A semi-discrete scheme for horizontal mean curvature flow in the Heisenberg group

Annalisa Cesaroni*, Matteo Novaga† and Paola Pozzi‡

Abstract

The horizontal mean curvature flow is a geometric evolution associated with the horizontal perimeter in sub-Riemannian spaces. In this paper, we introduce a minimizing movement scheme for this flow in the first Heisenberg group $\mathbb{H}$, and prove the existence of a flat flow which is locally 1/2-Hölder continuous in time, with respect to the L^1-distance. The construction applies to arbitrary bounded sets of finite horizontal perimeter and does not require smoothness or symmetry of the initial datum.

Contents

1 Introduction **1**

2 The Heisenberg group **4**
2.1 Metric structure 5
2.2 Differential structure 7

3 Horizontal perimeter and mean curvature **9**
3.1 Horizontal mean curvature 10
3.2 Horizontal mean convex sets 12
3.3 Horizontal perimeter and functions of bounded variations 14
3.4 Outward minimizing sets 17

4 The minimizing movement scheme **20**
4.1 Definition and basic properties 20
4.2 The scheme for unbounded sets 24
4.3 Density estimates 25
4.4 The flat flow 30
4.5 Remarks on the consistency of the scheme 32

1 Introduction

The Heisenberg group is the simplest noncommutative Carnot group and provides a basic setting for the study of geometric variational problems in sub-Riemannian geometry [8, 28, 29]. Its geometry is determined by a two-dimensional, nonintegrable horizontal distribution and by the associated Carnot–Carathéodory distance. The horizontal perimeter measures the variation of characteristic functions along the horizontal directions. The theory of sets of finite horizontal

*Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, via Trieste 63, 35131 Padova, Italy, annalisa.cesaroni@unipd.it

†Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy, matteo.novaga@unipi.it

‡Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Straße 9, 45127 Essen, Germany, paola.pozzi@uni-due.de

perimeter, including rectifiability and compactness, supplies the measure-theoretic framework for these problems [22, 8]. The group $\mathbb{H}$ has homogeneous dimension 4: the volume scales with degree 4, whereas the horizontal perimeter scales with degree 3.

For a smooth surface $\Sigma_t = \partial E_t$, horizontal mean curvature flow prescribes, at non-characteristic points, the velocity

$$\partial_t p(t) = -H_{\mathbb{H}}(p(t), E_t)\nu_{\mathbb{H}}(p(t)),$$

where $\nu_{\mathbb{H}}$ is the horizontal outer unit normal and $H_{\mathbb{H}} = \operatorname{div}_{\mathbb{H}}\nu_{\mathbb{H}}$ is its horizontal divergence. A distinctive difficulty is the presence of characteristic points, at which the tangent plane coincides with the horizontal distribution. At such points the horizontal normal is not defined, even when the surface is smooth in the Euclidean sense. Thus, besides the possible formation of singularities during the evolution, the geometry itself introduces a degeneracy in the classical motion law.

Related difficulties already arise in the stationary theory. Danielli, Garofalo, Nhieu and Pauls [13] show that stability restores a Bernstein-type rigidity: the only stable C^2 entire H-minimal graphs with empty characteristic locus are vertical planes. For the isoperimetric problem, existence of minimizers in Carnot groups was established by Leonardi and Rigot [25]. Pansu balls are the natural candidate minimizers in $\mathbb{H}$, and their role is supported by classification results under axial symmetry, Euclidean convexity, or C^2 regularity assumptions [30, 31, 36]. In particular, Monti and Rickly [31] characterize convex isoperimetric sets without imposing additional smoothness or symmetry. Calibration methods also yield isoperimetric comparisons with Pansu balls [35]. These results provide useful geometric models and variational tools for the evolution problem considered here.

Several weak approaches to horizontal mean curvature flow have been developed. In the level-set formulation, the evolving sets are represented as sublevel sets of a function satisfying, away from points where its horizontal gradient vanishes,

$$u_t = |\nabla_{\mathbb{H}} u| \operatorname{div}_{\mathbb{H}} \frac{\nabla_{\mathbb{H}} u}{|\nabla_{\mathbb{H}} u|}\,.$$

Capogna and Citti [7] extended the viscosity approach to Carnot groups, proving existence and comparison results under suitable restrictions. Ferrari, Liu and Manfredi [18] established existence, uniqueness and stability for axisymmetric viscosity solutions in $\mathbb{H}$, and obtained an explicit evolution starting from a Korányi sphere. Baspinar and Citti [4] proved comparison and uniqueness results in step-two Carnot groups and in the roto-translation group by relating viscosity solutions to Riemannian approximations; their comparison theorem is formulated for continuous solutions which are constant outside a compact set. A paper by Bombardieri, Fogagnolo and Franceschi [6] exploring qualitative properties of level set viscosity solutions with mean-convex initial data is in preparation. Stochastic control representations provide another approach to the level-set equation [16], with further developments in [15, 23] concerning optimal controls and approximating flows. More recently, Citti, Dirr, Dragoni and Grande [12] derived horizontal mean curvature motion through a formal asymptotic expansion of a nonlocal mean-field equation under Heisenberg dilations. Their analysis at non-characteristic points and numerical experiments provide a further approximation of the flow, motivated also by models of visual processing.

Our approach in this paper is variational. We adapt the time-discrete construction introduced in the Euclidean setting by Almgren, Taylor and Wang [1] and Luckhaus and Sturzenhecker [26]. This construction has been developed for a variety of geometric evolutions, including forced, anisotropic and inhomogeneous mean curvature flows; see, for instance, [5, 10]. Given a time step $h > 0$ and a bounded set E of finite horizontal perimeter, we consider minimizers of

$$J_h(F) := P_{\mathbb{H}}(F) + \frac{1}{h}\int_F d_E(p)\,dp,$$

where d_E denotes the signed Carnot–Carathéodory distance, negative in E. Up to a constant independent of F, this functional can be written as

$$P_{\mathbb{H}}(F) + \frac{1}{h}\int_{E\triangle F} |d_E(p)|\,dp.$$

The second term penalizes the displacement from the preceding set, while minimization decreases the horizontal perimeter. Iterating the scheme, with a consistent choice of maximal or minimal minimizers, defines a piecewise constant evolution $E_h(t)$.

The use of the Carnot–Carathéodory distance is closely related to the geometry of the prescribed motion. Arcozzi and Ferrari [2, 3] studied metric normals and the regularity and horizontal Hessian of the signed distance to a smooth surface. In particular, for a C^3 boundary, at non-characteristic points one has

$$\nabla_{\mathbb{H}} d_E = \nu_{\mathbb{H}}, \qquad \mathrm{div}_{\mathbb{H}} \nabla_{\mathbb{H}} d_E = H_{\mathbb{H}}.$$

These identities motivate the relation between the discrete variational problem and the horizontal mean curvature flow. They also indicate why consistency is delicate near characteristic points, where a regular tubular description by the signed distance is unavailable.

An important ingredient of our analysis is a class $\mathcal{B}$ of bounded C^2, axially symmetric sets which are strictly starshaped with respect to the group of dilations, horizontally mean convex, and have bounded horizontal mean curvature; see Definition 3.27. Extending the horizontal normal along the foliation generated by dilations, we construct a calibration and prove that these sets are outward minimizing. More precisely, if $H_{\mathbb{H}} \geq \delta \geq 0$ on the non-characteristic part of ∂B, $B \in \mathcal{B}$, Proposition 3.28 gives

$$P_{\mathbb{H}}(F) \geq P_{\mathbb{H}}(B) + \frac{\delta}{\lambda_F} |F \setminus B|, \qquad B \subseteq F,$$

where λ_F is the smallest dilation parameter needed to contain F, with $\lambda_F = +\infty$ for unbounded F. The class includes Korányi balls, with $\delta = 0$, and Pansu balls of parameter R, with $\delta = 2/R$. In particular, Pansu balls give uniform confinement of the discrete evolutions. We also establish order preservation for the extremal minimizers and preservation of outward minimality under the hypotheses of Proposition 4.4. For strictly mean convex members of $\mathcal{B}$, Proposition 4.5 gives a quantitative one-step contraction estimate. The symmetry assumptions enter these geometric constructions, but are not imposed on the sets evolved by the scheme.

We then prove estimates for general discrete minimizers $F \in T_h(E)$ of the functional J_h. The displacement bound

$$\sup_{E \triangle F} |d_E| \leq \gamma \sqrt{h}$$

is complemented by two-sided volume density estimates and horizontal perimeter bounds in Carnot–Carathéodory balls centred on the interface, at radii up to a fixed multiple of $\sqrt{h}$. Their respective orders, r^4 and r^3, reflect the homogeneous dimension of $\mathbb{H}$. Combining these bounds with a volume-distance comparison and the discrete perimeter dissipation yields compactness as $h \to 0$.

Our main result, Theorem 4.17, gives a sequence $h_i \to 0$ along which the discrete evolutions converge in L^1 at every positive time to a bounded set $E(t)$ of finite horizontal perimeter. The limiting evolution satisfies

$$|E(t) \triangle E(t')| \leq C|t - t'|^{1/2}, \qquad t, t' > 0, \quad |t - t'| < 1,$$

where C depends only on the initial horizontal perimeter and the isoperimetric constant. If the initial set also satisfies $|\partial E| = 0$, the same estimate holds up to $t = 0$, with $E(0) = E$. We call such a subsequential limit a flat flow. Its uniqueness and its agreement with smooth horizontal mean curvature flow or with the viscosity level-set evolution are not established here.

The paper is organized as follows. Section 2 recalls the geometry of $\mathbb{H}$ and its distance functions. Section 3 collects the relevant facts about horizontal perimeter and mean curvature and develops the calibration argument for outward minimizing sets. Section 4 introduces the scheme, proves the density and displacement estimates, and constructs the flat flow, before discussing the remaining consistency questions.

Aknowledgments A.C and M.N. are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

2 The Heisenberg group

We briefly recall relevant definitions and refer to [8, Chapter 2] for more details. The Heisenberg group $\mathbb{H}$ is the Lie group $\mathbb{R}^3$ with the usual differentiable structure and composition given by

$$(x_1, y_1, z_1) \circ (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)).$$

The dilations of parameter $\lambda > 0$ in $\mathbb{H}$ are defined as

$$\delta_\lambda(p) = \delta_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z).$$

The Haar measure on $\mathbb{H}$ is the Lebesgue measure in $\mathbb{R}^3$. The left translation $L_p : \mathbb{H} \to \mathbb{H}$ is defined simply as $L_p(q) = p \circ q = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1))$, where $p = (x_1, y_1, z_1)$, $q = (x_2, y_2, z_2)$. A straightforward calculation gives that for any p, q, $\bar{q}$ in $\mathbb{H}$, and $\lambda \in [0, 1]$, we have that $L_p(\lambda q + (1 - \lambda)\bar{q}) = \lambda L_p(q) + (1 - \lambda) L_p(\bar{q})$, yielding that Euclidean convex sets remain Euclidean convex after left-translation.

If $|E|$ denotes the Lebesgue measure of any measurable set E, we have $|L_p(E)| = |E|$. Moreover it holds $|\delta_\lambda(E)| = \lambda^4 |E|$. In the following we will denote

$$E^\lambda = \delta_\lambda(E).$$

We introduce the left invariant vector fields at every point $p = (x, y, z) \in \mathbb{H}$

$$X(p) = \left(1, 0, -\frac{y}{2}\right)^t \qquad Y(p) = \left(0, 1, \frac{x}{2}\right)^t \qquad T(p) = (0, 0, 1)^t$$

which will be identified with the differential operators

$$X(p) = \partial_x - \frac{y}{2}\partial_z \qquad Y(p) = \partial_y + \frac{x}{2}\partial_z \qquad T(p) = \partial_z. \tag{1}$$

The horizontal distribution $\mathcal{H}$ in every point $p = (x, y, z) \in \mathbb{H}$ is generated by the vector fields $X(p), Y(p)$, that is $\mathcal{H}(p) = \mathrm{Ker}[\partial_z - \frac{1}{2}(x\partial_y - y\partial_x)]$. The horizontal distribution is nonintegrable since the Lie brackets of X, Y satisfy $[X(p), Y(p)] = T(p)$.

A curve $\gamma : [0, 1] \to \mathbb{H}$ is horizontal if at every $s \in [0, 1]$, it holds

$$\gamma_3'(s) = -\frac{1}{2}\gamma_2(s)\gamma_1'(s) + \frac{1}{2}\gamma_1(s)\gamma_2'(s).$$

Hence, given a planar curve (γ_1, γ_2) in $\mathbb{R}^2$ and an initial point $z_0 = \gamma_3(0)$, there exists a unique lift to a horizontal curve $(\gamma_1, \gamma_2, \gamma_3) : [0, 1] \to \mathbb{H}$ where

$$\gamma_3(s) = \gamma_3(0) + \frac{1}{2}\int_0^s -\gamma_2(s)\gamma_1'(s) + \gamma_1(s)\gamma_2'(s) ds. \tag{2}$$

In particular note that the height $|\gamma_3(1) - \gamma_3(0)|$ equals the area spanned by the planar curve $(\gamma_1(s), \gamma_2(s))$ for $s \in [0, 1]$.

We define the Carnot–Carathéodory distance among 2 points $p, q \in \mathbb{H}$ as follows:

$$d_{CC}(p, q) = \inf\{T \ \mid \exists\, \gamma : [0, T] \to \mathbb{R}^3, \text{s.t.} \gamma(0) = p, \gamma(T) = q, \\ (\gamma_1')^2 + (\gamma_2')^2 \leq 1 \ \ \gamma_3' = \frac{1}{2}(-\gamma_2\gamma_1' + \gamma_1\gamma_2')\}. \tag{3}$$

It holds that $d_{CC}(p, q) = d_{CC}(w \circ p, w \circ q)$ for all $w \in \mathbb{H}$, so in particular $d_{CC}(p, q) = d_{CC}(q^{-1} \circ p, 0) = d_{CC}(0, p^{-1} \circ q)$ where $q^{-1} = (-x, -y, -z)$ if $q = (x, y, z)$, see [8, Section 2.2].

A sub-Riemannian metric on $\mathbb{H}$ is determined by any choice of inner product on the horizontal subbundle $\mathcal{H}$. In this way it is possible to define the length of horizontal curves and equip $\mathbb{H}$ with the structure of a metric length space.

We fix a Riemannian metric g on $\mathbb{H}$ such that the vector fields $X(p), Y(p), T(p)$ are orthonormal at every point $p = (x, y, z)$:

$$g_p = \begin{pmatrix} 1 + y^2/4 & -xy/4 & y/2 \\ -xy/4 & 1 + x^2/4 & -x/2 \\ y/2 & -x/2 & 1 \end{pmatrix}. \tag{4}$$

Note that $det g_p = 1$, so the volume form coincides with the standard one. Note also that $X^t g_p = (1, 0, 0)$ and $Y^t g_p = (0, 1, 0)$, whereas $T^t g_p = (\frac{y}{2}, -\frac{x}{2}, 1)$.

We define the length of a horizontal curve $\gamma : [0, 1] \to \mathbb{H}$ as

$$L(\gamma) = \int_0^1 \sqrt{(\gamma'(s) \cdot_g X(\gamma(s)))^2 + (\gamma'(s) \cdot_g Y(\gamma(s)))^2} ds = \int_0^1 \sqrt{(\gamma_1')^2 + (\gamma_2')^2} ds$$

where $\gamma' \cdot_g X(\gamma) = (\gamma')^t g_\gamma X(\gamma)$.

Remark 2.1 (Geodesics)**.** Recalling (2) if we fix $(0, 0, 0)$ and $p = (x_1, y_1, z_1)$ in $\mathbb{H}$, the problem of finding a horizontal curve of minimal length connecting 0 and p coincides with the problem of finding a curve in the plane of minimal length connecting $(0, 0)$ and (x_1, y_1) among the curves such that the signed area of the region S delimited by the curve and the segment joining $(0, 0)$ and (x_1, y_1) is fixed and equal to z_1. Therefore a length minimizing curve between 0 and p is the lift of a circular arc joining the origin with (x_1, y_1), whose convex hull has signed area z_1. If $p = (0, 0, z_1)$ there are infinitely many geodesics (which are all the lifts of circles of area $|z_1|$ passing through the origin), whereas if $x_1^2 + y_1^2 \neq 0$ it is possible to show that there exists a unique geodesic connecting 0 and p (see [8, §2.3] for further details and a representation formula).

