Mathematical aspects of statistical mechanics

From microscopic rules
to macroscopic order

PhD course in mathematics

November 2026 – January 2027

How can simple microscopic rules produce complex collective behavior?

Knowing the parts: do we know the whole?

If we know the laws governing every particle in a system, do we understand the behavior of the system as a whole?

A macroscopic sample contains an enormous number of interacting constituents.

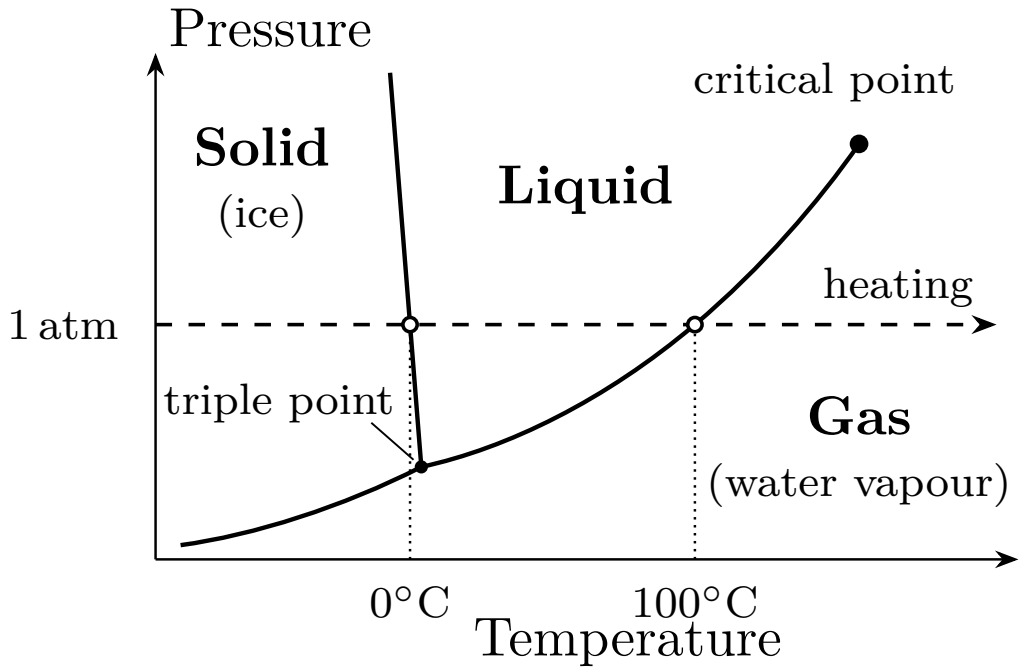
Following every microscopic trajectory is neither practical nor necessarily the most informative description.

We seek **stable, collective, macroscopic laws**.

What can the parts do together?

Water: familiar phenomena, a phase diagram

Ice, liquid water, water vapour: the same molecules.



What is a phase?

A macroscopically uniform form of matter at equilibrium.

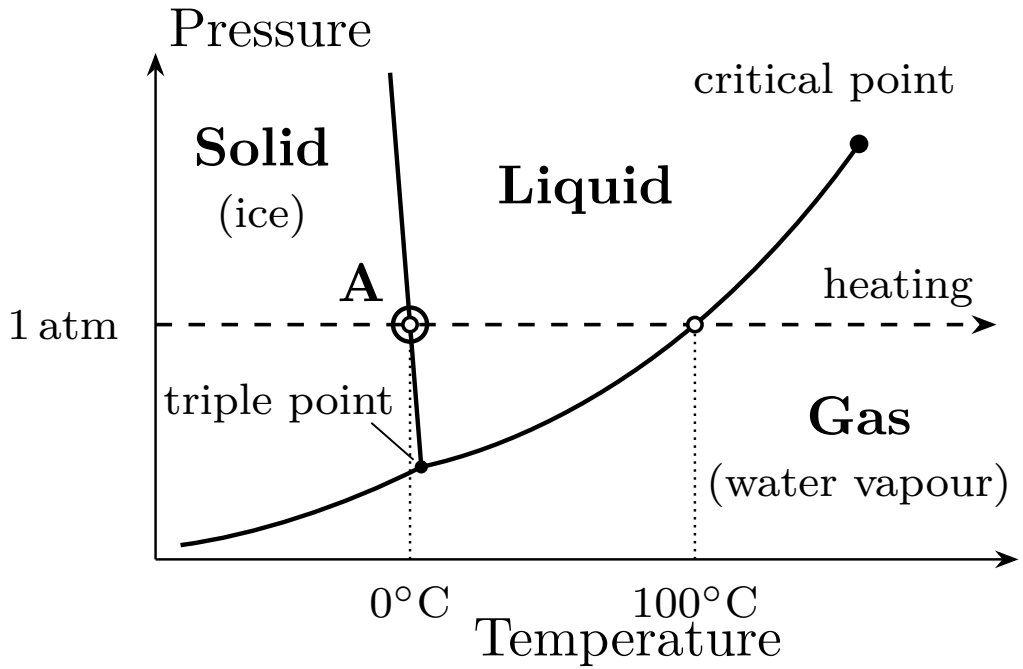
It has characteristic bulk properties, es: **density**.

