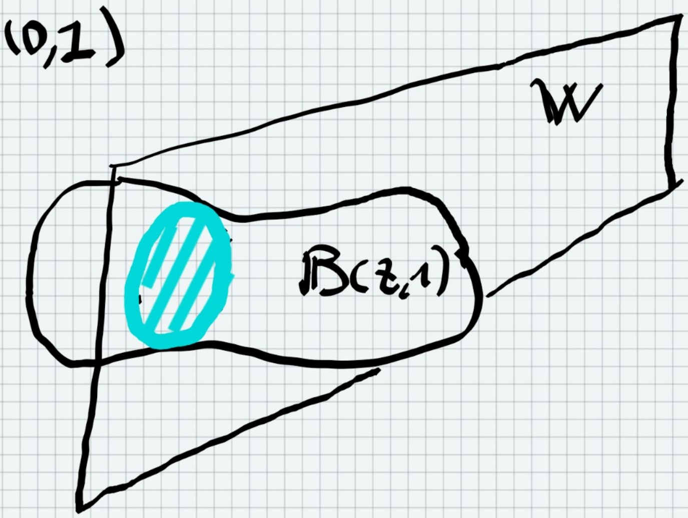
$\mathcal{H}^m_{1\cdot 1}$ is the m-dimensional Hausdorff measure in $\mathbb{G}$ with respect to the Euclidean distance

Spherical factor

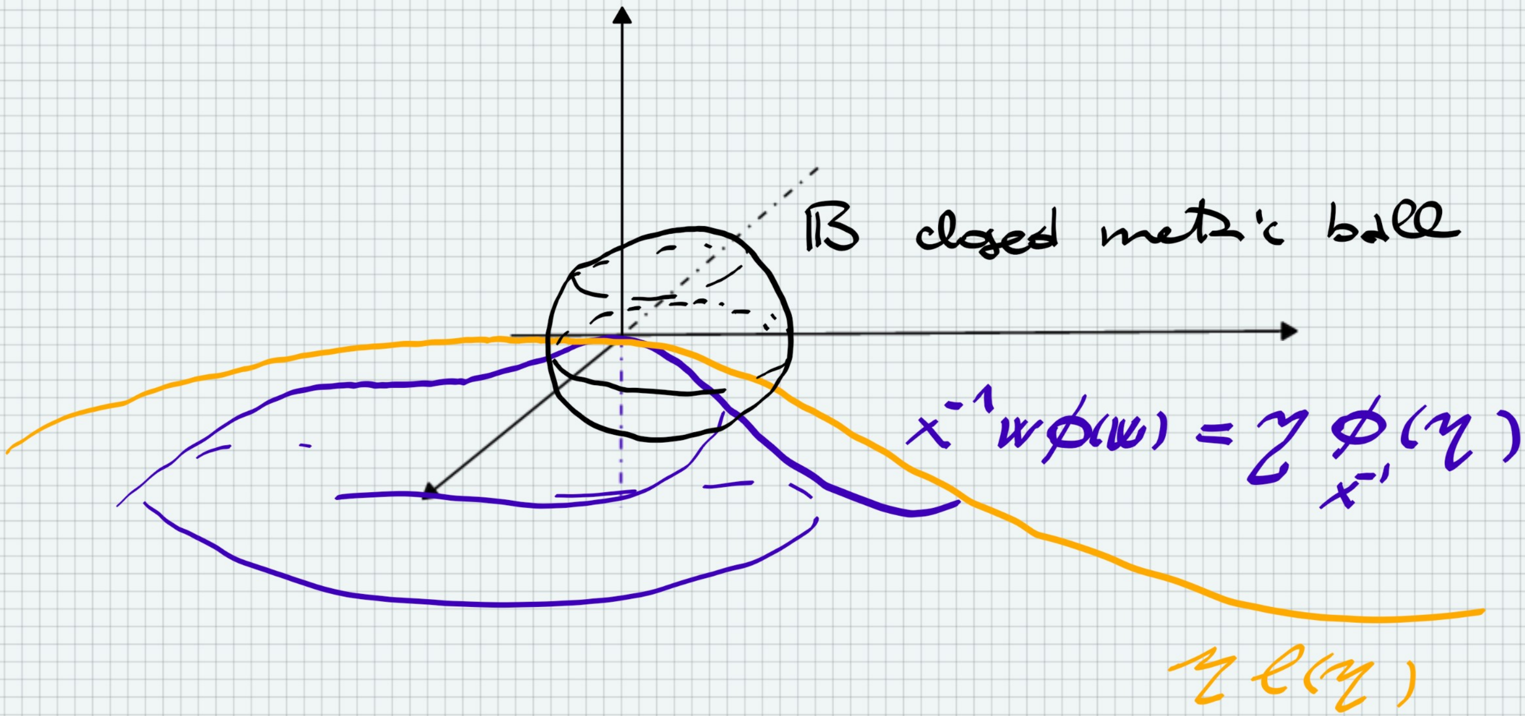
Consider a homogeneous subgroup $\mathbb{W} \subset \mathbb{G}$ and a homogeneous distance d on $\mathbb{G}$.