Definition 2.2 (Pansu sphere)**.** *We consider the geodesic* $\gamma : [0, 2\pi] \to \mathbb{H}$ *joining* 0 *and* $(0, 0, \pi R^2)$ *which are parametrized as*

$$\gamma(s) = (R(1 - \cos s), -R \sin s, R^2(s - \sin s)/2).$$

The curvature of the planar projection of the geodesic is $1/R$*. Rotating such a geodesic about the* z*-axis produces a surface of revolution which is called the Pansu sphere.*

The Pansu sphere is conjectured to be the isoperimetric set, see [33].

2.1 Metric structure

In the previous section we introduced the Carnot–Carathéodory distance on $\mathbb{H}$. There are other possible definitions of distance. For $p, q \in \mathbb{H}$ we have:

- Carnot–Carathéodory distance

$$d_{CC}(p, q) = \inf\{L(\gamma) \mid \gamma : [0, 1] \to \mathbb{H}^1 \text{ is horizontal}, \gamma(0) = p, \gamma(1) = q\}; \tag{5}$$

  This distance is left invariant and 1-homogeneous with respect to dilations.

- $d_\infty(p, q) := \|p^{-1} \circ q\|_\infty$, where the box norm $\|\cdot\|_\infty$ is given by

$$\|p\|_\infty = \max\{(x^2 + y^2)^{\frac{1}{2}}, 2|z|^{\frac{1}{2}}\};$$

- $d_{Kor}(p, q) := \|p^{-1} \circ q\|_{Kor}$, where the Korányi norm $\|\cdot\|_{Kor}$ is given by

$$\|p\|_{Kor} = (|x^2 + y^2|^2 + 16|z|^2)^{1/4}.$$

The above distances are all equivalent ([29, page 5]), i.e. there exists an absolute constant $C > 0$ such that

$$Cd.(p,q) \le d_{CC}(p,q) \le C^{-1} d.(p,q) \qquad \forall p,q \in \mathbb{H}, \qquad \text{for} \quad d. = d_\infty \text{ or } d. = d_{Kor}, \tag{6}$$

hence they induce the same topological structure.

Based on this equivalence, we have the following relation that will be useful later on: for every $\lambda > 0$,

$$d_{CC}(p, \delta_\lambda p) \le C\|p\|_\infty \max\left(|\lambda - 1|, \sqrt{|\lambda^2 - 1|}\right). \tag{7}$$

Indeed by 1-homogeneity and (6) we have, renaming the constant C from line to line,

$$\begin{aligned} d_{CC}(p, \delta_\lambda p) &= d_{CC}(0, p^{-1} \circ \delta_\lambda p) \le C\|p^{-1} \circ \delta_\lambda p\|_\infty = C\|(\lambda - 1)x, (\lambda - 1)y, (\lambda^2 - 1)z)\|_\infty \\ &= C \max\left(|\lambda - 1|\sqrt{x^2 + y^2}, 2\sqrt{|\lambda^2 - 1|}\sqrt{|z|}\right) \le C\|p\|_\infty \max\left(|\lambda - 1|, \sqrt{|\lambda^2 - 1|}\right) \end{aligned}$$

where $p = (x, y, z)$.

The relation to the Euclidean one is explained in the following

Proposition 2.3. *[22, Prop.2.1] For any bounded subset $\Omega \subset \mathbb{H}$ there exist positive constants $c_1(\Omega)$, $c_2(\Omega)$ such that*

$$c_1(\Omega)|p - q|_{\mathbb{R}^3} \le d_{CC}(p,q) \le c_2(\Omega)|p - q|_{\mathbb{R}^3}^{\frac{1}{2}}$$

for $p, q \in \Omega$. In particular, the topologies defined by d_{CC} and by the Euclidean distance coincide.

It is possible to associate to every distance above the corresponding family of balls. In particular we will consider the CC balls

$$B_{CC}(p,r) = \{q \in \mathbb{H} \,:\, d_{CC}(p,q) < r\}. \tag{8}$$

Note that

$$|B_{CC}(p,r)| = r^4 |B_{CC}(0,1)|$$

for any $p \in \mathbb{H}$ and $r > 0$. For $p \in \mathbb{H}$ and sets $A, B \subset \mathbb{H}$ let us introduce the distance between p and the set A and the Hausdorff distance between A, B:

$$\operatorname{dist}(p, A) := \inf_{q \in A} d_{CC}(p,q) \qquad \operatorname{dist}(A,B) = \max\left\{\sup_{a \in A} \operatorname{dist}(a,B), \sup_{b \in B} \operatorname{dist}(b,A)\right\}. \tag{9}$$

Given $E \subset \mathbb{H}$ open bounded set we define the <u>signed distance</u> to be

$$d_E(p) = \begin{cases} d_{CC}(p,E), & p \in \mathbb{H} \setminus E, \\ -d_{CC}(p, \mathbb{H} \setminus E), & p \in E. \end{cases} \tag{10}$$

The signed distance function is a Lipschitz map with respect to the metric structure in $\mathbb{H}$, $|d_E(p) - d_E(q)| \le C d_{CC}(p,q)$, hence only locally Hölder continuous with respect to the Euclidean distance by Prop. 2.3 and the equivalence of the above norms [29].

The metric space $(\mathbb{H}, d_{CC})$ is complete and locally compact. By the definition of d_{CC}, it is also a length space, and also a geodesic space, namely for all $p, q \in \mathbb{H}$ there exists a horizontal curve γ with $\gamma(0) = p$ and $\gamma(1) = q$, such that $d_{CC}(p,q) = L(\gamma)$, see [29], [8, Chapter 2].

For any measurable set F and $t \in [0,1]$ we define $F(t)$ to be the set of points where the volume density of F equals t, i.e.

$$F(t) = \left\{p \in \mathbb{H} \,:\, \lim_{r \to 0} \frac{|F \cap B_{CC}(p,r)|}{|B_{CC}(p,r)|} = t\right\}$$

In the following we identify any measurable set with the representative given by the set of Lebesgue points of the characteristic functions, i.e. with $F(1)$.

The essential boundary $\partial^e F$ of F is defined by $\partial^e F := \mathbb{H} \setminus (F(1) \cup F(0))$. A set B is said to be C^k if its boundary ∂B is C^k in the classical (euclidean) sense.

We introduce the definition of sets which are strictly starshaped with respect to the origin, which will be useful to construct foliations of the space $\mathbb{H}$, see [19, 17].

Definition 2.4. *We say that a C^1 bounded open set B which contains 0, is strictly starshaped with respect to 0 (with respect to the group of dilations) if*

$$\forall p \in \partial B \quad Z(p) \cdot N(p) > 0, \tag{11}$$

where $N(p)$ is the outer Euclidean normal, and for $p = (x, y, z)$, $Z(p) = (x, y, 2z)$ is the vectorial field generated by the dilations.

In particular strictly starshaped sets fibrate all the space by dilations $B^\lambda = \delta_\lambda B$.

Lemma 2.5. *Let B a C^1 bounded open set which contains 0 and which is strictly starshaped with respect to the origin according to the previous definition. Then $\mathbb{H} = \cup_{\lambda \geq 0} B^\lambda$ and for every $p \in \mathbb{H} \setminus \{0\}$ there exists a unique λ such that $p \in \partial B^\lambda$.*

Proof. Obviously $\delta_0 p = 0$ for every p.

Let us assume that $B = \{p,\ F(p) < 0\}$, $\partial B = \{p,\ F(p) = 0\}$ for some C^1 function $F : \mathbb{H} \to \mathbb{R}$ with $|\nabla F| \neq 0$ on ∂B. Therefore the condition that B is strictly starshaped reads $\nabla F(p) \cdot Z(p) > 0$ for all $p \in \partial B$.

First of all we observe that for every $q \in \mathbb{H} \setminus \{0\}$ there exist $\lambda > 0$ and $p \in \partial B$ such that $\delta_\lambda p = q$. Let us define $h(\lambda) := \delta_\lambda q$ for $\lambda \in [0, +\infty)$. Then $h(0) = 0 \in B$ and $h(1) = q$. Moreover since B is bounded there exists $\lambda_0 > 0$ such that $h(\lambda_0) \notin B$. Therefore $F(h(0)) < 0$, whereas $F(h(\lambda_0)) > 0$, and the function $\lambda \to F(h(\lambda))$ is continuous: this implies that there exists $\bar{\lambda} \in (0, \lambda_0)$ such that $F(h(\bar{\lambda})) = 0$. Hence $\delta_{\bar{\lambda}} q = p \in \partial B$, and $q = \delta_{\frac{1}{\bar{\lambda}}} p$.

Next we prove that such $\lambda > 0$ and $p \in \partial B$ are unique. Assume by contradiction that there exists $p_1, p_2 \in \partial B$, λ_1, λ_2 such that $q = \delta_{\lambda_1} p_1 = \delta_{\lambda_2} p_2$. Without loss of generality we may fix $0 < \frac{\lambda_1}{\lambda_2} \leq 1$. Note that $\delta_{\frac{\lambda_1}{\lambda_2}} p_1 = p_2$.

Now we claim that for every $p \in \partial B$ it holds that $\delta_\lambda p \in B$ for all $\lambda < 1$, that is $F(\delta_\lambda p) < 0$. If the claim is true, we conclude from $\delta_{\frac{\lambda_1}{\lambda_2}} p_1 = p_2$ with $p_1, p_2 \in \partial B$ and $\frac{\lambda_1}{\lambda_2} \leq 1$ that necessarily $\frac{\lambda_1}{\lambda_2} = 1$, and then also $p_1 = p_2$.

We are left to prove the claim. Let us define $g(\lambda) := F(\delta_\lambda p)$. We have that $g(1) = 0$ and $g(0) < 0$ (since 0 is in B). We compute $g'(\lambda) = \nabla F(\delta_\lambda p) \cdot \frac{1}{\lambda} Z(\delta_\lambda p)$. In particular $g'(1) > 0$. Therefore there exists $\hat{\delta} > 0$ such that $g(\lambda) < 0$ for every $\lambda \in (1 - \hat{\delta}, 1)$. Let's assume by contradiction that there exists $\lambda_0 = \sup\{\lambda \leq 1 \ \text{ s.t. } g(\lambda) \geq 0\}$. Then $0 < \lambda_0 \leq 1 - \hat{\delta}$. By continuity $g(\lambda_0) = 0$ and then $\delta_{\lambda_0} p \in \partial B$, and $g(\lambda) < 0$ for $\lambda \in (\lambda_0, 1)$. Recall that $g'(\lambda_0) = \nabla F(\delta_{\lambda_0} p) \cdot \frac{1}{\lambda_0} Z(\delta_{\lambda_0} p) > 0$ since $\delta_{\lambda_0} p \in \partial B$. This contradicts the fact that $g(\lambda_0) = 0$ and $g(\lambda) < 0$ for $\lambda > \lambda_0$ and so, we conclude that for every $p \in \partial B$, it holds that $\delta_\lambda p \in B$ for every $\lambda \in [0, 1)$.

□

2.2 Differential structure

We will indicate the Levi-Civita connection in $(\mathbb{H}, g)$ as D, see [8, Chapter 2]. Upon recalling (1) we have that

$$D_X(X) = 0 \quad D_X(Y) = \tfrac{1}{2} T \quad D_X(T) = -\frac{1}{2} Y \tag{12}$$

$$D_Y(X) = -\frac{1}{2} T \quad D_Y(Y) = 0 \quad D_Y(T) = \frac{1}{2} X \tag{13}$$

$$D_T(X) = -\frac{1}{2} Y \quad D_T(Y) = \tfrac{1}{2} X \quad D_T(T) = 0. \tag{14}$$

For a vector field U we define

$$J(U) := D_U(2T). \tag{15}$$

Hence by the Lie bracket condition we have that $J(X) = -Y$, $J(Y) = X$ and $J(T) = 0$. Thus J defines an isometry on the horizontal distribution and $J^2 = -Id$.

Given a C^1 function $f : \mathbb{H} \to \mathbb{R}$ we define the horizontal gradient at $p = (x, y, z)$ as

$$\nabla_{\mathbb{H}} f = XfX + YfY = (\partial_x f - \frac{y}{2}\partial_z f)X + (\partial_y f + \frac{x}{2}\partial_z f)Y. \tag{16}$$

We point out that

$$\nabla_{\mathbb{H}} f(p) = \begin{pmatrix} 1 & 0 & -y/2 \\ 0 & 1 & x/2 \end{pmatrix} \cdot \nabla f(p)$$

and that we are identifying a vector $\xi = (\xi_1, \xi_2)$ in the horizontal distribution $\mathcal{H}(p)$ with a vector $\tilde{\xi} \in \mathbb{R}^3$ through

$$\tilde{\xi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -y/2 & x/2 \end{pmatrix} \cdot \xi.$$

Given a C^1 vector field $U : \mathbb{H} \to \mathbb{H}$, we may express it using both the basis (X, Y, T) and the standard basis of vector fields in $\mathbb{R}^3$ (with $e_3 = T$) at every point $p \in \mathbb{H}$:

$$U(p) = u_1 X + u_2 Y + u_3 T = u_1(1, 0, 0)^t + u_2(0, 1, 0)^t + \left(-\frac{y}{2}u_1 + \frac{x}{2}u_2 + u_3\right)(0, 0, 1)^t.$$

We define the horizontal projection of U on the horizontal distribution $\mathcal{H}$, that is: at $p = (x, y, z)$, we have

$$U_{\mathbb{H}}(p) = u_1 X + u_2 Y. \tag{17}$$

Definition 2.6. *We have that a vector field is horizontal iff $U = U_{\mathbb{H}}$, that is $u_3 = 0$. In particular observe that every function in $C^1(\mathbb{R}^3, \mathbb{R}^2)$ can be associated in a unique way to a horizontal vector field in $\mathbb{H}$.*

We define the divergence in $\mathbb{H}$ as follows: given a C^1 vector field we have that

$$\mathrm{div}_g U = D_X U \cdot_g X + D_Y U \cdot_g Y + D_T U \cdot_g T.$$

Note that $D_X U \cdot_g X = D_X U \cdot_g X + U \cdot_g D_X X = X(U \cdot_g X) = X(u_1)$, and analogously $D_Y U \cdot_g Y = Y(u_2)$ and $D_T U \cdot_g T = T(u_3)$. Therefore

$$\mathrm{div}_g U = X(u_1) + Y(u_2) + T(u_3) = \partial_x u_1 + \partial_y u_2 - \frac{y}{2}\partial_z u_1 + \frac{x}{2}\partial_z u_2 + \partial_z u_3.$$

The horizontal divergence of a vector field will be given by

$$\mathrm{div}_{\mathbb{H}} U = X(u_1) + Y(u_2) \tag{18}$$

and it coincides with the divergence of its horizontal projection

$$\mathrm{div}_{\mathbb{H}} U = \mathrm{div}_g U_{\mathbb{H}} = \partial_x u_1 + \partial_y u_2 - \frac{y}{2}\partial_z u_1 + \frac{x}{2}\partial_z u_2.$$

In particular if a vector field is horizontal we have that $\mathrm{div}_g U = \mathrm{div}_{\mathbb{H}} U$. In terms of the standard Euclidean divergence $\mathrm{div}_{\mathbb{R}^3}$ observe that

$$\mathrm{div}_{\mathbb{R}^3} U_{\mathbb{H}} = \mathrm{div}_{\mathbb{R}^3}(u_1 X + u_2 Y) = X(u_1) + Y(u_2)$$

therefore we can also state that

$$\mathrm{div}_{\mathbb{R}^3} U_{\mathbb{H}} = \mathrm{div}_{\mathbb{H}} U_{\mathbb{H}} = \mathrm{div}_g U_{\mathbb{H}}. \tag{19}$$

In the following we denote the Euclidean divergence $\mathrm{div}_{\mathbb{R}^3}$ simply by div.

3 Horizontal perimeter and mean curvature

We collect here the most relevant notions on the geometry of manifolds in $\mathbb{H}$; for more details we refer to [8, Chapter 4] and [36]. We consider a C^1 surface Σ enclosing an open set E, i.e.

$$\partial E = \Sigma.$$

Moreover, we assume that it has regular points, that is, it can be locally written as the 0 level set of a C^1 function with non vanishing gradient, and $E = \{p \mid f(p) < 0\}$ for some $f \in C^1$.

