Analytic Aspects of Convex Geometry

Program

9:00-9:45 E. Werner: *Affine surface areas and approximation of convex bodies by polytopes*

9:50-10:20 A. Colesanti: *Convex bodies and partial differential equations: problems in between*

10:20-10:45 Coffee break

10:50-11:05 P. Goodey: *Mixed support functions*

11:10-11:25 L. Brandolini: *On the average decay of the characteristic function of a convex body*

11:30-11:45 E. L. Grinberg: *Inversion of the Radon transform via Gårding-Gindikin fractional integrals*

11:50-12:05 M. Ferri: *Bounds on the size functions of 3D convex objects from 2D projections*

12:10-12:25 P. M. Gruber: *Approximating of convex bodies in* $\mathbb{E}^3$ *by circumscribed polytopes*

12:30-12:45 P. Gronchi:*Revisited inequalities for convex sets*

13:00-13:45 Lunch

15:00-15:45 G. Bianchi: *Determining convex sets from their covariogram and related problems*

15:50-16:20 A. Koldobsky: *The Busemann-Petty problem via spherical harmonics*

16:20-16:45 Coffee break

16:50-17:05 R. Howard: *Estimating the distance to the convex hull*

17:10-17:25 C. Peri: *Point* $X$*-rays of convex bodies in planes of constant curvature*

17:30-17:45 D. Klain: *The Minkowski problem for polytopes*

17:50-18:05 P. Salani: *Minkowski sum of quasi-convex functions and* $p$*-capacity of convex sets*

18:10-18:25 R. Schneider: *Stability results involving surface area measures of convex bodies*

18:30-18:45 C. Zanco: *Tilings of normed spaces: The state of the art*

**Determining convex sets from their covariogram and related problems**

Gabriele Bianchi
*Università degli Studi di Firenze, Italy*
e-mail `gabriele.bianchi@unifi.it`

Let $K$ be a convex body in $\mathbb{R}^n$. The *covariogram* $g_K(x)$ of $K$ is the function

$$g_K(x) = V(K \cap (K + x)),$$

where $x \in \mathbb{R}^n$ and $V$ denotes volume in $\mathbb{R}^n$. The covariogram is clearly unchanged by a translation or a reflection. The following problem was posed by Matheron: *Does the covariogram determine a convex body, among all convex bodies, up to translation and reflection?*

This problem has been posed independently also in other contexts, like in probability and in stochastic geometry. The covariogram of $K$ is, up to a multiplicative factor, the probability density of the difference of two independent random variables $X - Y$ which are uniformly distributed over $K$. The knowledge of $g_K$ is equivalent to the knowledge of the length distribution of the chords of $K$ parallel to $u$, for each direction $u$.

The covariogram problem is also equivalent to the phase retrieval problem in Fourier analysis, restricted to the class of characteristic functions of convex bodies.

We will present recent results obtained on this problem. The answer is known to be positive for planar convex bodies with piecewise $C^2$ boundary (plus an extra minor condition), while it has been proved that in $\mathbb{R}^n$, $n \geq 4$, there are convex bodies which are not determined by their covariogram. There are currently attempts in solving the problem both for polyhedrons and for bodies with $C^2_+$ boundary in $\mathbb{R}^3$.

## On the average decay of the characteristic function of a convex body

Luca Brandolini
*Università di Bergamo ,Italy*
e-mail: brandolini@unibg.it

An important question in the study of the Fourier transform of the Lebesgue measure on a convex bounded set in $\mathbb{R}^d$ is whether the $L^2$-average decay rate is optimal in the sense that

$$\left( \int_{S^{d-1}} |\widehat{\chi}_B(R\omega)|^2 d\omega \right)^{\frac{1}{2}} \leq CR^{-\frac{d+1}{2}},$$

which is exactly the point-wise decay rate in the case when $B$ is the ball. This estimate has extensive applications in numerous problems in harmonic analysis and convexity including the distribution of lattice points in convex domains, irregularity of distribution, generalized Radon transform and others. We present some results that hold under minimal regularity assumption.

# Convex bodies and partial differential equations: problems in between

Andrea Colesanti
Università degli Studi di Firenze – Italy
e–mail: andrea.colesanti@math.unifi.it

I would like to present some problems, and related results, which are connected with both the classic theory of convex bodies and elliptic partial differential equations (PDE's) of second order.