The spherical factor of d with respect to $\mathbb{W}$ is the number

$$\beta_d(\mathbb{W}) = \max_{z \in \mathbb{B}(0,1)} \mathcal{H}^m_{|\cdot|}(\mathbb{B}(z,1) \cap \mathbb{W}).$$

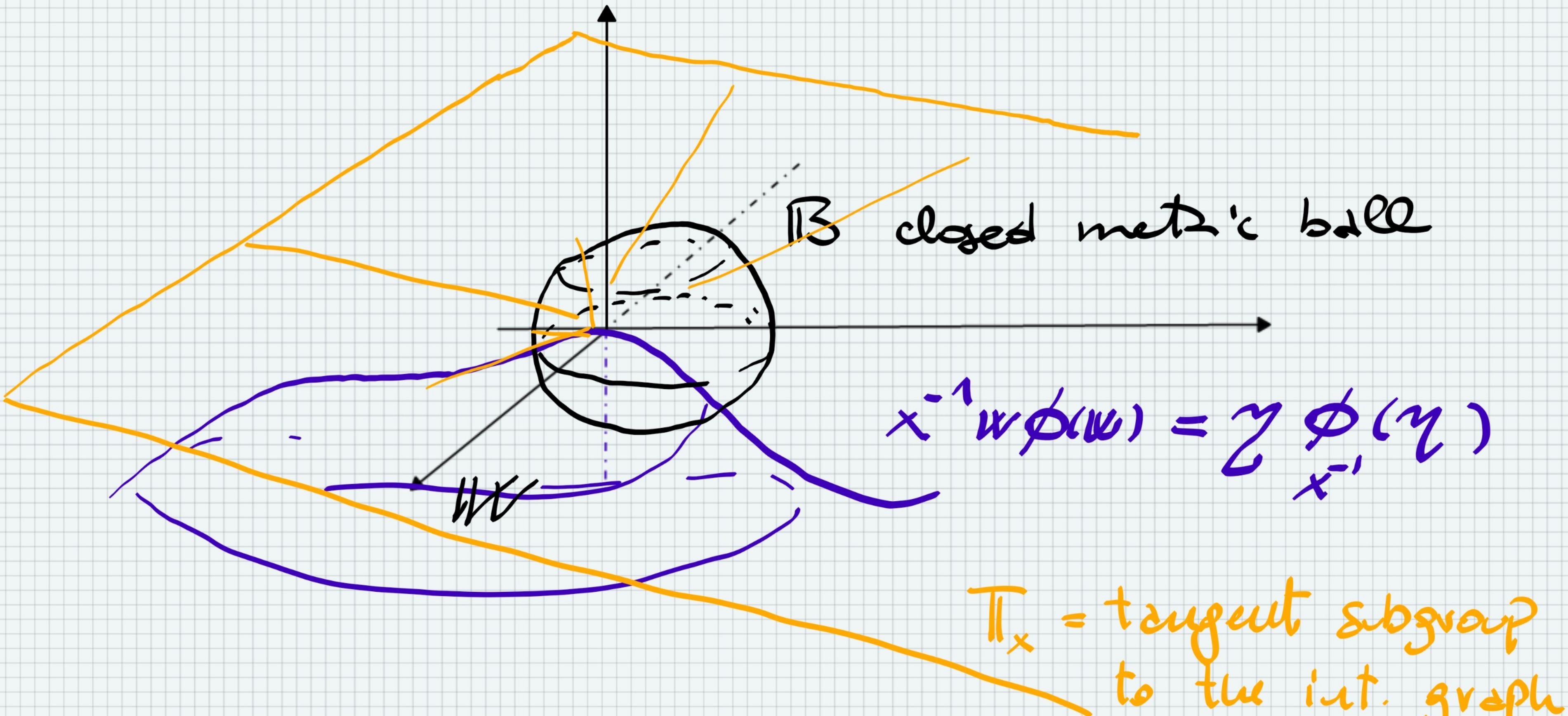


Blow-up of the intrinsic graph



we put the measure $\mu(B) = \int_{\Phi^{-1}(B)} J\phi(x) \, d\mathcal{H}^m_{|\cdot|}(x)$

on the intrinsic graph.



Definition (Tangent subgroup). Let $A \subset \mathbb{W}$ be an open set and let $\phi : A \to \mathbb{V}$ be a function. Let us fix $\bar{w} \in A$ and consider a point $\bar{x} = \bar{w}\phi(\bar{w}) \in \text{graph}(\phi)$. We say that a homogeneous subgroup $\mathbb{T}$ of $\mathbb{G}$ is the *tangent subgroup* to $\text{graph}(\phi)$ at $\bar{x}$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that we have

$$\text{graph}(\phi_{\bar{x}^{-1}}) \cap \{x \in \mathbb{G} : \|\pi_{\mathbb{W}}(x)\| < \delta\} \subset X(0, \mathbb{T}, \varepsilon).$$

- The domain of the graph map of the intrinsic differential is

$$\pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}}(\mathbb{B} \cap \mathbb{T}_x)$$

- Its $\mathcal{H}^m_H$-measure is related to $\mathcal{H}^m_H(\mathbb{B} \cap \mathbb{T}_x)$ by the algebraic lemma.

- maximizing $\mathcal{H}^m_H(\mathbb{B} \cap \mathbb{T}_x)$ in $\mathbb{B}$ we get the spherical factor

Let $\mathbb{G}$ be a stratified group.

F. Corni & V. Magnani, ArXiv 2023