We denote by $N(p) = \frac{\nabla f(p)}{|\nabla f(p)|}$ the euclidean exterior normal to the surface Σ at a point $p = (x, y, z)$. We recall that at p there holds

$$Xf = \partial_x f - \frac{y}{2}\partial_z f \qquad Yf = \partial_y f + \frac{x}{2}\partial_z f \qquad Tf = \partial_z f.$$

Following [8, Chapter 4], we define:

$$e_1 := \frac{Yf}{\sqrt{(Xf)^2 + (Yf)^2}}X - \frac{Xf}{\sqrt{(Xf)^2 + (Yf)^2}}Y \tag{20}$$

$$e_2 := \frac{Tf}{\sqrt{(Xf)^2 + (Yf)^2 + (Tf)^2}}\frac{Xf}{\sqrt{(Xf)^2 + (Yf)^2}}X + \frac{Tf}{\sqrt{(Xf)^2 + (Yf)^2 + (Tf)^2}}\frac{Yf}{\sqrt{(Xf)^2 + (Yf)^2}}Y - \frac{\sqrt{(Xf)^2 + (Yf)^2}}{\sqrt{(Xf)^2 + (Yf)^2 + (Tf)^2}}T. \tag{21}$$

We observe that

$$Xf = X(p) \cdot \nabla f(p) \qquad Yf = Y(p) \cdot \nabla f(p) \qquad Tf = T(p) \cdot \nabla f(p).$$

Using this it is straightforward to check that

$$e_1 \cdot N(p) = \frac{1}{\sqrt{(Xf)^2 + (Yf)^2}|\nabla f|}\left[Yf(X \cdot \nabla f) - Xf(Y \cdot \nabla f)\right] = 0$$

$$e_2 \cdot N(p) = \frac{1}{\sqrt{(Xf)^2 + (Yf)^2 + (Tf)^2}|\nabla f|}\left[Tf\frac{(Xf)^2}{\sqrt{(Xf)^2 + (Yf)^2}} + Tf\frac{(Yf)^2}{\sqrt{(Xf)^2 + (Yf)^2}} + -\sqrt{(Xf)^2 + (Yf)^2}Tf\right] = 0.$$

Therefore $\{e_1, e_2\}$ is a basis of the tangent space to Σ at the point p. Moreover one can verify that the basis $\{e_1, e_2\}$ is orthonormal with respect to the Riemannian metric g, that is $e_1 \cdot_g e_1 = 1 = e_2 \cdot_g e_2$, and $e_1 \cdot_g e_2 = 0$.

We define the Riemannian normal at p to Σ as follows:

$$N_g(p) = \frac{Xf}{\sqrt{(Xf)^2 + (Yf)^2 + (Tf)^2}}X + \frac{Yf}{\sqrt{(Xf)^2 + (Yf)^2 + (Tf)^2}}Y + \frac{Tf}{\sqrt{(Xf)^2 + (Yf)^2 + (Tf)^2}}T.$$

Note that $N_g \cdot_g e_1 = 0 = N_g \cdot_g e_2$. So (N_g, e_1, e_2) form a basis of the tangent bundle of $(\mathbb{H}, g)$ at p, whereas (e_1, e_2) form a basis of the tangent bundle of Σ at p. Let

$$N_{\mathbb{H}}(p) = N_g(p) - (N_g(p) \cdot_g T(p))T(p) = \frac{Xf}{\sqrt{(Xf)^2 + (Yf)^2 + (Tf)^2}}X + \frac{Yf}{\sqrt{(Xf)^2 + (Yf)^2 + (Tf)^2}}Y. \tag{22}$$

Definition 3.1. *The singular points in Σ are the points p such that $N_{\mathbb{H}}(p) = 0$, that is the points where $N(p) \cdot X(p) = 0$ and $N(p) \cdot Y(p) = 0$. This implies that $N(p)$ is orthogonal to the horizontal bundle $\mathcal{H}(p)$ and therefore the horizontal bundle $\mathcal{H}(p)$ and the euclidean tangent space to Σ at p coincide. We denote the set of singular points as Σ_0.*

Remark 3.2. The set Σ_0 is closed and has empty interior in Σ. Moreover, for a C^2 surface Σ, it is known that the Hausdorff dimension of Σ_0 with respect to the Riemannian distance in $\mathbb{H}$ is less than or equal to 1. We refer to [36, Section 2] for more details and references.

The exterior horizontal normal to Σ at $p \in \Sigma \setminus \Sigma_0$ is the normalization of $N_{\mathbb{H}}$ with respect to the g metric:

$$\nu_{\mathbb{H}}(p) = \frac{Xf}{\sqrt{(Xf)^2 + (Yf)^2}} X + \frac{Yf}{\sqrt{(Xf)^2 + (Yf)^2}} Y = \frac{\nabla_{\mathbb{H}} f(p)}{|\nabla_{\mathbb{H}} f(p)|_g} = \frac{N_{\mathbb{H}}(p)}{|N_{\mathbb{H}}|_g(p)}. \tag{23}$$

Note that for every $p \in \Sigma \setminus \Sigma_0$

$$e_1(p) = J(\nu_{\mathbb{H}}(p)) \tag{24}$$

$$e_2(p) = (N_g \cdot_g T)\ \nu_{\mathbb{H}}(p) - |N_{\mathbb{H}}|_g T. \tag{25}$$

In particular $\{e_1, \nu_{\mathbb{H}}\}$ forms an orthonormal basis of the horizontal bundle $\mathcal{H}(p)$ at every $p \in \Sigma \setminus \Sigma_0$.

Remark 3.3 (Horizontal normal of planes)**.** We consider the euclidean plane $\{(x, y, z) \mid z = k\}$. It is easy to see that the only characteristic point is $(0, 0, k)$ and that the horizontal normal coincides with

$$\nu_{\mathbb{H}}(p) = \frac{-y}{\sqrt{y^2 + x^2}} X + \frac{x}{\sqrt{y^2 + x^2}} Y.$$

More generally if we consider a non-vertical plane $\{(x, y, z) \mid ax + by + z = k\}$ then the only characteristic point is $(-2b, 2a, k)$ and the horizontal normal is given by

$$\nu_{\mathbb{H}}(p) = \frac{(a - y/2)X + (b + x/2)Y}{\sqrt{(a - y/2)^2 + (b + x/2)^2}}.$$

Finally the vertical planes $\{(x, y, z) \mid ax + by = k\}$ have no singular points and

$$\nu_{\mathbb{H}}(p) = \frac{aX + bY}{\sqrt{a^2 + b^2}}.$$

3.1 Horizontal mean curvature

To introduce the notion of mean curvature we assume now that the surface $\Sigma = \partial E$ is of class C^2. Recalling (24), we have that

$$\{\nu_{\mathbb{H}}(p), e_1(p), T\}$$

forms an orthonormal basis of the tangent bundle of the Riemannian space $(\mathbb{H}, g)$ at every $p \in \Sigma \setminus \Sigma_0$.

Moreover, we already observed that $e_1(p)$ belongs to the tangent space to Σ. The integral curves of e_1 in $\Sigma \setminus \Sigma_0$ are called characteristic curves and foliate the regular part of Σ.

Since $|\nu_{\mathbb{H}}|_g = 1$, it holds $D_{\nu_{\mathbb{H}}} \nu_{\mathbb{H}} \cdot_g \nu_{\mathbb{H}} = 0$, and since $\nu_{\mathbb{H}} \cdot_g T = 0$ and $D_T T = 0$, we get that $D_T \nu_{\mathbb{H}} \cdot_g T = -D_T T \cdot_g \nu_{\mathbb{H}} = 0$. Therefore

$$\mathrm{div}_g \nu_{\mathbb{H}}(p) = D_{\nu_{\mathbb{H}}} \nu_{\mathbb{H}} \cdot_g \nu_{\mathbb{H}} + D_T \nu_{\mathbb{H}} \cdot_g T + D_{e_1} \nu_{\mathbb{H}} \cdot_g e_1 = D_{e_1} \nu_{\mathbb{H}} \cdot_g e_1 = -D_{e_1} e_1 \cdot_g \nu_{\mathbb{H}}. \tag{26}$$

Recall that an equivalent formulation of this divergence using the basis $\{X, Y, T\}$ is given by

$$\begin{aligned} \mathrm{div}_g \nu_{\mathbb{H}}(p) &= D_X \nu_{\mathbb{H}} \cdot_g X + D_T \nu_{\mathbb{H}} \cdot_g T + D_Y \nu_{\mathbb{H}} \cdot_g Y \\ &= X\left(\frac{Xf}{\sqrt{(Xf)^2 + (Yf)^2}}\right) + Y\left(\frac{Yf}{\sqrt{(Xf)^2 + (Yf)^2}}\right) = \mathrm{div}_{\mathbb{H}} \nu_{\mathbb{H}}(p). \end{aligned} \tag{27}$$

We consider the vector e_2 in the tangent bundle, defined in (21). As observed in the previous section the tangent bundle of Σ at p has orthonormal basis given by $\{e_1, e_2\}$. Then the Riemannian mean curvature of Σ at $p \in \Sigma$ is defined as the tangential divergence of the normal N_g:

$$H_g(p, \Sigma) = \mathrm{div}_\Sigma(N_g) = D_{e_1} N_g \cdot_g e_1 + D_{e_2} N_g \cdot_g e_2. \tag{28}$$

We compute now the tangential divergence of the horizontal normal $\nu_{\mathbb{H}}$, Note that $D_{e_1}T = \frac{1}{2}J(e_1) = -\frac{1}{2}\nu_{\mathbb{H}}$, therefore $D_{e_1}T \cdot_g e_1 = 0$. So, recalling the definition of $\nu_{\mathbb{H}}$, we get

$$|N_{\mathbb{H}}|_g D_{e_1}\nu_{\mathbb{H}} \cdot_g e_1 = D_{e_1}N_g \cdot_g e_1 - (T \cdot_g N_g)(D_{e_1}T \cdot_g e_1) = D_{e_1}N_g \cdot_g e_1.$$

On the other hand, since $\{e_1, T, \nu_{\mathbb{H}}\}$ forms an orthonormal basis of the tangent bundle of $(\mathbb{H}, g)$, and $D_{e_2}\nu_{\mathbb{H}} \cdot_g \nu_{\mathbb{H}} = 0$ (due to the fact that $|\nu_{\mathbb{H}}|_g = 1$) we have that

$$D_{e_2}\nu_{\mathbb{H}} = (D_{e_2}\nu_{\mathbb{H}} \cdot_g e_1)e_1 + (D_{e_2}\nu_{\mathbb{H}} \cdot_g T)T.$$

Observe that, due to the definition of J,

$$D_{e_2}\nu_{\mathbb{H}} \cdot_g T = -D_{e_2}T \cdot_g \nu_{\mathbb{H}} = -\frac{1}{2}J(e_2) \cdot_g \nu_{\mathbb{H}} = -\frac{1}{2}e_2 \cdot_g J(\nu_{\mathbb{H}}) = -\frac{1}{2}e_2 \cdot_g e_1 = 0.$$

Therefore $D_{e_2}\nu_{\mathbb{H}} = (D_{e_2}\nu_{\mathbb{H}} \cdot_g e_1)e_1$, and then

$$\mathrm{div}_\Sigma(\nu_{\mathbb{H}}) = D_{e_1}\nu_{\mathbb{H}} \cdot_g e_1 + D_{e_2}\nu_{\mathbb{H}} \cdot_g e_2 = D_{e_1}\nu_{\mathbb{H}} \cdot_g e_1 = \frac{1}{|N_{\mathbb{H}}|_g}D_{e_1}N_g \cdot_g e_1.$$

We define the horizontal mean curvature of Σ at $p \in \Sigma \setminus \Sigma_0$ as the tangential divergence of the horizontal normal, written above, that is

$$H_{\mathbb{H}}(p, E) = \mathrm{div}_\Sigma(\nu_{\mathbb{H}}) = \frac{1}{|N_{\mathbb{H}}|_g}D_{e_1}N_g \cdot_g e_1 = D_{e_1}\nu_{\mathbb{H}} \cdot_g e_1 = \mathrm{div}_g\nu_{\mathbb{H}}(p). \tag{29}$$

It coincides with the divergence of $\nu_{\mathbb{H}}$ (see (27)).

Remark 3.4. The horizontal mean curvature is -1 homogeneous with respect to dilations and constant with respect to left translations, i.e. $H_{\mathbb{H}}(\delta_\lambda(p), \delta_\lambda(E)) = \frac{1}{\lambda}H_{\mathbb{H}}(p, E)$ and that for every $q \in \mathbb{H}$, $H_{\mathbb{H}}(L_q(p), L_q(E)) = H_{\mathbb{H}}(p, E)$.

We recall from [2], [3], the following result relating the horizontal gradient and horizontal divergence of the signed distance function from a smooth set to the horizontal normal and horizontal curvature.

Theorem 3.5. *Let $E \subseteq \mathbb{H}$ be a bounded C^k open set (in Euclidean sense), for $k \geq 2$, and let d_E be the signed distance function from E as defined in* (10) *(with respect to the Carnot–Carathéodory distance d_{CC}). Then the following hold:*

1. *d_E and $\nabla_{\mathbb{H}}d_E$ are of class C^{k-1} in Euclidean sense in an open neighborhood of $\partial E \setminus \Sigma_0$, where Σ_0 is the set of singular points in ∂E. Moreover at every $p \in \partial E \setminus \Sigma_0$, there holds $\nabla_{\mathbb{H}}d_E(p) = \nu_{\mathbb{H}}(p)$ and $|\nabla_{\mathbb{H}}d_E(p)| = 1$.*
2. *Let $k \geq 3$. At every $p \in \partial E \setminus \Sigma_0$, it holds that*

$$\nabla_{\mathbb{H}}d_E(p) = \nu_{\mathbb{H}}(p) \quad \text{and} \quad \mathrm{div}_{\mathbb{H}}\nabla_{\mathbb{H}}d_E(p) = H_{\mathbb{H}}(p, E).$$

Proof. For the proof we refer to [2, Theorem 1.1, Corollary 6.2], [3, Theorem 1.1, Corollary 4.1]. □

Remark 3.6. Using (29) and (27), the horizontal mean curvature of $\Sigma = \{f = 0\}$, boundary of $E = \{f \leq 0\}$, at $p \in \Sigma \setminus \Sigma_0$ can be expressed as

$$\begin{aligned} H_{\mathbb{H}}(p, E) &= X\left(\frac{Xf}{\sqrt{(Xf)^2 + (Yf)^2}}\right) + Y\left(\frac{Yf}{\sqrt{(Xf)^2 + (Yf)^2}}\right) \\ &= \frac{1}{|\nabla_{\mathbb{H}}f|_g} tr\left[\left(I_2 - \frac{\nabla_{\mathbb{H}}f \otimes \nabla_{\mathbb{H}}f}{|\nabla_{\mathbb{H}}f|_g^2}\right)(\nabla^2_{\mathbb{H}}f)^*\right], \end{aligned}$$

where I_2 is the identity matrix in $\mathbb{R}^2$, $|\nabla_{\mathbb{H}} f|_g = \sqrt{(Xf)^2 + (Yf)^2}$, $\nabla_{\mathbb{H}} f = (Xf, Yf)$ (recall (16)) and $(\nabla^2_{\mathbb{H}} f)^*$ is the matrix

$$(\nabla^2_{\mathbb{H}} f)^* := \begin{pmatrix} XXf & \frac{1}{2}(XYf + YXf) \\ \frac{1}{2}(XYf + YXf) & YYf \end{pmatrix}.$$

Moreover, with the notation $X_1 := X$, $X_2 := Y$ we can also write

$$H_{\mathbb{H}}(p, E)|\nabla_{\mathbb{H}} f|_g = \sum_{i,j=1}^{2} \left(\delta_{ij} - \frac{(X_i f)(X_j f)}{|\nabla_{\mathbb{H}} f|_g^2} \right) X_i X_j f.$$

For axially symmetric sets we have a more explicit formula for the horizontal mean curvature.