The first problem is to prove (or disprove) that a smooth convex function $u$ of $n$ variables, defined in $\mathbf{R}^n$, such that one of the elementary symmetric functions $S_i(D^2u(x))$, $i = 1, 2, \ldots n$, of the Hessian matrix $D^2u(x)$ is equal to some positive constant for every $x$, is a quadratic polynomial. This problem may be regarded as a counterpart, in the field of PDE's, of the following: prove that a convex body in $\mathbf{R}^n$ such that one of its area measures is proportional to the $(n-1)$–dimensional Hausdorff measure on $S^{n-1}$, is a ball. The latter problem is completely settled, while the corresponding one in PDE's is solved for $i = 1$ and $i = n$. For arbitrary $i$, I proved that the claim is true if a condition on the growth of $u$ at infinity is added.

The second problem regards capacity of $n$–dimensional convex bodies; this was studied by myself and P. Salani. C. Borell proved that the classical electrostatic capacity, or 2–capacity, of convex bodies satisfies a Brunn–Minkowski type inequality. We generalized this result to the $p$–capacity, for $1 < p \leq n$. The notion of $p$–capacity arises in the theory of Sobolev spaces, and one of the equivalent definition involves the solution of an elliptic PDE.

Finally, together with D. Hug, we proved the following result: let $u$ be a convex function defined in $\mathbf{R}^n$, and assume that the Hessian measure of order one of $u$ has a $p$–integrable density with respect to the $n$–dimensional Lebesgue measure, for some $p > 1$. Then one can deduce absolute continuity results on Hessian measures of higher order (depending on $p$). This follows from classical results for functions whose Laplacian is $p$–integrable. As a consequence we obtain a new proof of a result by W. Weil, regarding absolute continuity of area measures of convex bodies.

# Bounds on the size functions of 3D convex objects from 2D projections

Massimo Ferri
*ARCES "E. De Castro", Univ. di Bologna, Italia*
e-mail `ferri@dm.unibo.it`

Patrizio Frosini
*ARCES "E. De Castro", Univ. di Bologna, Italia*
e-mail `frosini@dm.unibo.it`

Size functions are modular shape descriptors of a geometrical–topological nature. Technically, they are invariants of a pair consisting of a manifold and of a continuous real function (called *measuring function*) defined on it. Whatever the input pair, a size function is always defined on $\mathbb{R}^2$, with values in the extended integers. Size functions have already been applied to several pattern recognition and database searching tasks.

The problem of deducing information on the shape of a 3D object from 2D projections is well–known and is a relevant one. A way to face the problem is obviously to get several projections and to integrate information as, e.g., in Computer Assisted Tomography, but this is not always feasible. In computer vision, it is frequent to have access to just one view of an object.

Size functions can describe shape of objects of any dimension; however, no effort has been done so long, in order to relate these descriptors applied to a 3D object and a 2D image of it.

The present research deals with this problem, restricted to the wide and significant class of convex bodies. Given a bound on the difference between the measuring function defined in 3D, and the one defined on the projected points, the size function of the 3D object at a given real pair $(x, y)$ is proved to be bound from both sides by the size function of the image at nearby pairs. Specific examples are given.

**Presented by Massimo Ferri**

# Mixed Support Functions

Paul Goodey
*University of Oklahoma, USA*
e-mail `pgoodey@ou.edu`

Wolfgang Weil
*Universität Karlsruhe, Germany*
e-mail `weil@ma2geo.mathematik.uni-karlsruhe.de`

For convex bodies $K$ and $M$ in $\mathbb{E}^d$, we will describe a construction which shows how the $j$-th surface area measure of $K$ can be combined with the $(d-j+1)$-st surface area measure of $M$ to form the first surface area measure of a mixed convex body $T_j(K, M)$. The Minkowski sum of all these mixed bodies, for $j = 1, \ldots, d$, is the (centred) sum of all intersections of $K$ with translations of $M$. The latter observation follows from an integral translative formula of W. Weil which was the motivation for this study. Our construction relies on a result of R. Schneider which characterizes first surface area measures of polytopes.