Theorem *Let $\Omega \subset \mathbb{G}$ be an open set and let $f \in C^1_h(\Omega, \mathbb{M})$. Consider $\Sigma = f^{-1}(0)$ and assume that there exist an open set $\Omega' \subset \Omega$ and a homogeneous subgroup $\mathbb{V} \subset \mathbb{G}$ of topological dimension p such that $J_{\mathbb{V}} f(y) > 0$ for any $y \in \Sigma \cap \Omega'$. Let $\mathbb{W} \subset \mathbb{G}$ be a homogeneous subgroup complementary to $\mathbb{V}$ and consider the unique map $\phi : A \to \mathbb{V}$, whose graph mapping $\Phi : A \to \mathbb{G}$ satisfies $\Sigma \cap \Omega' = \Phi(A)$, where $A \subset \mathbb{W}$ is an open set. If $\mathbf{V}$ is an orienting unit p-vector of $\mathbb{V}$ and $\mathbf{W}$ is an orienting unit $(q-p)$-vector of $\mathbb{W}$, then we have*

$$\int_B \beta_d(\mathbb{T}_x)\, d\mathcal{S}^{Q-P}(x) = |\mathbf{V} \wedge \mathbf{W}| \int_{\Phi^{-1}(B)} \frac{J_H f(\Phi(n))}{J_{\mathbb{V}} f(\Phi(n))}\, d\mathcal{H}^{q-p}_{|\cdot|}(n)$$

for every Borel set $B \subset \Sigma \cap \Omega'$, where $\mathbb{T}_x$ is the tangent subgroup of Σ at x.

$$\Uparrow$$

$$J\Phi(n) = |\mathbf{V} \wedge \mathbf{W}|\, \frac{J_H f(\Phi(n))}{J_{\mathbb{V}} f(\Phi(n))}$$

Symmetric distances

Let $\mathcal{F}$ be a family of homogeneous subgroups.
A homogeneous distance d is rotationally symmetric with respect to $\mathcal{F}$ if

$$\beta_d(\mathbb{W}) \equiv \omega_d(\mathcal{F}) \quad \forall\, \mathbb{W} \in \mathcal{F}.$$