Remark 3.7 (Horizontal mean curvature for axially symmetric sets)**.** Let Σ be an axially symmetric surface, that is $\Sigma = \{(x, y, z) \mid |z| = f(\sqrt{x^2 + y^2})\}$ for some $f \in C^2$. The singular set is given by $\Sigma_0 \subseteq \{(0, 0, \pm f(0))\}$. The horizontal outer normal to $E = \{(x, y, z) \mid |z| < f(\sqrt{x^2 + y^2})\}$ can be easily computed at a point $p = (x, y, z)$, with $r = \sqrt{x^2 + y^2} > 0$ and $z > 0$, as

$$\nu_{\mathbb{H}}(p) = \frac{1}{\sqrt{(f'(r))^2 + r^2/4}} \left[\left(-f'(r)\frac{x}{r} - \frac{y}{2} \right) X + \left(-f'(r)\frac{y}{r} + \frac{x}{2} \right) Y \right]. \tag{30}$$

So the horizontal mean curvature at $p \in \Sigma \setminus \Sigma_0$ (with $z > 0$) can be computed as follows (for details see [8, Section 4.3]):

$$H_{\mathbb{H}}(p, E) = -\frac{\frac{1}{4} r^2 f''(r) + \frac{1}{r}(f'(r))^3}{[(f'(r))^2 + r^2/4]^{3/2}} \tag{31}$$

For a comparison, the euclidean counterparts, again at a point $p = (x, y, z)$, with $r = \sqrt{x^2 + y^2} > 0$ and $z > 0$ are given by the euclidean outer normal

$$N(p) = \frac{1}{\sqrt{r^2 + r^2 (f'(r))^2}} (-f'(r)x, -f'(r)y, r), \tag{32}$$

and euclidean principal curvatures and euclidean mean curvature

$$k_1(p, E) = -\frac{f''(r)}{(1 + (f'(r))^2)^{\frac{3}{2}}}, \qquad k_2(p, E) = -\frac{f'(r)}{r(1 + (f'(r))^2)^{\frac{1}{2}}}, \quad H(p, E) = k_1(p) + k_2(p). \tag{33}$$

3.2 Horizontal mean convex sets

We introduce the definition of horizontal mean convex set and horizontal strictly mean convex set.

Definition 3.8. *We say that a C^2 set B is horizontal mean convex with constant $\delta \geq 0$, if $H_{\mathbb{H}}(p, B) \geq \delta$ for every $p \in \partial B \setminus (\partial B)_0$, where $(\partial B)_0$ denotes the set of singular points of the boundary.*

If $\delta > 0$, then B is called horizontal strictly mean convex with constant δ.

Remark 3.9. If E is an axially symmetric set (as described in Remark 3.7) and it is (euclidean) convex, then it is horizontal mean convex. Indeed, since $f'' \leq 0$ and $f' \leq 0$, we deduce using (31) that $H_{\mathbb{H}}(p, E) \geq 0$.

In the following we collect some examples of horizontal mean convex and strictly mean convex sets.

Remark 3.10. [Hyperplanes] Recalling the formulas for the horizontal normal provided in Remark 3.3 we apply (27) and (29) away from Σ_0 to show that every plane has zero horizontal mean curvature. Every horizontal plane $\{p \mid z = k\}$ has zero horizontal mean curvature since

$$H_{\mathbb{H}}(p, \{z \leq k\}) = \partial_x \left(\frac{-y}{\sqrt{x^2+y^2}} \right) + \partial_y \left(\frac{x}{\sqrt{x^2+y^2}} \right) = 0.$$

The same argument gives that the planes $\{p \mid ax + by + z = k\}$ has zero mean curvature since

$$\partial_x \left(\frac{a - y/2}{\sqrt{(a-y/2)^2 + (b+x/2)^2}} \right) + \partial_y \left(\frac{b + x/2}{\sqrt{(a-y/2)^2 + (b+x/2)^2}} \right) = 0.$$

The vertical planes $\{p \mid ax + by = k\}$ have constant horizontal normal, so they have trivially zero horizontal mean curvature.

Therefore all hyperplanes are horizontal mean convex sets.

Remark 3.11. [Balls] We compute the horizontal mean curvature of some different balls, based on the previous formula (31), see [8, Section 4.3].

1. Let us consider the boundary of euclidean ball $B_E(0,R) = \{(x,y,z) \,|\, x^2+y^2+z^2 < R^2\}$. Then
$$H_{\mathbb{H}}(p, B_E(0,R)) = \frac{2(4+R^2)}{\sqrt{x^2+y^2}(4+z^2)^{3/2}}.$$
Note that the horizontal curvature explodes at the singular points $(0,0,\pm R)$. In particular a euclidean ball is a horizontal strictly mean convex with constant $\frac{2}{R\sqrt{4+R^2}}$.

2. The Korányi ball $B_K(0,R) = \{(x,y,z) \,|\, (x^2+y^2)^2 + 16z^2 < R^4\}$ has curvature at $p \in \partial B_K(0,R) \setminus \{(0,0,\pm R^2/4)$ given by
$$H_{\mathbb{H}}(p, B_K(0,R)) = \frac{3\sqrt{x^2+y^2}}{R^2}. \tag{34}$$
The Korányi ball is convex and moreover it is horizontal mean convex (but not strictly mean convex) since $0 \leq H_{\mathbb{H}}(p, B_K(0,R)) \leq \frac{3}{R}$ for all $p \in \partial B_K(0,R) \setminus (\partial B_K(0,R))_0$.
Note that the euclidean principal curvatures (see (33)) are given by
$$k_1(p, B_K(0,R)) = \frac{4r^2(3R^4 - r^4)}{(4(R^4-r^4)+r^6)^{\frac{3}{2}}}, \qquad k_2(p, B_K(0,R)) = \frac{r^2}{(4(R^4-r^4)+r^6)^{\frac{1}{2}}}$$
where $r = \sqrt{x^2+y^2}$. The euclidean principal curvatures vanish for $r \to 0$.

3. The boundary of the Carnot–Carathéodory ball centered at 0, that is $\partial B_{CC}(0,R)$, can be parametrized recalling that a length minimizing curve between 0 and (x,y,z) is the lift of a circular arc joining the origin in $\mathbb{R}^2$ with (x,y), whose convex hull has area z (see Remark 2.1). The parametrization is given by
$$\sqrt{x^2+y^2} = \frac{2}{c}\sin(cR/2) \qquad z = \frac{cR - \sin cR}{2c^2}.$$
The points with maximal and minimal height (that is the points such that $\frac{dz}{dc}(c) = 0$), correspond to $cR = \pm\pi$ and have coordinates (x,y,z) with $\sqrt{x^2+y^2} = \frac{2R}{\pi} < R$, and $z = \pm\frac{R^2}{2\pi}$. The points on the plane $\{z = 0\}$ corresponds to $c = 0$, and satisfy $\sqrt{x^2+y^2} = R$, whereas the points on the z-axis correspond to $c = \pm(2\pi)/R$ and have coordinate $(0,0,\pm R^2/(4\pi))$. It is possible to compute explicitly
$$H_{\mathbb{H}}(p, B_{CC}(0,R)) = \frac{1}{2}\frac{c}{2\sin(cR/2)}\frac{\sin cR - cR\cos cR}{\sin(cR/2) - cR\cos(cR/2)/2}. \tag{35}$$

Note that if $p = (x, y, 0) \in \partial B_{CC}(0, R)$, then $c = 0$ and $H_{\mathbb{H}}(p, B_{CC}(0, R)) = \frac{4}{R}$. At points of maximal and minimal height, $cR = \pm\pi$ and $H_{\mathbb{H}}(p, B_{CC}(0, R)) = \frac{\pi^2}{4R}$. On the other hand, if $p = (x, y, z) \in \partial B_{CC}(0, R)$ satisfies $\sqrt{x^2 + y^2} \to 0$, then $c \to \left(\frac{2\pi}{R}\right)^-$ and we obtain $H_{\mathbb{H}}(p, B_{CC}(0, R)) \to -\infty$.

Thus, the Carnot–Carathéodory ball is not horizontal mean convex.

3.2.1 Pansu ball

The Pansu sphere, as introduced in Definition 2.2 is the surface obtained by rotating around the z-axis the geodesics connecting 0 with $(0, 0, \pi R^2)$. In order to center the sphere in the origin, the set is vertically translated and dilated. So, we will consider the Pansu sphere of parameter R as the surface obtained by rotating around the z-axis the geodesics connecting $(0, 0, -\pi R^2/8)$ with $(0, 0, \pi R^2/8)$, which coincides with the singular points of the sphere. By recalling the description of geodesic in Remark 2.1, it is possible to show (see e.g. [35]) that the Pansu ball is defined as

$$B_P^R = \{(x, y, z) \mid |z| < f_R(\sqrt{x^2 + y^2})\} \tag{36}$$

where

$$f_R(r) = \frac{1}{4}R^2 \arccos\frac{r}{R} + \frac{1}{4}r\sqrt{R^2 - r^2} = \frac{1}{4}R^2\left(\arccos\frac{r}{R} + \frac{r}{R}\sqrt{1 - \frac{r^2}{R^2}}\right).$$

The horizontal mean curvature of ∂B_P is constant:

$$H_{\mathbb{H}}(p, B_P^R) = \frac{2}{R}. \tag{37}$$

Note that $\delta_{1/R} B_P^R = B_P^1$ or equivalently $B_P^R = \delta_R B_P^1$ and the boundary of the Pansu ball is C^2 but not C^3 ([8]). The Pansu ball is a convex set which is horizontal strictly mean convex.

Observe that for the Pansu ball the euclidean principal curvatures can be computed using (33) obtaining

$$k_1(p, B_P^R) = \frac{4r(2R^2 - r^2)}{(4(R^2 - r^2) + r^4)^{\frac{3}{2}}}, \qquad k_2(p, B_P^R) = \frac{r}{(4(R^2 - r^2) + r^4)^{\frac{1}{2}}}$$

where $r = \sqrt{x^2 + y^2}$. In particular the euclidean principal curvatures vanish for $r \to 0$.

In [36, Theorem 6.1] an Alexandrov theorem in $\mathbb{H}$ is proved about the Pansu spheres: they are the only C^2 immersed compact connected surfaces with positive constant horizontal mean curvature and the regular part of every Pansu sphere is foliated by horizontal geodesics of curvature equal to the constant curvature of the sphere.

Remark 3.12. Observe that by the Ball-box theorem there exists $\lambda_1 < 1$, $\lambda_2 \geq 1$ such that $B_P^{\lambda_1 R} \subseteq B_{CC}(0, R) \subseteq B_P^{\lambda_2 R}$.

3.3 Horizontal perimeter and functions of bounded variations

It is possible to adapt the definition of functions of bounded variation to the case of $\mathbb{H}$, see [22], [8, Chapter 5]. We define the notion of horizontal variation for a measurable function $f : \mathbb{H} \to \mathbb{R}$:

$$\begin{aligned} \mathrm{Var}_{\mathbb{H}}(f; \Omega) &:= \sup\left\{\int_\Omega f \mathrm{div}_{\mathbb{H}}\phi dp \mid \phi \in C_c^1(\Omega, \mathbb{R}^2), \sqrt{(\phi_1)^2 + (\phi_2)^2} \leq 1\right\} \\ &= \sup\left\{\int_\Omega f \mathrm{div}_g \phi_{\mathbb{H}} dp \mid \phi \in C_c^1(\Omega, \mathbb{R}^3), |\phi_{\mathbb{H}}|_g \leq 1\right\}, \end{aligned}$$

where $\Omega \subseteq \mathbb{H}$ denotes an open set. Recall that for $\phi \in C^1(\Omega, \mathbb{R}^2)$, it holds $\mathrm{div}_{\mathbb{H}}\phi = X(\phi_1) + Y(\phi_2)$, and for $\phi \in C^1(\Omega, \mathbb{R}^3)$ it holds $\mathrm{div}_g \phi_{\mathbb{H}} = \mathrm{div}\phi_{\mathbb{H}}$ (cp. with (19)).

Definition 3.13. *Let $\Omega \subseteq \mathbb{H}$ be an open set and $f : \Omega \subset \mathbb{H} \to \mathbb{R}$. Then $f \in BV_{\mathbb{H}}(\Omega)$ if $f \in L^1(\Omega)$ and if $\mathrm{Var}_{\mathbb{H}}(f;\Omega) < +\infty$.*

Let $\mathcal{M}$ be the class of all measurable sets in $\mathbb{H}$. A set $E \in \mathcal{M}$ has finite $\mathbb{H}$ perimeter in Ω if $P_{\mathbb{H}}(E,\Omega) := \mathrm{Var}_{\mathbb{H}}(\chi_E;\Omega) < +\infty$, where χ_E is the characteristic function. More explicitly:

$$P_{\mathbb{H}}(E;\Omega) = \sup_{\phi \in C_c^1(\Omega,\mathbb{R}^2), |\phi| \le 1} \int_E \mathrm{div}_{\mathbb{H}}\phi \, dp. \tag{38}$$

If $\Omega = \mathbb{H}$ we denote $P_{\mathbb{H}}(E;\mathbb{H}) = P_{\mathbb{H}}(E)$. Moreover E is a $\mathbb{H}$-Caccioppoli set in Ω if $\chi_E \in BV(\mathbb{H})(\Omega)$. We will denote

$$\mathcal{P} := \{F \in \mathcal{M} \,:\, \chi_F \in BV(\mathbb{H})\}. \tag{39}$$

The perimeter functional $P_{\mathbb{H}}$ is lower semicontinuous with respect to the L^1_{loc} convergence ([8, Proposition 5.6]).

Remark 3.14. *If F is a set of finite perimeter then $\mathcal{H}^3_{CC}(\partial^e F \setminus F(1/2)) = 0$, where $\mathcal{H}^s_{CC}$ denotes the s-dimensional Hausdorff measure obtained from d_{CC} (see [25, Proposition 1.23]).*
If E, E' are measurable sets, then

$$P_{\mathbb{H}}(E,\Omega) = P_{\mathbb{H}}(\mathbb{H} \setminus E,\Omega), \tag{40}$$

$$P_{\mathbb{H}}(E,\Omega) = P_{\mathbb{H}}(E',\Omega) \qquad \text{whenever } |(E \triangle E') \cap \Omega| = 0 \tag{41}$$

where $E \triangle E' = (E \setminus E') \cup (E' \setminus E)$ is the symmetric difference between E and E'.

The sets E and E' are said to be equivalent if $|E \triangle E'| = 0$.

For regular sets, the horizontal perimeter has an explicit integral representation in terms of the underlying Riemannian geometry. We recall the divergence theorem for the Riemannian space $(\mathbb{H}, g)$. Let $M \subseteq \mathbb{H}$ a C^1 submanifold with boundary, and U a C^1 vector field then

$$\int_M \mathrm{div}_g U \, dp = \int_{\partial M} U \cdot_g N_g d\sigma, \tag{42}$$

where N_g is the Riemannian outer normal at ∂M and $d\sigma$ is the Riemannian area element on ∂M.

Proposition 3.15 (Area of a surface)**.** *If E is set of class C^1, then*

$$P_{\mathbb{H}}(E;\Omega) = \sup_{\phi \in C_c^1(\Omega,\mathbb{R}^2), |\phi| \le 1} \int_{\partial E} \phi \cdot_g N_g d\sigma = \int_{\partial E \cap \Omega} \nu_{\mathbb{H}} \cdot_g N_g d\sigma = \int_{\partial E \cap \Omega} |N_{\mathbb{H}}|_g d\sigma \tag{43}$$

where $N_{\mathbb{H}}$ is defined in (22) and $d\sigma$ is the Riemannian area element on ∂E.

In terms of Euclidean unit normal $N = \frac{\nabla f}{|\nabla f|}$ and standard 2-dimensional Hausdorff measure $\mathcal{H}^2$ we have

$$P_{\mathbb{H}}(E;\Omega) = \int_{\partial E \cap \Omega} \nu_{\mathbb{H}} \cdot N(p) d\mathcal{H}^2(p) = \int_{\partial E \cap \Omega} \frac{\sqrt{(Xf)^2 + (Yf)^2}}{|\nabla f|} d\mathcal{H}^2. \tag{44}$$

For the proof we refer to [8, Proposition5.7] or [29, Proposition 3.1].

Remark 3.16 (Invariance properties of the perimeter)**.** The horizontal perimeter by its very definition is invariant with respect to any isometry of $\mathbb{H}$ which leaves invariant the horizontal distribution. So in particular it is invariant with respect to left translations that is

$$P_{\mathbb{H}}(E;\Omega) = P_{\mathbb{H}}(L_p(E); L_p(\Omega))$$

and rotations around the z-axis . Indeed since X, Y are left invariant, then $(\mathrm{div}_{\mathbb{H}}\phi) \circ L_p = (\mathrm{div}_{\mathbb{H}}\phi \circ L_p)$. Moreover, let $\theta \in (0, 2\pi)$ and R_θ the associated rotation around the z-axis of angle θ, then

$$P_{\mathbb{H}}(E) = P_{\mathbb{H}}(R_\theta E).$$

Remark 3.17. The horizontal perimeter is homogeneous of degree 3 with respect to dilations of sets, see [8, Lemma 5.5] that is for all $\lambda > 0$,

$$P_{\mathbb{H}}(\delta_\lambda E; \delta_\lambda \Omega) = \lambda^3 P_{\mathbb{H}}(E; \Omega).$$

The horizontal perimeter is also subadditive, see [8]. The proof of this fact follows the same arguments as for the standard perimeter.