**Presented by Paul Goodey**

### Inversion of the Radon transform via Gårding-Gindikin fractional integrals

Eric L. Grinberg
*Department of Mathematics, Temple University, Philadelphia PA 19122, USA*
e-mail `grinberg@math.temple.edu`

Boris Rubin
*Institute of Mathematics, Hebrew University, Jerusalem 91904, ISRAEL*
e-mail `boris@math.huji.ac.il`

The Funk-Radon transform has figured prominently in convex geometry during the past decade and inversion formulas have played an important role. There are many standard inversions, some involving differential operators and others involving fractional integrals. Here we consider a natural generalization of the Radon transform, which acts on functions on Grassmannians to produce functions on other Grassmannians. These transforms also occur in convex geometry and have been studied and inverted before, e.g., by I.M. Gelfand and his collaborators. We establish a connection between the Radon transform and Gårding-Gindikin fractional integrals associated to the cone of positive definite matrices, and use it to obtain Abel type representations and inversion formulae on various function spaces, including $L^p$.

**Presented by Eric Grinberg**

## Revisited inequalities for convex sets

Stefano Campi
*Dipartimento di Matematica Pura e Applicata "G. Vitali",*
*Università degli Studi di Modena e Reggio Emilia, ITALIA*
e-mail campi@unimo.it

Paolo Gronchi
*Istituto per le Applicazioni del Calcolo,*
*Consiglio Nazionale delle Ricerche, ITALIA*
e-mail paolo@iaga.fi.cnr.it

A significant part of the theory of convex bodies is based on geometric inequalities. These inequalities are usually expressed in terms of extrema of affine functionals and from this point of view an important role is played by ellipsoids and their characterizations.

We present some new applications of a result by Rogers and Shephard on continuous movements of convex sets. Classical inequalities of Convex Geometry are rediscovered and connections between open problems are drafted.

**Presented by Paolo Gronchi**

**Approximating of convex bodies in $\mathbb{E}^3$ by circumscribed polytopes**

Peter M. Gruber
*Technische Universität Wien, Austria*
e-mail `peter.gruber@tuwien.ac.at`

In this lecture we describe the asymptotics, as $n \to \infty$, of the minimum volume difference of a smooth convex body in $\mathbb{E}^3$ and its circumscribed convex polytopes with $n$ facets. In addition, a rough description of the form of the best approximating polytopes is given. Some remarks on the mathematical context of this problem are made.

# Estimating the distance to the convex hull

S. J. Dilworth
*Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA*
e-mail `dilworth@math.sc.edu`

Ralph Howard
*Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA*
e-mail `howard@math.sc.edu`

James W. Roberts
*Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA*
e-mail `roberts@math.sc.edu`

Let $A$ be a bounded subset of $\mathbf{R}^n$ and let $\operatorname{co}(A)$ be the convex hull of $A$. It is natural to estimate how much larger $\operatorname{co}(A)$ is than $A$. One easily stated result in this direction is

**Theorem** *Assume that for all $a, b \in A$ that $\operatorname{dist}((a+b)/2, A) \leq \delta$. Then there is a constant $C(n)$, only depending on the dimension, so that*

$$x \in \operatorname{co}(A) \quad \text{implies} \quad \operatorname{dist}(x, A) \leq C(n)\delta.$$

The sharp constant $C(n)$ is given. Therefore there is a sharp bound for the distance of $A$ to $\operatorname{co}(A)$ in terms of the distance from $A$ to the set of all midpoints of segments with endpoints in $A$. Surprisingly, the examples giving the sharp constant are fractals. The proof involves a detailed study of *almost convex functions*, defined by $f((x+y)/2) \leq (f(x)+f(y))/2 + 1$, and how well they can be approximated by convex functions. Several generalizations are given.

**Presented by Ralph Howard**

# The Minkowski Problem for Polytopes

Dan Klain
*University of Massachusetts Lowell, USA*
e-mail Daniel_Klain@uml.edu

The traditional solution to the Minkowski problem for polytopes involves two steps. First, the existence of a polytope satisfying given boundary data is demonstrated. In the second step, the uniqueness of that polytope (up to translation) is then shown to follow from the equality conditions of Minkowski's inequality, an inequality for mixed volumes that is typically proved in a separate context. In recent work the speaker has adapted the classical argument to prove both the existence theorem of Minkowski *and* his mixed volume inequality simultaneously, thereby demonstrating the equiprimordial relationship between these two fundamental theorems of modern convex geometry.