F. Corni & V. Magnani, ArXiv 2023

Theorem *Let $(\mathbb{W}, \mathbb{V})$ be a couple of complementary subgroups of $\mathbb{G}$ and denote by m and M the topological and the Hausdorff dimensions of $\mathbb{W}$, respectively. We consider an open set $A \subset \mathbb{W}$ and a mapping $\phi : A \to \mathbb{V}$. We assume that ϕ is intrinsically differentiable at any point of A and that $d\phi : A \to \mathcal{IL}(\mathbb{W}, \mathbb{V})$ is continuous. We set $\Sigma = \Phi(A)$ and suppose that d is rotationally symmetric with respect to*

$$\mathcal{F}_{\mathbb{V}} = \{\mathbb{W} \subset \mathbb{G} : \mathbb{W} \text{ homogeneous subgroup complementary to } \mathbb{V}\}.$$

Thus, setting $\mathcal{S}^M_d = \omega_d(\mathcal{F}_{\mathbb{V}})\mathcal{S}^M$, for every Borel set $B \subset \Sigma$ we have

$$\mathcal{S}^M_d \llcorner \Sigma(B) = \int_{\Phi^{-1}(B)} J\Phi(n)\, d\mathcal{H}^m_{|\cdot|}(n). \tag{7}$$

An even more explicit Jacobian

In the special case where $\phi : A \subset \mathbb{W} \longrightarrow \mathbb{V}$

and $\mathbb{V}$ is horizontal, by a result of

Antonelli - Di Donato - Don - Le Donne we have

$$D^{\phi}_{X_j} \tilde{\phi}_i = (\nabla^{\phi}\phi)_{ij}$$

where we have defined

Definition (Projected vector fields). Let $A \subset \mathbb{W}$ be an open set and let $\phi : A \to \mathbb{V}$ be a continuous function. Let Φ be the graph map of ϕ. For $j = p+1, \ldots, q$ we define the *continuous projected vector field* $D^{\phi}_{X_j}$ on $\mathbb{W}$ as

$$\left(D^{\phi}_{X_j}\right)_w (f) = \left(X_j\right)_{\Phi(w)} (f \circ \pi_{\mathbb{W}})$$

for every $w \in A$ and $f \in C^{\infty}(\mathbb{W})$.

F. Corni & V. Magnani, ArXiv 2023

Definition Let $A \subset \mathbb{W}$ be an open set, let $w \in A$ and consider a map $\phi : A \to \mathbb{V}$ intrinsically differentiable at w. We introduce the *intrinsic Jacobian of* $\widetilde{\phi}$ *at* $\widetilde{w}$ as

$$J^{\phi}\widetilde{\phi}(\widetilde{w}) = \sqrt{1 + \sum_{\ell=1}^{\min\{p, n_1 - p\}} \sum_{I \in \mathcal{I}_\ell} (M_I^{\widetilde{\phi}}(\widetilde{w}))^2},$$

where $\mathcal{I}_\ell$ is the set of multiindexes

$$\{(i_1, \dots, i_\ell, j_1, \dots, j_\ell)) \in \mathbb{N}^{2\ell} : p + 1 \le i_1 < i_2 < \dots < i_\ell \le n_1,\ 1 \le j_1 < j_2 \dots < j_\ell \le p\}.$$

We have also introduced the minors

$$M_I^{\widetilde{\phi}}(\widetilde{w}) = \det \begin{pmatrix} D^{\phi}_{X_{i_1}} \widetilde{\phi}_{j_1}(\widetilde{w}) & \dots & D^{\phi}_{X_{i_\ell}} \widetilde{\phi}_{j_1}(\widetilde{w}) \\ \dots & \dots & \dots \\ D^{\phi}_{X_{i_1}} \widetilde{\phi}_{j_\ell}(\widetilde{w}) & \dots & D^{\phi}_{X_{i_\ell}} \widetilde{\phi}_{j_\ell}(\widetilde{w}) \end{pmatrix}.$$

Finally we prove that $J\phi(w) = J^{\phi}\widetilde{\phi}(\widetilde{w})$,

which is not algebraically easy.

In this case $\mathbb{W}$ is assumed to be orthogonal to $\mathbb{V}$.

Thank You for Your Attention!