Lemma 3.18. *For $E, F \in \mathcal{P}$ we have*

$$P_{\mathbb{H}}(E \cap F) + P_{\mathbb{H}}(E \cup F) \leq P_{\mathbb{H}}(E) + P_{\mathbb{H}}(F). \tag{45}$$

Also in the Heisenberg group we may formulate the classical isoperimetric problem:

$$\min_{E \in \mathcal{M}, |E|=1} P_{\mathbb{H}}(E). \tag{46}$$

Existence of isoperimetric sets is proved in [25] (see also [21], [29]).

Theorem 3.19 (Isoperimetric inequality)**.** *If F is any set with finite $\mathbb{H}$-perimeter then either F or $\mathbb{H} \setminus F$ is a $\mathbb{H}$-Caccioppoli set and there exists a universal constant $C > 0$ such that*

$$P_{\mathbb{H}}(F) \geq C \min(|F|, |\mathbb{H} \setminus F|)^{3/4}. \tag{47}$$

Proof. See [25, Theorem 2.2]. □

Remark 3.20. The Pansu sphere B_P^R as introduced in Definition 2.2 is conjectured to be the solution to the isoperimetric problem [33]. We have that

$$|B_P^R| = \frac{3}{16}\pi^2 R^4 \qquad P_{\mathbb{H}}(B_P^R) = \frac{1}{2}\pi^2 R^3$$

so the isoperimetric constant is conjectured to be

$$C := \frac{4\sqrt{\pi}}{3^{3/4}}.$$

The conjecture was proved in the class of axially symmetric sets in [30], in the convex setting in [31], in the smooth setting [36], and in the axially symmetric cylinders [35].

Theorem 3.21 (CC-Balls are domains of isoperimetry)**.** *There exists a constant $C > 0$ such that for any measurable set E and any ball $B_{CC}(p, r)$ (as in (8)) such that $\emptyset \neq E \cap B_{CC}(p, r)$ there holds*

$$P_{\mathbb{H}}(E \cap B_{CC}(p, r), B_{CC}(p, r)) \geq C \min\{|E \cap B_{CC}(p, r)|, |B_{CC}(p, r) \setminus E|\}^{3/4}.$$

Proof. See [25, Theorem 2.15 and Definition 2.13]. □

Finally let us recall an important compactness theorem.

Theorem 3.22 (Compactness in $BV_{\mathbb{H}}$)**.** *Let (E_i) be a sequence of measurable set with $\sup_i P_{\mathbb{H}}(E_i) < \infty$. Then, up to passing to a subsequence, (E_i) converges in $L^1_{loc}(\mathbb{H})$ to a measurable set E_∞ with finite $\mathbb{H}$-perimeter.*

Moreover, if there exists a bounded open set Ω such that $E_i \subseteq \Omega$ for all i, then E_i converges up to subsequences in $L^1(\mathbb{H})$ to $E_\infty \subseteq \overline{\Omega}$ with finite $\mathbb{H}$-perimeter.

Proof. See [8, Theorem 8.4]. □

3.4 Outward minimizing sets

In this section we introduce the notion of outward minimizing sets and we provide some relevant examples. Upon recalling (39) we give the following

Definition 3.23. *Let $\Omega \subset \mathbb{H}^1$ be an open set. We say that a set $E \in \mathcal{P}$, $E \subset \Omega$, is* <u>*outward minimizing*</u> *in Ω if*

$$P_{\mathbb{H}}(E) \leq P_{\mathbb{H}}(F) \qquad \text{for all } F \in \mathcal{P} \text{ with } E \subset F \subset \Omega. \tag{48}$$

An equivalent characterization of the outward minimality is the following:

Lemma 3.24. *The set $E \in \mathcal{P}$ is outward minimizing in Ω if and only if*

$$P_{\mathbb{H}}(E \cap G) \leq P_{\mathbb{H}}(G) \qquad \text{for all } G \subset \Omega, G \in \mathcal{P}. \tag{49}$$

Moreover if E_n is a sequence of outward minimizing sets in Ω, and $E_n \to E$ in $L^1(\Omega)$, then E is outward minimizing in Ω.

Proof. One can argue exactly as in [14, Lemma 2.5] using (45). For sake of completeness we report it here. If condition (49) holds, then we apply it to $G \supseteq E$ and get immediately $P_{\mathbb{H}}(E) \leq P_{\mathbb{H}}(G)$. On the other hand, assume E outward minimizing. Since $E \cup G \supseteq E$ by outward minimality $P_{\mathbb{H}}(E) \leq P_{\mathbb{H}}(E \cup G)$. By (45), $P_{\mathbb{H}}(E \cap G) + P_{\mathbb{H}}(E \cup G) \leq P_{\mathbb{H}}(G) + P_{\mathbb{H}}(E)$. So, putting together the two inequalities we get $P_{\mathbb{H}}(E \cap G) + P_{\mathbb{H}}(E) \leq P_{\mathbb{H}}(E \cap G) + P_{\mathbb{H}}(E \cup G) \leq P_{\mathbb{H}}(G) + P_{\mathbb{H}}(E)$, and so (49).

The stability under L^1 convergence is a consequence of the characterization of outward minimality given in (49) and of the lower semicontinuity of $P_{\mathbb{H}}$ with respect to L^1 convergence, see [9, Proposition 2.7]. □

We recall that outward minimizing sets satisfy an upper density estimate.

Lemma 3.25 (Upper density estimate)**.** *Let $E \in \mathcal{P}$ be an outward minimizing set in $\mathbb{H}$. There exist a constant $\theta > 0$ such that for all $p \in \mathbb{H}$, $r > 0$ there holds*

$$P_{\mathbb{H}}(E, B_{CC}(p, r)) \leq \theta r^3$$

where $B_{CC}(p, r)$ is defined in (8)*.*

Proof. The proof follows exactly as in [25, Lemma 4.1] by substituting the property that E is an isoperimetric set with the property that $P_{\mathbb{H}}(E) \leq P_{\mathbb{H}}(E \cup B_{CC}(p, r))$ for all $r > 0$. □

We provide next some relevant examples of outward minimizing sets in the Heisenberg group, using a calibration argument (see [35], and [36]).

First of all we recall the definition of area-minimizing surface.

Definition 3.26. *A surface Σ is area-minimizing if any region $M \subseteq \Sigma$ satisfies $P_{\mathbb{H}}(M) \leq P_{\mathbb{H}}(M')$ for any other C^1 compact surface M' with $\partial M = \partial M'$.*

In [36, Theorem 4.17] it is proved that if an immersed surface is area-minimizing then its horizontal mean curvature is zero (or constant under a volume constraint) and the characteristic curves (with are the integral curves of $J(\nu_{\mathbb{H}})$) meet orthogonally the singular curves (if there are any). Moreover it is proved that every non vertical plane in $\mathbb{H}$ is globally area minimizing, whereas in [13] vertical Euclidean planes are characterized as the only C^2 area-minimizing entire graphs in $\mathbb{H}$ with empty singular set. In particular, by arguing first for smooth sets and then proceeding by regularization, this implies that for every set $E \in \mathcal{P}$ with finite $\mathbb{H}$ perimeter and for every half-space Π it holds

$$P_{\mathbb{H}}(E \cap \Pi) \leq P_{\mathbb{H}}(E).$$

3.4.1 A family of outward minimizing sets

We introduce a class of smooth sets, and we show, by using a calibration type argument, that they are outward minimizers.

Definition 3.27. *We denote by $\mathcal{B}$ the class of open bounded sets $B \subseteq \mathbb{H}$ of class C^2 which are axially symmetric (with respect to the z-axis), strictly starshaped w.r.t. 0 according to Definition 2.4, horizontal mean convex according to Definition 3.8, and moreover satisfy the following regularity condition: there exists $H_0 = H_0(B) > 0$ such that*

$$H_{\mathbb{H}}(p, B) \leq H_0 \qquad \forall\, p \in \partial B \setminus (\partial B)_0.$$

The most important example in the class $\mathcal{B}$ is the Pansu ball of parameter $R > 0$, centered at the origin, described in Section 3.2.1. Moreover by Remark 3.11, Kòranyi balls with radius R are in the class $\mathcal{B}$. On the other hand the Euclidean ball of radius R is strictly starshaped with respect to 0 and horizontal strictly mean convex with constant $\delta = \frac{2}{R\sqrt{4+R^2}}$, but does not satisfy the upper bound on the curvature.

Proposition 3.28. *Let $B \in \mathcal{B}$, horizontal mean convex with constant $\delta \geq 0$. Let $B^\lambda := \delta_\lambda B$, for $\lambda > 0$. Then B is an outward minimizing set in $\mathbb{H}$, that is for any $F \in \mathcal{P}$ such that $B \subseteq F$ it holds*

$$P_{\mathbb{H}}(F) \geq P_{\mathbb{H}}(B) + \frac{\delta}{\lambda_F}|F \setminus B| \tag{50}$$

where $\lambda_F := \inf\{\lambda \,|\, F \subseteq B^\lambda\} \geq 1$, with $\lambda_F = +\infty$ if F is unbounded. Moreover for general $F \subseteq \mathbb{H}$

$$P_{\mathbb{H}}(F) \geq P_{\mathbb{H}}(F \cap B) + \frac{\delta}{\max\{\lambda_F, 1\}}|F \setminus B| \tag{51}$$

Remark 3.29. Proposition 3.28 applies to the Pansu ball B_P^R with $\delta = \frac{2}{R}$ and to the Kòranyi ball with radius R with $\delta = 0$.

Proof. First of all we observe that if B is C^2 and axially symmetric, then the singular set $(\partial B)_0$ is contained in the z-axis as proved in [37, Section 4]. We divide the proof of the first statement in several steps.

Step 1: definition of the homogeneous gauge. Since B is strictly starshaped w.r.t. 0 as in Definition 2.4 then the family of dilations $\delta_\lambda B := B^\lambda$ for $\lambda \geq 0$ produce a foliation of $\mathbb{H}$ as proved in Lemma 2.5. We may associate to B an homogeneous gauge: for $p \neq 0$ let us define

$$\rho_B(p) \text{ be the unique positive number such that } p \in \partial B^{\rho_B(p)}, \tag{52}$$

and set $\rho_B(0) = 0$. Thus $B^\lambda = \{\rho_B < \lambda\}$ and $\rho_B(\delta_\lambda p) = \lambda \rho_B(p)$. The function ρ_B is a continuous strictly positive function, homogeneous of degree 1 with respect to dilations. By compactness of the unit euclidean sphere we have that the map $p \mapsto \frac{1}{\rho_B(p)}$ is locally integrable at 0, by passing to spherical coordinates and using the fact that the homogeneous dimension in $\mathbb{H}$ is 4, so greater than 1.

Step 2: extension of the outer horizontal unit normal. Away from the z-axis define the horizontal vector field

$$U_B(p) = \nu_{\mathbb{H}}^{B^{\rho_B(p)}}(p) \tag{53}$$

where $\nu_{\mathbb{H}}^{B^{\rho_B(p)}}(q)$ is the horizontal exterior normal to $B^{\rho_B(p)}$ at $q \in \partial B^{\rho_B(p)}$. An explicit expression for U_B can be computed using (30). In particular this vector field is C^1 away from the z-axis. Therefore, away from the z-axis we have, recalling (29) and Remark 3.4,

$$|U_B|_g = 1, \qquad \frac{\delta}{\rho_B(p)} \leq \operatorname{div}_{\mathbb{H}} U_B(p) = H_{\mathbb{H}}(p, B^{\rho_B(p)}) \leq \frac{H_0}{\rho_B(p)}. \tag{54}$$

We observe that $p \mapsto \operatorname{div}_{\mathbb{H}} U(p)$ is locally integrable at 0.

Step 3: calibration argument: for every $G \in \mathcal{P}$ there holds

$$P_{\mathbb{H}}(G) \geq \int_G \mathrm{div}_{\mathbb{H}}\, U_B(p)dp \geq \int_G \frac{\delta}{\rho_B(p)}\, dp. \tag{55}$$

Moreover equality holds when $G = B^\lambda$, $\lambda > 0$:

$$P_{\mathbb{H}}(B^\lambda) = \int_{B^\lambda} \mathrm{div}_{\mathbb{H}}\, U_B(p)dp. \tag{56}$$

The calibration argument is based on [35, Theorem 3.1]. We consider two family of smooth functions $\phi_\epsilon(p), \psi_\epsilon(p) : \mathbb{H} \to [0,1]$, defined as follows.

Let us fix a smooth function $\phi_\varepsilon : \mathbb{R}^2 \to [0,1]$ such that

$$\begin{aligned}
\phi_\epsilon(x,y) &\equiv 0 \quad \text{for } \sqrt{x^2+y^2} \leq \epsilon^2 \text{ and } \sqrt{x^2+y^2} \geq \tfrac{1}{\sqrt{\epsilon}} + 1,\\
\phi_\epsilon(x,y) &\equiv 1 \quad \text{for } \epsilon \leq \sqrt{x^2+y^2} \leq \tfrac{1}{\sqrt{\epsilon}},\\
|\nabla \phi_\epsilon| &\leq 2/\epsilon \quad \text{for } \epsilon^2 \leq \sqrt{x^2+y^2} \leq \epsilon,\\
|\nabla \phi_\epsilon| &\leq 2 \quad \text{elsewhere.}
\end{aligned}$$

It is straightforward to observe that we may extend ϕ_ε to be a smooth function on $\mathbb{H}$, by setting $(x,y,z) \in \mathbb{H} \mapsto \phi_\varepsilon(x,y)$. Moreover $\nabla \phi_\varepsilon = \nabla_{\mathbb{H}} \phi_\varepsilon$ and $|\nabla_{\mathbb{H}} \phi_\varepsilon| = |\nabla \phi_\varepsilon|_g$.

Now we consider a smooth function $\psi_\varepsilon : \mathbb{R} \to [0,1]$ such that

$$\psi_\epsilon(z) \equiv 1, \text{ for } |z| \leq \frac{1}{\sqrt{\epsilon}} \quad \psi_\epsilon(z) \equiv 0, \text{ for } |z| \geq \frac{1}{\sqrt{\epsilon}} + 1$$

with $|\psi'_\epsilon| \leq 2$. Also in this case we may extend ψ_ε to be a smooth function on $\mathbb{H}$, by setting $(x,y,z) \in \mathbb{H} \mapsto \psi_\varepsilon(z)$. Moreover $|\nabla_{\mathbb{H}} \psi_\varepsilon|_g = |\psi'_\varepsilon| \sqrt{\frac{x^2+y^2}{4}}$.

Therefore, the vector field $U_B \phi_\epsilon \psi_\epsilon$ is still horizontal, and it is C^1, compactly supported, identically 0 on the singular set $\mathbb{R}(0,0,1)$. Therefore by definition of perimeter (38) we obtain that for every $G \in \mathcal{P}$:

$$P_{\mathbb{H}}(G) \geq \int_G \mathrm{div}_{\mathbb{H}}(U_B \phi_\epsilon \psi_\epsilon)(p)dp.$$

At any point $p = (x,y,z)$ with $z \neq 0$ we have

$$\begin{aligned}
&\mathrm{div}_{\mathbb{H}}(U_B \phi_\epsilon \psi_\epsilon)(p) \qquad (57)\\
&\quad = \phi_\varepsilon \psi_\varepsilon \mathrm{div}_{\mathbb{H}} U_B + \psi_\varepsilon[(U_B)_1 \partial_x \phi_\varepsilon + (U_B)_2 \partial_y \phi_\varepsilon] + \phi_\varepsilon \left[-\tfrac{y}{2}(U_B)_1 + \tfrac{x}{2}(U_B)_2\right] \psi'_\varepsilon(z).
\end{aligned}$$

First of all we assume $G \subseteq \mathcal{C}_s = \{(x,y,z) \,|\, x^2+y^2 \leq s^2, |z| \leq s\}$ for some $s > 0$. We claim that

$$\lim_{\epsilon \to 0} \int_G \mathrm{div}_{\mathbb{H}}(U_B \phi_\epsilon \psi_\epsilon)dp = \lim_{\epsilon \to 0} \int_G \phi_\epsilon \psi_\epsilon \mathrm{div}_{\mathbb{H}} U_B dp. \tag{58}$$

Note that if the claim holds true, we infer (55). Indeed $P_{\mathbb{H}}(G) \geq \lim_{\epsilon \to 0} \int_G \mathrm{div}_{\mathbb{H}}(U_B \phi_\epsilon \psi_\epsilon)dp$ and moreover $\lim_{\epsilon \to 0} \int_G \phi_\epsilon \psi_\epsilon \mathrm{div}_{\mathbb{H}} U_B dp = \int_G \mathrm{div}_{\mathbb{H}} U_B dp$ by the dominated convergence theorem since $\chi_G \mathrm{div}_{\mathbb{H}} U_B \in L^1$.