Each region: conditions where one phase is stable.

What happens where the regions meet?

Water: familiar phenomena, a phase diagram

Ice, liquid water, water vapour: the same molecules.



At A: coexistence

Same temperature and pressure; different density and molecular organization.

Ice and liquid remain **two distinct phases**.

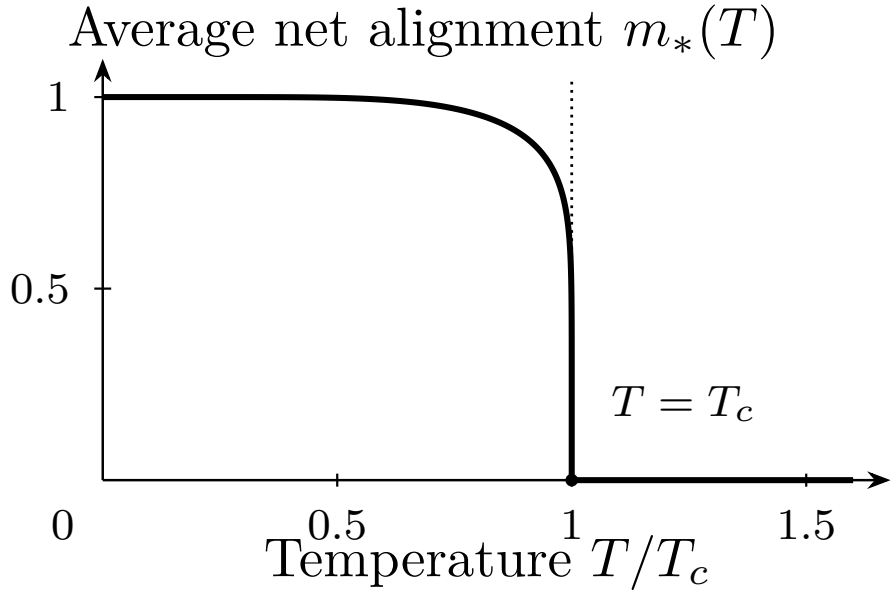
How can we define phases mathematically?

And prove that different phases can coexist?

What changes at a transition?

Model the material's tiny magnets by arrows: ↑ or ↓.

Each two-valued orientation is called a **spin**.



Magnetization is the average up-minus-down imbalance per arrow.

$m_* = 1$: all point up.

$m_* = 0$: no net preference.

Heating to T_c:

net alignment disappears.

Continuous transition: $m_*(T)$ reaches zero without a jump.

It distinguishes the regimes: an **order parameter**.

Can we derive this change from microscopic rules?

Exact 2D Ising prediction; infinite volume; positive phase at zero field.

Universality: why simple models can work

In the previous example, **continuous** means that the order parameter reaches zero **without a jump**.

Different systems can approach their transition according to the **same mathematical laws**, for example the same power-law exponents.

This shared behavior is **universality**. Systems with the same leading critical laws form a **universality class**.

Liquid–gas critical endpoint	$\longleftrightarrow$	Three-dimensional Ising model

Symmetry, dimension and interaction range help determine the class.

A simple model can explain more than one material.

From the diagram to the mathematics

What is a phase, mathematically?

How can finite systems, each described by a probability measure, produce distinct macroscopic equilibrium states in the thermodynamic limit?

What happens between phases?

How do coexistence and singular behavior arise, and which mathematical tools let us **prove that they occur**?

$$\text{Microscopic rules} \longrightarrow \text{probability measures}$$

$$\longrightarrow \text{infinite-volume limits} \longrightarrow \text{phases and transitions}$$

The microscopic rules behind the curve

On a finite box $\Lambda \subset \mathbb{Z}^d$, encode each arrow by a spin:
$\sigma_x = +1$ for $\uparrow$, $\sigma_x = -1$ for $\downarrow$. (The plot used $d = 2$.)

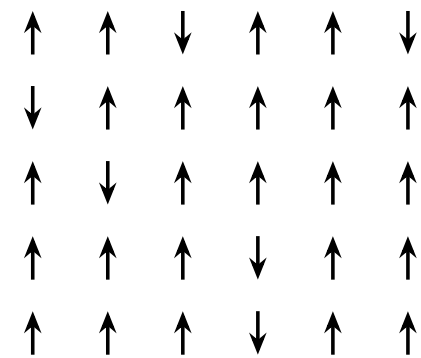
$$H_\Lambda(\sigma) = -J \sum_{\langle x,y \rangle \subset \Lambda} \sigma_x \sigma_y - h \sum_{x \in \Lambda} \sigma_x, \qquad J > 0.$$

The first sum runs over nearest-neighbor pairs.

Neighbors prefer to align.

The external field h favors one orientation.

Thermal fluctuations compete with ordering.



A schematic configuration

Simple local rules; what about collective order?