**The Busemann-Petty problem via spherical harmonics**

Alexander Koldobsky
*University of Missouri-Columbia, USA*
e-mail `koldobsk@math.missouri.edu`

The Busemann-Petty problem asks whether symmetric convex bodies in $R^n$ with smaller central hyperplane sections necessarily have smaller $n$-dimensional volume. The solution has recently been completed, and the answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. In this article we present a short proof of the affirmative result and its generalization using the Funk-Hecke formula for spherical harmonics.

**Point $X$-rays of convex bodies in planes of constant curvature**

Paolo Dulio
*Politecnico di Milano, Italy*
e-mail `paodul@mate.polimi.it`

Carla Peri
*Università Cattolica, Italy*
e-mail `cperi@mi.unicatt.it`

The $X$-ray of a set $A$ from a point $p$ is the function giving the total length of the intersection with $A$ of each ray issuing from $p$. In a previous paper we showed that $X$-rays from two points are enough, with some exceptions, to distinguish between any two convex bodies in non-Euclidean spaces such that the line joining the two points intersects the interior of the bodies. Here we prove that any convex body in a plane of constant curvature is uniquely determined by the $X$-rays from four points in general positions, so extending results obtained by R. Gardner and A. Volčič in Euclidean and projective plane.

**Presented by Carla Peri**

## Minkowski sum of quasi-convex functions and p-capacity of convex sets

Andrea Colesanti
*Dipartimento di Matematica "U. Dini", Universitá di Firenze, Italy*
e-mail `colesant@math.unifi.it`

Paolo Salani
*Dipartimento di Matematica "U. Dini", Universitá di Firenze, Italy*
e-mail `salani@math.unifi.it`

A function $u$ is said *quasi-convex* (*quasi-concave*) if its sublevel (superlevel) sets are convex. If $u_0$ and $u_1$ are quasi-convex (quasi-concave) functions, for every $t \in [0,1]$ we define the Minkowski sum of $u_0$ and $u_1$ as the function $u_t$ whose sublevel (superlevel) sets are the Minkowski sums of the corresponding sublevel (superlevel) sets of $u_0$ and $u_1$.

We investigate the relationship between $\nabla u_t$, $D^2 u_t$ (eventually intended in the viscosity sense) and $\nabla u_i$, $D^2 u_i$, $i = 0, 1$, proving that some relevant differential inequalities satisfied by the original functions $u_0$ and $u_1$ are inherited by $u_t$.

Applying this to quasi-concave functions in convex rings, we obtain a new and simple proof of the Brunn-Minkowski inequality for electrostatic capacity of convex sets and we are able to extend the result to the case of $p$-capacity.

**Presented by Paolo Salani**

# Stability results involving surface area measures of convex bodies

Daniel Hug
*Universität Freiburg, Germany*
hug@email.mathematik.uni-freiburg de

Rolf Schneider
*Universität Freiburg, Germany*
rschnei@uni-freiburg.de

We report on stability versions of uniqueness results for convex bodies in $\mathbb{R}^d$, where the uniqueness is implied by assumptions on area measures of order $d-1$ or lower, or by assumptions on geometrically significant integral transforms of such measures. For example, a stability result is obtained for Lutwak's extension of the Aleksandrov-Fenchel-Jessen theorem to the Brunn-Minkowski-Firey theory. In a different direction, a method used by Bourgain and Lindenstrauss in the case of projection bodies is extended. It yields some stability theorems for not necessarily symmetric convex bodies.

**Presented by Rolf Schneider**

**Affine surface areas and approximation of convex bodies by polytopes**

Elisabeth Werner
*Case Western Reserve University, USA*
e-mail `emw2@po.cwru.edu`

We discuss the classical affine surface area and its extensions, the $p$-affine surface areas, for $-\infty \leq p \leq \infty$. The 1-affine surface area is just the classical affine surface area.

We show the role it plays in convexity theory in general, and in particular in the approximation of convex bodies by polytopes. Various types of approximation will be considered, so for instance best approximation and random approximation.

It is natural that the affine surface areas should appear in this context as they "measure" the boundary structure of a convex body.

## Tilings of normed spaces: The state of the art

Clemente Zanco
*Università degli Studi, Milano, Italy*
e-mail `zanco@mat.unimi.it`

We illustrate the main results available up to now concerning tilings of infinite-dimensional normed spaces. In particular, we discuss the possibility of getting tilings with uniformly bounded tiles from above and/or from below.