To prove the claim (58) we recall the formula (57). First of all we observe that

$$\lim_{\varepsilon \to 0} \int_G \phi_\varepsilon \left[-\frac{y}{2}(U_B)_1 + \frac{x}{2}(U_B)_2\right] \psi'_\varepsilon(z)dp = 0 \tag{59}$$

due to the dominated convergence theorem, since $\chi_G \in L^1$, $\phi_\varepsilon \left[-\frac{y}{2}(U_B)_1 + \frac{x}{2}(U_B)_2\right] \psi'_\varepsilon(t)\chi_G$ is uniformly bounded since $G \subseteq \mathcal{C}_s$ and is converging to 0. Next, we have that

$$\left| \int_G [(U_B)_1 \partial_x \phi_\varepsilon + (U_B)_2 \partial_y \phi_\varepsilon] \psi_\epsilon dp \right| \leq \int_G |\nabla(\phi_\epsilon)|\, \psi_\epsilon dp \leq \int_G \chi_{A_\varepsilon} \frac{2}{\epsilon} \psi_\epsilon dp + \int_G \chi_{B_\varepsilon} 2\psi_\epsilon dp \tag{60}$$

where $A_\varepsilon = \{p \mid \epsilon^2 \le \sqrt{x^2+y^2} \le \epsilon\}$ and $B_\varepsilon = \{p \mid \frac{1}{\sqrt{\epsilon}} \le \sqrt{x^2+y^2} \le \frac{1}{\sqrt{\epsilon}} + 1\}$. Since $2\chi_{B_\varepsilon}\psi_\varepsilon \to 0$ and $\chi_G \in L^1$, we infer by dominated convergence that

$$\lim_{\varepsilon\to 0} \int_G \chi_{B_\varepsilon} 2\psi_\epsilon dp = 0. \tag{61}$$

On the other hand passing to cylindrical coordinates

$$\lim_{\varepsilon\to 0} \int_G \chi_{A_\varepsilon} \frac{2}{\epsilon}\psi_\epsilon dp \le \int_{\varepsilon^2}^{\varepsilon} 2\pi r \int_{-\frac{1}{\sqrt{\varepsilon}}-1}^{\frac{1}{\sqrt{\varepsilon}}+1} \frac{2}{\varepsilon} dz dr = \lim_{\varepsilon\to 0} \int_{\varepsilon^2}^{\varepsilon} r \frac{8\pi}{\varepsilon}\left(\frac{1}{\sqrt{\varepsilon}}+1\right) dr = 0. \tag{62}$$

Therefore, by (57), (59),(60), (61), (62) the claim (58) and thus (55) is proved for G bounded. If G is not bounded, observe that by coarea formula [8, Theorem 6.11] there exists $s_k \to +\infty$ such that $P_{\mathbb{H}}(G_k) \to P_{\mathbb{H}}(G)$, where $G_k = G \cap B_{CC}(0, s_k)$ (an outline of this step is given for instance in [27, Remark 3.12]). Since $\lim_{s_k} \int_{G_k} \operatorname{div}_{\mathbb{H}} U_B(p)dp \ge \int_G \operatorname{div}_{\mathbb{H}} U_B(p)dp$ by Fatou's lemma, and $P_{\mathbb{H}}(G_k) \to P_{\mathbb{H}}(G)$, we conclude (55) also for G.

On the other hand, equality (56) is obtained by the Gauss Green formula for C^2 sets (42), and the area formula (43), recalling (19) and that $U_B = \nu_{\mathbb{H}}$ on $\partial B^\lambda \setminus \mathbb{R}(0,0,1)$ for every $\lambda > 0$:

$$\lim_{\epsilon\to 0} \int_{B^\lambda} \operatorname{div}_g(U_B\phi_\epsilon\psi_\epsilon)dp = \lim_{\epsilon\to 0} \int_{\partial B^\lambda} \psi_\epsilon\phi_\epsilon\nu_{\mathbb{H}} \cdot_g N_g d\sigma = \int_{\partial B^\lambda} |N_{\mathbb{H}}| d\sigma = P_{\mathbb{H}}(B^\lambda). \tag{63}$$

Step 4: B is outward minimizing and (50) **holds**. Let us define $\lambda_F := \inf\{\lambda \mid F \subseteq B^\lambda\}$, with $\lambda_F = +\infty$ if such λ does not exist.

Recall that away from the z-axis we have that $\operatorname{div}_{\mathbb{H}} U_B(p) \ge \frac{\delta}{\rho_B(p)}$. Therefore, if $F \subseteq B^\lambda$, then for every $p \in F$ which is not on the z-axis, since $\rho_B(p) \le \lambda$, there holds $\operatorname{div}_{\mathbb{H}} U_B(p) \ge \frac{\delta}{\rho_B(p)} \ge \frac{\delta}{\lambda}$.

Let now $F \in \mathcal{P}$ be such that $F \supseteq B$. Then by (55), (56) and recalling that $\operatorname{div}_{\mathbb{H}} U_B(p) \ge \frac{\delta}{\lambda_F}$ for $p \in F \setminus B$, we obtain

$$P_{\mathbb{H}}(F) \ge \int_F \operatorname{div}_{\mathbb{H}} U_B(p)dp = \int_B \operatorname{div}_{\mathbb{H}} U_B(p)dp + \int_{F\setminus B} \operatorname{div}_{\mathbb{H}} U_B(p)dp \ge P_{\mathbb{H}}(B) + \frac{\delta}{\lambda_F}|F \setminus B|.$$

The second statement (51) follows easily using the submodularity (45). If $F \supseteq B$, then (51) coincides with (50). Assume that $F \not\supseteq B$. Let $E = F \cup B$. Then $\lambda_E = \max\{\lambda_F, 1\}$, so we apply (50) to E observing that $|E \setminus B| = |F \setminus B|$ and obtain

$$\begin{aligned} P_{\mathbb{H}}(F) + P_{\mathbb{H}}(B) &\ge P_{\mathbb{H}}(F \cap B) + P_{\mathbb{H}}(E) \ge P_{\mathbb{H}}(F \cap B) + P_{\mathbb{H}}(B) + \frac{\delta}{\lambda_E}|E \setminus B| \\ &= P_{\mathbb{H}}(F \cap B) + P_{\mathbb{H}}(B) + \frac{\delta}{\max\{\lambda_F, 1\}}|F \setminus B|. \end{aligned}$$

□

4 The minimizing movement scheme

4.1 Definition and basic properties

We introduce the following set valued operator T_h for $h > 0$: given a measurable set $\emptyset \ne E \in \mathcal{M}$ let

$$E \to T_h(E) = \arg\min\{J_h(F)\ ,\ F \in \mathcal{M}\}$$

where

$$J_h(F) := P_{\mathbb{H}}(F) + \frac{1}{h}\int_F d_E(p)dp, \tag{64}$$

and d_E is the signed distance introduced in (10) associated with the Carnot–Carathéodory distance d_{CC} defined in (5).

Remark 4.1. Note that the results presented in Sections 4.1 and 4.2 hold for any choice of distance function presented in Section 2.1. Nonetheless, from now on we will use only the Carnot–Carathéodory distance d_{CC}. This choice is motivated by the analysis we will present in Section 4.3.

Notice that in view of (45) the functional J_h is submodular, meaning that

$$J_h(F_1 \cap F_2) + J_h(F_1 \cup F_2) \leq J_h(F_1) + J_h(F_2)$$

for any $F_1, F_2 \in \mathcal{P}$.

In general the minimizers of the functional J_h in (64) are not unique (see Proposition 4.3 below). With a slight abuse of notation we will denote by $T_h(E)$ a generic minimizer of J_h, i.e. an element $F \in T_h(E)$.

Remark 4.2. Observe that

$$\int_F d_E(p)dp - \int_E d_E(p)dp = \int_{F\setminus E} d_E(p)dp - \int_{E\setminus F} d_E(p)dp = \int_{E\triangle F} dist(p, \partial E)dp.$$

In particular

$$J_h^a(F) := P_{\mathbb{H}}(F) + \frac{1}{h}\int_{E\triangle F} dist(p, \partial E)dp = J_h(F) - \frac{1}{h}\int_E d_E(p)dp, \tag{65}$$

so $J_h(\cdot)$ and $J_h^a(\cdot)$ share the same minimizers.

We verify some standard properties of the operator T_h (cp. with [11, Section 6] where the euclidean case is treated).

Proposition 4.3. *Let $E \in \mathcal{P}$ be bounded, so that $P_{\mathbb{H}}(E) < \infty$ and $E \subseteq B_P^R$ for some $R > 0$, where B_P^R is the Pansu ball of parameter R.*

1. *There exists at least one bounded $\mathbb{H}$-Caccioppoli set which solves the minimization problem $\min_{F\in\mathcal{M}} J_h(F)$. Moreover every minimizer is contained in $\overline{B_P^R}$ and there exists a minimal $T_h^-(E)$ and a maximal solution $T_h^+(E)$ (with respect to inclusion) and $T_h^+(E) \subseteq \overline{B_P^R}$.*
2. *If $E \subset E' \subset \mathbb{H}$ for $E' \in \mathcal{P}$ bounded set, then for all $F \in T_h(E)$ there exists $F' \in T_h(E')$ such that $F \subseteq F'$. Moreover: $T_h^{\pm}(E) \subset T_h^{\pm}(E')$.*
3. *If $F \in T_h(E)$ and $R_\theta E = E$ with R_θ a rotation around the z-axis of angle θ, then also $R_\theta F \in T_h(E)$. Therefore, if E is rotationally symmetric with respect to the z-axis, i.e. $R_\theta E = E$ for every $\theta \in (0, 2\pi)$ then $T_h^+(E)$ is rotationally symmetric with respect to the z-axis, that is $T_h^+(E) = R_\theta T_h^+(E)$ for all θ.*
4. *If $F \in T_h(E)$, then $P_{\mathbb{H}}(F) \leq P_{\mathbb{H}}(E)$.*

Proof. 1. By Proposition 3.28 for every $F \subseteq \mathcal{P}$ there holds $P_{\mathbb{H}}(F \cap B_P^R) \leq P_{\mathbb{H}}(F)$. Moreover, since $d_E(x) > 0$ for $x \in \mathbb{H} \setminus B_P^R$, we obtain that $\int_F d_E(p)dp \geq \int_{F\cap B_P^R} d_E(p)dp$. Therefore $J_h(F) \geq J_h(B_P^R \cap F)$, with strict inequality if $|F \setminus B_P^R| > 0$.

This implies that every minimizer of J_h necessarily satisfies $|F \setminus B_P^R| = 0$, and so $F \subseteq \overline{B_P^R}$. Now, we are reduced to minimize $J_h(F)$ on sets $F \subseteq \overline{B_P^R}$, $F \in \mathcal{P}$. First of all we note that $\min_{B_P^R} d_E(x) = -k \leq 0$, so for every $F \subseteq \overline{B_P^R}$, $J_h(F) \geq \frac{1}{h}\int_F d_E(p)dp \geq -\frac{k|F|}{h} \geq -\frac{k|B_P^R|}{h}$.

We consider a minimizing sequence F_n: in particular for every n, since $d_E \geq 0$ on $\mathbb{H} \setminus E$ and $d_E \geq -k$ on E, it holds

$$\infty > J_h(E) \geq J_h(F_n) \geq P_{\mathbb{H}}(F_n) + \frac{1}{h}\int_{E\cap F_n} d_E(p)dp \geq P_{\mathbb{H}}(F_n) - \frac{k}{h}|E|.$$

Therefore $P_{\mathbb{H}}(F_n)$ are equibounded and by the compactness theorem in the space of $BV_{\mathbb{H}}$ functions (see Theorem 3.22), we have that up to passing to a subsequence $F_n \to F$ in L^1 (which means, $\chi_{F_n} \to \chi_F$ in L^1), and $\liminf_n P_{\mathbb{H}}(F_n) \geq P_{\mathbb{H}}(F)$ since the perimeter functional is lower semicontinuous with respect to L^1_{loc} convergence of (characteristic functions of) sets.

Note that up to passing to a subsequence we have $\chi_{F_n} \to \chi_F$ for a.e. p.

Then observe that by Fatou lemma, it holds

$$\liminf_n \int_{\mathbb{H}\setminus E} \chi_{F_n} d_E dp \geq \int_{\mathbb{H}\setminus E} \chi_F d_E dp \qquad \liminf_n \int_E (d_E \chi_{F_n} + k) dp \geq \int_E (d_E \chi_F + k) dp$$

Therefore we conclude that F is a minimizer.

If we consider $F_i \in T_h(E)$, for $i = 1, 2$, then $F_1 \cup F_2$ and $F_1 \cap F_2$ are both minimizers, by the submodularity of J_h.

For the last statement see [11, Proposition 6.1].

In the rest of the proof with a slight abuse of notation we will denote as $T_h(E)$ a minimizer of J_h.

2. Assume $E \subset\subset E'$, so that $d_E(p) > d_{E'}(p)$ for all $p \in \mathbb{H}$.

   By minimality of $T_h(E)$ we have

$$P_{\mathbb{H}}(T_h(E) \cap T_h(E')) + \frac{1}{h} \int_{T_h(E)\cap T_h(E')} d_E(p) dp \geq P_{\mathbb{H}}(T_h(E)) + \frac{1}{h} \int_{T_h(E)} d_E(p) dp. \tag{66}$$

   In a similar way, by minimality of $T_h(E')$ we have

$$P_{\mathbb{H}}(T_h(E) \cup T_h(E')) + \frac{1}{h} \int_{T_h(E)\cup T_h(E')} d_{E'}(p) dp \geq P_{\mathbb{H}}(T_h(E')) + \frac{1}{h} \int_{T_h(E')} d_{E'}(p) dp. \tag{67}$$

   Summing the two inequalities and using the sub-additivity of the perimeter (45) we obtain

$$\int_{T_h(E)\cap T_h(E')} d_E(p) dp + \int_{T_h(E)\cup T_h(E')} d_{E'}(p) dp \geq \int_{T_h(E)} d_E(p) dp + \int_{T_h(E')} d_{E'}(p) dp$$

   from which we deduce that

$$\int_{T_h(E)\setminus T_h(E')} \big(d_{E'}(p) - d_E(p)\big) dp \geq 0$$

   but, since $d_{E'}(p) - d_E(p) < 0$ then $T_h(E) \setminus T_h(E') = \emptyset$. Hence $T_h(E) \subset T_h(E')$.

   The statement for $E \subset E'$ is achieved by a perturbation argument as explained in [11, Lemma 6.2].

3. By Remark 3.16, we know that

$$P_{\mathbb{H}}(F) = P_{\mathbb{H}}(R_\theta F).$$

   Moreover, we have that

$$\begin{aligned}
\int_{R_\theta F} d_E(p) dp &= -\int_E \chi_{R_\theta F}(p) \mathrm{dist}(p, \mathbb{H} \setminus E) dp + \int_{\mathbb{H}\setminus E} \chi_{R_\theta F}(p) \mathrm{dist}(p, E) dp \\
&= -\int_E \chi_F(q) \mathrm{dist}(R_\theta q, R_\theta(\mathbb{H} \setminus E)) dq + \int_{\mathbb{H}^1\setminus E} \chi_F(q) \mathrm{dist}(R_\theta q, R_\theta E) dq \\
&= \int_F d_E(q) dq
\end{aligned}$$

   where we have used that $\mathrm{dist}(R_\theta p, R_\theta q) = \mathrm{dist}(p, q)$.

4. Using minimality of $T_h(E)$ and E as a competitor we obtain

$$P_{\mathbb{H}}(T_h(E)) + \frac{1}{h}\int_{T_h(E)} d_E(p)dp \leq P_{\mathbb{H}}(E) + \frac{1}{h}\int_E d_E(p)dp.$$

Observe that

$$\frac{1}{h}\int_E d_E(p)dp - \frac{1}{h}\int_{T_h(E)} d_E(p)dp = \int_{E\setminus T_h(E)} d_E(p)dp - \int_{T_h(E)\setminus E} d_E(p)dp \leq 0.$$

from which we conclude.