From energies to probability: Gibbs measures

The energy determines an equilibrium probability distribution:

$$\mu_{\Lambda,\beta,h}(\sigma) = \frac{e^{-\beta H_\Lambda(\sigma)}}{Z_\Lambda(\beta,h)}, \qquad \beta = (k_B T)^{-1},$$

$$Z_\Lambda(\beta,h) = \sum_{\sigma\in\{-1,+1\}^\Lambda} e^{-\beta H_\Lambda(\sigma)}.$$

Z_Λ normalizes the probabilities.

The observable behind the plot:

$$m_\Lambda(T,h) = \mathbb{E}_{\mu_{\Lambda,\beta,h}}\left[\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma_x\right].$$

Correlations describe how different sites behave collectively.
Low energy is favored, but **all configurations contribute**.

The thermodynamic limit: a mathematical surprise

For a finite system, the partition function is a finite sum.

$$p_\Lambda = \frac{1}{|\Lambda|} \log Z_\Lambda, \qquad m_\Lambda(T,h) = \frac{1}{\beta}\, \partial_h p_\Lambda(\beta,h).$$

Both functions are analytic for real $\beta > 0$ and $h \in \mathbb{R}$.

In the infinite-volume limit,

$$p(\beta,h) = \lim_{\Lambda \uparrow \mathbb{Z}^d} p_\Lambda(\beta,h)$$

it may be **nonanalytic**. The curve on slide 4 is an infinite-volume result.

How can a limit of analytic functions
become nonanalytic?

A theorem we will prove

Equilibrium: in every finite region, the conditional law given the exterior is Gibbs (the **DLR condition**).

A **pure phase** is not a mixture of other equilibrium laws.

Theorem: low-temperature phase coexistence

For the nearest-neighbor Ising model on $\mathbb{Z}^d$, $d \geq 2$, with $J > 0$ and $h = 0$, at sufficiently large β:

All-up and all-down boundaries give limiting states μ_β^+ and μ_β^- with

$$\mathbb{E}_{\mu_\beta^+}[\sigma_0] = -\mathbb{E}_{\mu_\beta^-}[\sigma_0] > 0, \qquad \mu_\beta^+ \neq \mu_\beta^- .$$

Spin reversal preserves the interaction but exchanges the phases:

spontaneous symmetry breaking.

A boundary sent to infinity still selects the bulk phase.

From the theorem to the course

Two models: Ising spins and the quantum Bose gas.

Construct infinite systems

Thermodynamic limits: prove that macroscopic quantities and equilibrium states exist.

Prove collective order

Correlation inequalities compare states. Peierls' argument balances the energy cost and the number of interfaces.

Characterize equilibrium and phases

Gibbs measures and ergodic states; the KMS condition for quantum dynamics.

Analysis and probability, with geometry and combinatorics; operator algebras for quantum equilibrium.

Beyond classical spins: the quantum Bose gas

Bosons can share one-particle states. **Bose–Einstein condensation**: one state holds a macroscopic fraction of all particles.

Why a different mathematical language?

Quantum measurements can be incompatible: position and momentum do not have simultaneous sharp values.

Their **observables** (measurable quantities) are operators with $AB \neq BA$, rather than commuting real-valued random variables.

For fixed particle number in finite volume, Gibbs probabilities become a **density operator**:

$$\rho_{\Lambda,\beta} = \frac{e^{-\beta H_\Lambda}}{\operatorname{Tr} e^{-\beta H_\Lambda}}, \qquad \langle A \rangle = \operatorname{Tr}(\rho_{\Lambda,\beta} A).$$

H_Λ and A are operators; Tr sums diagonal matrix elements.
KMS: an equilibrium condition on observables and their time evolution, meaningful even without a global density operator.

Where these methods lead: Yang–Mills

Constructive field theory: build quantum models rigorously from well-defined approximations.

On a finite lattice, field configurations U have a Gibbs-type law:

$$d\mu_{a,L}(U) = \frac{e^{-S_{a,L}(U)}}{Z_{a,L}}\, dU.$$

S: action, analogous to energy; dU: reference measure.

Two limits: system size $L \to \infty$ and lattice spacing $a \downarrow 0$.

Estimates on correlations must survive both.

Yang–Mills existence and mass gap

A Millennium Problem: construct a nontrivial quantum theory on $\mathbb{R}^4$ and prove a **positive gap above its lowest energy**.

Our starting tools: probability measures, limits, correlations.

Practical information and references

Format

PhD / Master's course

6 CFU for master's students

30 hours
November 2026 to January 2027

Starting on Monday 16/11, 11.30-13.30 (Aula Seminari)

Wednesday 18/11, 11.00-13.00 (Saletta Riunioni)

Oral exam

Prerequisites

No previous statistical mechanics.

Solid analysis and probability.

Basic quantum mechanics helpful.

Needed results will be recalled.

Main references

[1] S. Friedli and Y. Velenik, *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction* (2017).

[2] D. Ruelle, *Statistical Mechanics: Rigorous Results* (1969).

Further references

[3] G. Gallavotti (1999)

[4] J. Glimm and A. Jaffe (1987)

[5] B. Simon (1993)

[6] A. Ya. Khinchin (1949)

The course does not follow a single textbook.

How do microscopic interactions
produce macroscopic order?

And how can we **prove it mathematically**?

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