□

We provide a description of the scheme for outward minimizing sets.

Proposition 4.4. *Let $E \in \mathcal{P}$ be a bounded outward minimizing in $\mathbb{H}$ with $|\partial E| = 0$. Then*

1. *$T_h(E) \subseteq E$, that is every minimizer $F \in T_h(E)$ satisfies $F \subseteq E$,*
2. *every minimizer in $T_h(E)$ is outward minimizing in $\mathbb{H}$.*

Proof. Note that if $|\partial E| = 0$, then $d_E(p) > 0$ for a.e. $p \in \mathbb{H} \setminus E$.

1. Let F be a minimizer of J_h in (64). If E is an outward minimizing set, then by the characterization in Lemma 3.24 we have that $P_{\mathbb{H}}(F \cap E) \leq P_{\mathbb{H}}(F)$. Moreover

$$\int_{F\cap E} d_E(p)dp \leq \int_F d_E(p)dp$$

with strict inequality if $F \setminus E$ has positive measure. Therefore if $|F \setminus E| > 0$, it holds that $J_h(F \cap E) < J_h(F)$, which implies that F cannot be a minimizer of J_h. This implies that necessarily every minimizer $T_h(E)$ is contained in E.

2. Let $G \in \mathcal{P}$ with $F \subset G$, where $F \in T_h(E)$. Then, using $G \cap E$ as a competitor we get, recalling that $F \subseteq E \cap G$ and $d_E \leq 0$ in E,

$$\begin{aligned} P_{\mathbb{H}}(F) &\leq P_{\mathbb{H}}(E\cap G) + \frac{1}{h}\int_{E\cap G} d_E(p)dp - \frac{1}{h}\int_F d_E(p)dp \\ &= P_{\mathbb{H}}(E\cap G) + \frac{1}{h}\int_{E\cap G\setminus F} d_E(p)dp \leq P_{\mathbb{H}}(E\cap G) \leq P_{\mathbb{H}}(G) \end{aligned}$$

where the last inequality is a consequence of the fact that E is outward minimizing in $\mathbb{H}$ and of the characterization proved in Lemma 3.24.

□

We present an estimate on the one time step of the scheme for sets B in the class $\mathcal{B}$ introduced in Definition 3.27 and which are also horizontal strictly mean convex. The following Proposition 4.5 applies in particular to Pansu balls. Observe that $T_h(B)$ might no longer belong to $\mathcal{B}$. For $\lambda > 0$ recall that $B^\lambda := \delta_\lambda(B)$.

Proposition 4.5. *Let $B \in \mathcal{B}$ be horizontal strictly mean convex with constant $\delta > 0$, and let*

$$\lambda^0 = \inf\{\lambda > 0 \ s.t. \ |T_h(B) \setminus B^\lambda| = 0\}\}.$$

Then there exist $h_0 = h_0(diam(B), \delta)$, $C = C(diam(B), \delta)$, such that for all $0 < h \leq h_0$ there holds

$$\lambda^0 \leq \sqrt{1 - Ch^2}.$$

Proof. First of all we observe that by Proposition 3.28 the set B is outward minimizing, therefore by Proposition 4.4 we have that $T_h(B) \subseteq B$. Let $\lambda \in (0,1)$ and $B^\lambda \subseteq B$: note that B^λ is still open, bounded, C^2, axially symmetric and horizontal mean convex by Remark 3.4 with $H_{\mathbb{H}}(p, B^\lambda) \geq \frac{\delta}{\lambda}$. Since $T_h(B) \subseteq B$ obviously $\lambda^0 \leq 1$. Using Proposition 3.28 applied to B^λ and to $T_h(B)$, and the fact that $T_h(B) \subseteq B$, we obtain that

$$P_{\mathbb{H}}(T_h(B) \cap B^\lambda) \leq P_{\mathbb{H}}(T_h(B)) - \delta |T_h(B) \setminus B^\lambda|. \tag{68}$$

By minimality of $T_h(B)$, using as a competitor $T_h(B) \cap B^\lambda$ we obtain

$$P_{\mathbb{H}}(T_h(B)) \leq P_{\mathbb{H}}(T_h(B) \cap B^\lambda) - \frac{1}{h} \int_{T_h(B) \setminus B^\lambda} d_B(p) dp. \tag{69}$$

Then by (68), (69) we infer that

$$\delta |T_h(B) \setminus B^\lambda| \leq -\frac{1}{h} \int_{T_h(B) \setminus B^\lambda} d_B(p) dp = \frac{1}{h} \int_{T_h(B) \setminus B^\lambda} \mathrm{dist}(p, \partial B) dp \tag{70}$$

where in the last equality we used the fact that $T_h(B) \subseteq B$.
Note that if $p \in T_h(B) \setminus B^\lambda$ then $\delta_{1/\lambda} p \notin B$, and so, recalling (7) we have

$$\mathrm{dist}(p, \partial B) \leq d_{CC}(p, \delta_{1/\lambda} p) \leq C\, diam(B) \max\left(|\lambda^{-1} - 1|, \sqrt{|\lambda^{-2} - 1|}\right).$$

We substitute in (70) and we get, for every $\lambda \in (0,1)$ such that $|T_h(B) \setminus B^\lambda| \neq 0$,

$$\delta \leq \frac{C diam(B)}{h} \max\left(|\lambda^{-1} - 1|, \sqrt{|\lambda^{-2} - 1|}\right) = \frac{C diam(B)}{h\lambda} \max\left((1-\lambda), \sqrt{(1-\lambda^2)}\right). \tag{71}$$

Note that if $\lambda < \lambda^0$ then $|T_h(B) \setminus B^\lambda| \neq 0$ and (71) holds. We have two possibilities:

(i)$\lambda^0 \leq \frac{1}{2}$: In this case

$$T_h(B) \subseteq B^{1/2}$$

and we conclude.

(ii)$\lambda^0 > \frac{1}{2}$: For $\frac{1}{2} < \lambda < \lambda^0$, (71) reads

$$\sqrt{\frac{3}{4}} \geq \sqrt{(1-\lambda^2)} = \max\left((1-\lambda), \sqrt{(1-\lambda^2)}\right) \geq \frac{h\delta}{2C diam(B)}. \tag{72}$$

In light of the previous inequality we set $h_0 < 2\sqrt{\frac{3}{4}} C \frac{diam(B)}{\delta}$. Furthermore we conclude by letting $\lambda \to \lambda^0$ in (72) that $\sqrt{(1-(\lambda^0)^2)} \geq \frac{h\delta}{2C diam(B)}$, that is $(\lambda^0)^2 \leq 1 - \frac{h^2\delta^2}{4C^2 diam(B)^2}$.

□

4.2 The scheme for unbounded sets

Here we extend the time discrete scheme to the case of unbounded sets E with compact boundaries. More generally we assume that the set E in (64) is such that $P_{\mathbb{H}}(E) < \infty$ and E^c is bounded. Recalling (40), the idea is to define the evolution as the complement of the evolution of the complementary set. More precisely, using Proposition 4.3, we set

$$T_h^{\pm}(E) := \mathbb{H} \setminus T_h^{\mp}(E^c).$$

In particular note that if E_1, $E_2 \in \mathcal{M}$ are such that $P_{\mathbb{H}}(E_i) < \infty$, and E_i^c are bounded, then it follows from Proposition 4.3(2.) that

$$E_1 \subset E_2 \Longrightarrow T_h^{\pm}(E_1) \subset T_h^{\pm}(E_2).$$

Lemma 4.6. *Let E_1, $E_2 \in \mathcal{M}$ be such that $P_{\mathbb{H}}(E_i) < \infty$, $i = 1, 2$. If $E_1 \subset E_2$ and $E_2^c \cup E_1 \subset B_P^R$ for a sufficiently large $R > 0$, then $T_h^{\pm}(E_1) \subset T_h^{\pm}(E_2)$.*

Proof. Since Pansu balls are outward minimizing sets we have by Proposition 4.4 (and Proposition 4.3) that

$$T_h^{\pm}(E_2^c) \subset T_h^{\pm}(B_P^R) \subset B_P^R.$$

The proof now follows the arguments given in [11, Lemma 6.10], which we report here for completeness. Since $T_h^-(E_2^c)$ is the minimal solution of

$$\min\left\{P_{\mathbb{H}}(F) + \frac{1}{h}\int_F d_{E_2^c}dp \,:\, F \in \mathcal{M}\right\},$$

we have (using the change of variable $\tilde{F} := \mathbb{H} \setminus F = F^c$, the fact that $P_{\mathbb{H}}(\mathbb{H} \setminus F) = P_{\mathbb{H}}(F)$, and $d_{E_2^c} = -d_{E_2}$) that $T_h^+(E_2) = \mathbb{H} \setminus T_h^-(E_2^c)$ is the maximal solution of

$$\min\left\{P_{\mathbb{H}}(\tilde{F}) + \frac{1}{h}\int_{B_P^R} d_{E_2}dp - \frac{1}{h}\int_{\tilde{F}^c} d_{E_2}dp \,:\, \tilde{F} \in \mathcal{M}\right\} - \frac{1}{h}\int_{B_P^R} d_{E_2}dp.$$

Note that if $\tilde{F}^c \subset B_R^P$, then $\int_{B_P^R} d_{E_2}dp - \int_{\tilde{F}^c} d_{E_2}dp = \int_{\tilde{F}} d_{E_2}\chi_{B_P^R}dp$. Therefore, since $T_h^+(E_2)^c \subset B_R^P$, we conclude that $T_h^+(E_2)$ is also the maximal solution of

$$\min\left\{P_{\mathbb{H}}(\tilde{F}) + \frac{1}{h}\int_{\tilde{F}} d_{E_2}\chi_{B_P^R}dp \,:\, \tilde{F} \in \mathcal{M},\, \tilde{F}^c \subset B_R^P\right\}. \tag{73}$$

Analogously one can show that $T_h^-(E_2)$ is the minimal solution of (73).
Since $d_{E_2}\chi_{B_P^R} \le d_{E_1}$ and since $T_h^{\pm}(E_1) \cup T_h^{\pm}(E_2)$, $T_h^{\pm}(E_1) \cap T_h^{\pm}(E_2)$ are admissible sets in (73), one can argue as in Proposition 4.3 (2.) to infer that $T_h^{\pm}(E_1) \subset T_h^{\pm}(E_2)$. □

4.3 Density estimates

In this section, we provide density estimates for minimizing solutions to (64). The arguments are quite classical (we refer to [26], [32], [10] and reference therein) and we provide sketch of the main proofs with adaptations to our setting.

We start recalling the following important result, which provides a relation between the $\mathbb{H}$-perimeter of a set F on the boundary of a ball and the derivative of its volume.

Lemma 4.7. *([25, Lemma 1.19]) Let F be a set with finite $\mathbb{H}$-perimeter, $p \in \mathbb{H}$ fixed, and define*

$$\begin{cases} P_{\mathbb{H}}(F \cap B_{CC}(p,r), \partial B_{CC}(p,r)) := P_{\mathbb{H}}(F \cap B_{CC}(p,r)) - P_{\mathbb{H}}(F, B_{CC}(p,r)) \\ P_{\mathbb{H}}(F \setminus B_{CC}(p,r), \partial B_{CC}(p,r)) := P_{\mathbb{H}}(F \setminus B_{CC}(p,r)) - P_{\mathbb{H}}(F, \mathbb{H} \setminus \overline{B_{CC}}(p,r)). \end{cases} \tag{74}$$

For $r > 0$ let $m(r) := |F \cap B_{CC}(p,r)|$. Then for a.e. $r > 0$, there holds

$$P_{\mathbb{H}}(F \cap B_{CC}(p,r), \partial B_{CC}(p,r)) \le m'(r),$$
$$P_{\mathbb{H}}(F \setminus B_{CC}(p,r), \partial B_{CC}(p,r)) \le m'(r).$$

We recall a technical Lemma (see e.g. [32, Lemma 4.4]). We provide a short sketch of the proof.

Lemma 4.8. *Let F be a set of finite $\mathbb{H}$-perimeter.*

(i) Assume that $|B_{CC}(\hat{p}, r) \setminus F| > 0$ for all $r > 0$ and that there exist $R > 0$, $\mu \ge 0$ such that

$$P_{\mathbb{H}}(F) \le P_{\mathbb{H}}(F \cup B_{CC}(\hat{p}, r)) + \mu|B_{CC}(\hat{p}, r) \setminus F| \quad \forall \quad 0 < r < R. \tag{75}$$

Then, there exist $\sigma > 0, C > 0$ depending only on the isoperimetric constant (47) such that

$$|B_{CC}(\hat{p}, r) \setminus F| \ge \sigma r^4 \quad \forall \quad 0 < r < \min\left\{R, \frac{C}{\mu}\right\}. \tag{76}$$

(ii) Assume that $|B_{CC}(\hat{p},r)\cap F|>0$ for all $r>0$ and that there exist $R>0$, $\mu\geq 0$ such that

$$P_{\mathbb{H}}(F)\leq P_{\mathbb{H}}(F\setminus B_{CC}(\hat{p},r))+\mu|B_{CC}(\hat{p},r)\cap F|\quad\forall\quad 0<r<R. \tag{77}$$

Then, there exist $\sigma>0, C>0$ depending only on the isoperimetric constant (47) *such that*

$$|B_{CC}(\hat{p},r)\cap F|\geq\sigma r^4\quad\forall\quad 0<r<\min\left\{R,\frac{C}{\mu}\right\}. \tag{78}$$

For $\mu=0$ in (77), (78), *we set* $\min\left\{R,\frac{C}{\mu}\right\}=R$.

Proof. (i) Recalling (40) and (74) for F^c, since $\mathbb{H}\setminus(F^c\cap B_{CC}(\hat{p},s))=F\cup B_{CC}(\hat{p},s)$ and $F^c\cap B_{CC}(\hat{p},s)=B_{CC}(\hat{p},s)\setminus F$, we observe that

$$\begin{aligned}
&P_{\mathbb{H}}(F\cup B_{CC}(\hat{p},s))=P(F,\mathbb{H}\setminus\overline{B_{CC}(\hat{p},s)})+P_{\mathbb{H}}(B_{CC}(\hat{p},s)\setminus F,\partial B_{CC}(\hat{p},s))\\
\leq&P_{\mathbb{H}}(F,\mathbb{H})-P_{\mathbb{H}}(F,B_{CC}(\hat{p},s))+P_{\mathbb{H}}(B_{CC}(\hat{p},s)\setminus F,\partial B_{CC}(\hat{p},s))\\
=&P_{\mathbb{H}}(F)-P_{\mathbb{H}}(F^c,B_{CC}(\hat{p},s))+P_{\mathbb{H}}(B_{CC}(\hat{p},s)\setminus F,\partial B_{CC}(\hat{p},s))\\
=&P_{\mathbb{H}}(F)-P_{\mathbb{H}}(F^c\cap B_{CC}(\hat{p},s))+P_{\mathbb{H}}(F^c\cap B_{CC}(\hat{p},s),\partial B_{CC}(\hat{p},s))+P_{\mathbb{H}}(B_{CC}(\hat{p},s)\setminus F,\partial B_{CC}(\hat{p},s))\\
=&P_{\mathbb{H}}(F)-P_{\mathbb{H}}(B_{CC}(\hat{p},s)\setminus F)+2P_{\mathbb{H}}(B_{CC}(\hat{p},s)\setminus F,\partial B_{CC}(\hat{p},s)).
\end{aligned} \tag{79}$$

From (79) and (75) we obtain for all $s<R$,

$$P_{\mathbb{H}}(B_{CC}(\hat{p},s)\setminus F)\leq 2P_{\mathbb{H}}(B_{CC}(\hat{p},s)\setminus F,\partial B_{CC}(\hat{p},s))+\mu|B_{CC}(\hat{p},s)\setminus F|.$$

We set $|B_{CC}(\hat{p},s)\setminus F|=|B_{CC}(\hat{p},s)\cap F^c|=:m(s)$. So, by Lemma 4.7, and the isoperimetric inequality (47) we get for a.e. $s<R$:

$$Cm(s)^{\frac{3}{4}}\leq 2m'(s)+\mu m(s).$$

Since $0<m(s)\leq|B_{CC}(0,1)|s^4$ we get $m(s)\leq|B_{CC}(0,1)|^{\frac{1}{4}}s\,m(s)^{1-\frac{1}{4}}$: then substituting in the previous inequality we conclude

$$\frac{C}{2}m(s)^{\frac{3}{4}}\leq\left(C-s\mu|B_{CC}(0,1)|^{\frac{1}{4}}\right)m(s)^{\frac{3}{4}}\leq 2m'(s)$$

for $s\leq\min\{R,\frac{C}{2|B_{CC}(0,1)|^{\frac{1}{4}}}\frac{1}{\mu}\}$, when $\mu>0$, for $s\leq R$ when $\mu=0$. By direct integration, we get the claim.

(ii) Recalling (74), we have that

$$\begin{aligned}
&P_{\mathbb{H}}(F\setminus B_{CC}(\hat{p},s))=P_{\mathbb{H}}(F,\mathbb{H}\setminus\overline{B_{CC}(\hat{p},s)})+P_{\mathbb{H}}(F\setminus B_{CC}(\hat{p},s),\partial B_{CC}(\hat{p},s))\\
&\leq P_{\mathbb{H}}(F,\mathbb{H})-P_{\mathbb{H}}(F,B_{CC}(\hat{p},s))+P_{\mathbb{H}}(F\setminus B_{CC}(\hat{p},s),\partial B_{CC}(\hat{p},s))\\
&=P_{\mathbb{H}}(F)-P_{\mathbb{H}}(F\cap B_{CC}(\hat{p},s))+P_{\mathbb{H}}(F\cap B_{CC}(\hat{p},s),\partial B_{CC}(\hat{p},s))\\
&\quad+P_{\mathbb{H}}(F\setminus B_{CC}(\hat{p},s),\partial B_{CC}(\hat{p},s)).
\end{aligned} \tag{80}$$

The claim follows in a similar way to part (i), using now (80) and (77).

□

The next result is an adaptation to our setting of [10, Lemma 3.5], and it is an estimate independent of the time step h.

Lemma 4.9. *Let $E\in\mathcal{P}$ be bounded. Let $F\in T_h(E)$. Then there exists a constant $\sigma>0$ (depending only on the isoperimetric constant in* (47)*) with the following property:*

(i) if $|F\cap B_{CC}(\hat{p},s)|>0$ for all $s>0$ and $E\cap B_{CC}(\hat{p},r_0)=\emptyset$ for some r_0 then

$$|F\cap B_{CC}(\hat{p},s)|\geq\sigma s^4\qquad\forall s\leq r_0. \tag{81}$$

(ii) if $|B_{CC}(\hat{p},s)\setminus F|>0$ for all $s>0$ and $B_{CC}(\hat{p},r_0)\subset E$ for some r_0, then

$$|B_{CC}(\hat{p},s)\setminus F|\geq \sigma s^4 \qquad \forall s\leq r_0. \tag{82}$$

Proof. (i) From the minimality of F we obtain for any $s>0$ that

$$P_{\mathbb{H}}(F)+\frac{1}{h}\int_F d_E(p)\,dp\leq P_{\mathbb{H}}(F\setminus B_{CC}(\hat{p},s))+\frac{1}{h}\int_{F\setminus B_{CC}(\hat{p},s)} d_E(p)\,dp.$$

Using that d_E is positive on $F\cap B_{CC}(\hat{p},s)\subset E^c$ we infer that

$$P_{\mathbb{H}}(F)\leq P_{\mathbb{H}}(F\setminus B_{CC}(\hat{p},s)).$$

The claim now follows from Lemma 4.8(ii) with $\mu=0$.

To prove (ii) one argues in a similar way: first use the minimality of F and test against $F\cup B_{CC}(\hat{p},s)$. Since d_E is negative on $B_{CC}(\hat{p},s)\subset E$ (for $s\in(0,r_0)$) we obtain

$$P_{\mathbb{H}}(F)\leq P_{\mathbb{H}}(F\cup B_{CC}(\hat{p},s)).$$

Application of Lemma 4.8(i) with $\mu=0$ yields the claim. □

We provide some uniform bounds between the boundary of a minimizing set and the boundary of the reference set.

Proposition 4.10. *($L^\infty-$estimate) There exists a constant $\gamma>0$ depending only on the isoperimetric constant such that for any bounded $E\in\mathcal{P}$ and $F\in T_h(E)$ there holds*

$$\sup_{E\triangle F}|d_E|\leq \gamma\sqrt{h}. \tag{83}$$

Proof. The proof follows [32, Proposition 3.2]. We set $\gamma=\max\left\{\frac{4P_{\mathbb{H}}(B_{CC}(0,1))}{\sigma},2\right\}$, where σ is the constant depending only on the isoperimetric constant appearing in Lemma 4.8.

We argue by contradiction.

Let $\hat{p}\in E\setminus F$, and assume that

$$d_E(\hat{p})<-\gamma\sqrt{h}. \tag{84}$$

This implies that $B_{CC}(\hat{p},r)\subset E$ for all $r\leq\frac{\gamma}{2}\sqrt{h}$. Using the minimality of F we obtain

$$P_{\mathbb{H}}(F)\leq P_{\mathbb{H}}(F\cup B_{CC}(\hat{p},r))+\frac{1}{h}\int_{B_{CC}(\hat{p},r)\setminus F} d_E(p)\,dp$$

From (84) and $r\leq\frac{\gamma}{2}\sqrt{h}$ we infer that

$$\frac{1}{h}\int_{B_{CC}(\hat{p},r)\setminus F} d_E(p)\,dp\leq\frac{1}{h}(r-\gamma\sqrt{h})|B_{CC}(\hat{p},r)\setminus F|\leq-\frac{\gamma}{2\sqrt{h}}|B_{CC}(\hat{p},r)\setminus F|$$

so that we obtain

$$P_{\mathbb{H}}(F)\leq P_{\mathbb{H}}(F\cup B_{CC}(\hat{p},r))-\frac{\gamma}{2\sqrt{h}}|B_{CC}(\hat{p},r)\setminus F|. \tag{85}$$

Since $\hat{p}\notin F$, then $|B_{CC}(\hat{p},r)\setminus F|>0$ for all r and we may apply Lemma 4.8 (i) with $\mu=0$ to get

$$|B_{CC}(\hat{p},r)\setminus F|\geq\sigma r^4 \qquad \text{for } 0<r<\frac{\gamma\sqrt{h}}{2}.$$

On the other hand, from (85) and (45) we have

$$\begin{aligned}\frac{\gamma}{2\sqrt{h}}|B_{CC}(\hat{p},r)\setminus F| &\le P_{\mathbb{H}}(F\cup B_{CC}(\hat{p},r)) - P_{\mathbb{H}}(F)\\ &\le P_{\mathbb{H}}(B_{CC}(\hat{p},r)) - P_{\mathbb{H}}(F\cap B_{CC}(\hat{p},r))\\ &\le P_{\mathbb{H}}(B_{CC}(\hat{p},r)) = P_{\mathbb{H}}(B_{CC}(0,1))r^3.\end{aligned}$$

Combining the last two inequalities and recalling the choice of γ, we get for $0<r<\frac{\gamma\sqrt{h}}{2}$

$$\sigma r^4 \le |B_{CC}(\hat{p},r)\setminus F| \le P_{\mathbb{H}}(B_{CC}(0,1))\frac{2\sqrt{h}}{\gamma}r^3 \le \frac{\sigma\sqrt{h}}{2}r^3$$

which gives a contradiction as soon as $\frac{\sqrt{h}}{2}\le\frac{\gamma\sqrt{h}}{4}<r<\frac{\gamma\sqrt{h}}{2}$.

The case $\hat{p}\in F\setminus E$ is treated in a similar way. Assume by contradiction that $|d_E(\hat{p})| = d_E(\hat{p}) > \gamma\sqrt{h}$, so that $B_{CC}(\hat{p},r)\subset\mathbb{H}\setminus E$ for any $r\le\frac{\gamma}{2}\sqrt{h}$. Using the minimality of F and testing against $F\setminus B_{CC}(\hat{p},r)$ we obtain for $r\le\frac{\gamma}{2}\sqrt{h}$:

$$P_{\mathbb{H}}(F)\le P_{\mathbb{H}}(F\setminus B_{CC}(\hat{p},r)) - \frac{1}{h}\int_{F\cap B_{CC}(\hat{p},r)} d_E(p)\,dp \le P_{\mathbb{H}}(F\setminus B_{CC}(\hat{p},r)) - \frac{\gamma}{2\sqrt{h}}|F\cap B_{CC}(\hat{p},r)|.$$

Using Lemma 4.8(ii) with $\mu=0$ we have

$$|F\cap B_{CC}(\hat{p},r)|\ge\sigma r^4 \qquad \text{for } 0<r<\frac{\gamma}{2}\sqrt{h}.$$

On the other hand, as above by the previous inequality and (45),

$$\begin{aligned}\frac{\gamma}{2\sqrt{h}}|F\cap B_{CC}(\hat{p},r)| &\le P_{\mathbb{H}}(F\setminus B_{CC}(\hat{p},r)) - P_{\mathbb{H}}(F) = P_{\mathbb{H}}(F^c\cup B_{CC}(\hat{p},r)) - P_{\mathbb{H}}(F^c)\\ &\le P_{\mathbb{H}}(B_{CC}(\hat{p},r)) - P_{\mathbb{H}}(F^c\cap B_{CC}(\hat{p},r)) \le P_{\mathbb{H}}(B_{CC}(0,1))r^3.\end{aligned}$$

As before, we get a contradiction as soon as $\frac{\sqrt{h}}{2}\le\frac{\gamma\sqrt{h}}{4}<r<\frac{\gamma\sqrt{h}}{2}$. □

Finally we provide density estimates, using Lemma 4.9 and the L^∞ estimate (83).

Proposition 4.11. *Let $E\in\mathcal{P}$ be bounded and let $F\in T_h(E)$.*

There exists positive constants σ, θ, Θ, depending only on the isoperimetric constant (47) *such that the following holds: for any $\hat{p}\in\partial^e F$ there holds*

$$\min\{|B_{CC}(\hat{p},r)\setminus F|, |B_{CC}(\hat{p},r)\cap F|\}\ge\sigma r^4, \tag{86}$$

$$\theta r^3 \le P_{\mathbb{H}}(F, B_{CC}(\hat{p},r)) \le \Theta r^3 \tag{87}$$

for all $r\in(0,C\sqrt{h})$, where C only depends on the isoperimetric constant.

Remark 4.12. Note that by the density estimate, for every $F\in T_h(E)$,

$$F(1)=\{p\in\mathbb{H},\exists r>0, |B_{CC}(p,r)\setminus F|=0\} \qquad F(0)=\{p\in\mathbb{H},\exists r>0, |B_{CC}(p,r)\cap F|=0\},$$

so they are open set, moreover there holds that $|\partial^e F|=0$ and $\partial^e F=\partial F(1)=\partial F(0)$, see [25, Theorem 3.4]. Therefore for every $F\in T_h(E)$ we have an open representative which is $F(1)$ and a closed representative which is $\overline{F(1)}$. From now on we choose the maximal solution of (64) to be closed and the minimal to be open.

Proof. Since $\hat{p}\in\partial^e F=\mathbb{H}\setminus(F(1)\cup F(0))$ we have that

$$\min\{|B_{CC}(\hat{p},r)\setminus F|, |B_{CC}(\hat{p},r)\cap F|\}>0 \qquad \forall r>0,$$

so that Lemma 4.8 can be applied. Moreover, by (83) we have that $|d_E(\hat{p})| \leq \gamma\sqrt{h}$ (indeed if for some $\hat{p} \in \partial^e F$, by contradiction we have $|d_E(\hat{p})| > \gamma\sqrt{h}$, then in every neighborhood of $\hat{p}$ we would have a subset of $E\Delta F$ of positive measure where (83) does not hold).

Let us fix $r \leq \gamma\sqrt{h}$. Now we use the minimality of F, testing against the competitor $F \cup B_{CC}(\hat{p}, r)$ and $F \setminus B_{CC}(\hat{p}, r)$ to obtain

$$\begin{cases} P_{\mathbb{H}}(F) & \leq P_{\mathbb{H}}(F \cup B_{CC}(\hat{p}, r)) + \frac{1}{h}\int_{B_{CC}(\hat{p},r)\setminus F} d_E dp \leq P_{\mathbb{H}}(F \cup B_{CC}(\hat{p}, r)) + \frac{2\gamma}{\sqrt{h}}|B_{CC}(\hat{p}, r) \setminus F| \\ P_{\mathbb{H}}(F) & \leq P_{\mathbb{H}}(F \setminus B_{CC}(\hat{p}, r)) - \frac{1}{h}\int_{B_{CC}(\hat{p},r)\cap F} d_E dp \leq P_{\mathbb{H}}(F \setminus B_{CC}(\hat{p}, r)) + \frac{2\gamma}{\sqrt{h}}|B_{CC}(\hat{p}, r) \cap F|. \end{cases}$$

Therefore by Lemma 4.8 with $\mu = \frac{2\gamma}{\sqrt{h}}$, we conclude (86) for every $r < \frac{C}{2\gamma}\sqrt{h}$.

The first inequality in (87) follows from Theorem 3.21.

To show the last inequality we proceed as follows. From (74) we deduce

$$\begin{aligned} P_{\mathbb{H}}(F \cup B_{CC}(\hat{p}, r)) &= P_{\mathbb{H}}(F, \mathbb{H} \setminus \overline{B_{CC}(\hat{p}, r)}) + P_{\mathbb{H}}(F \cup B_{CC}(\hat{p}, r), \partial B_{CC}(\hat{p}, r)) \\ &\leq P_{\mathbb{H}}(F, \mathbb{H}) - P_{\mathbb{H}}(F, B_{CC}(\hat{p}, r)) + P_{\mathbb{H}}(F \cup B_{CC}(\hat{p}, r), \partial B_{CC}(\hat{p}, r)). \end{aligned}$$

and from the inequality we conclude that

$$P_{\mathbb{H}}(F, B_{CC}(\hat{p}, r)) \leq P_{\mathbb{H}}(F \cup B_{CC}(\hat{p}, r), \partial B_{CC}(\hat{p}, r)) + \frac{2\gamma}{\sqrt{h}}|B_{CC}(\hat{p}, r) \setminus F|.$$

On the other hand by (40) and Lemma 4.7 we deduce that for a.e. $r > 0$:

$$\begin{aligned} &P_{\mathbb{H}}(F \cup B_{CC}(\hat{p}, r), \partial B_{CC}(\hat{p}, r)) = P_{\mathbb{H}}(F^c \setminus B_{CC}(\hat{p}, r), \partial B_{CC}(\hat{p}, r)) \\ &\leq \frac{d}{dr}\left(|F^c \cap B_{CC}(\hat{p}, r)|\right) = \frac{d}{dr}\Big(|B_{CC}(\hat{p}, r)| - |(F \cap B_{CC}(\hat{p}, r))|\Big) \\ &= 4r^3|B_{CC}(0,1)| - \frac{d}{dr}\left(|(F \cap B_{CC}(\hat{p}, r))|\right) \leq 4r^3|B_{CC}(0,1)| \end{aligned}$$

since the map $r \to |(F \cap B_{CC}(\hat{p}, r))|$ is nondecreasing in r. It follows that,

$$P_{\mathbb{H}}(F, B_{CC}(\hat{p}, r)) \leq 4r^3|B_{CC}(0,1)| + \frac{2\gamma}{\sqrt{h}}r^4|B_{CC}(0,1)|$$

and the claim follows using $r < \frac{C}{2\gamma}\sqrt{h}$. □

We are going now to recall an L^1 estimate that holds for sets which satisfy density estimates.

Lemma 4.13 (Volume-distance comparison)**.** *Let $F \in \mathcal{P}$ bounded and with the following property: there exists $\theta > 0$ and $r_0 < 1$ such that for all $p \in \partial(F(1))$ and $r \leq r_0$ there holds*

$$P_{\mathbb{H}}(F, B_{CC}(p, r)) = P_{\mathbb{H}}(F(1), B_{CC}(p, r)) \geq \theta r^3. \tag{88}$$

Then for any $G \in \mathcal{P}$ and any $l > 0$ we have

$$|F \triangle G| \leq \omega_{CC}\frac{5^4}{\theta} l \max\left\{1, \frac{l^3}{r_0^3}\right\} P_{\mathbb{H}}(F) + \frac{2}{l}\int_{F\triangle G} d_{CC}(p, \partial F(1))\, dp,$$

where $\omega_{CC} := |B_{CC}(p, 1)|$.

Remark 4.14. Note that due to Proposition 4.11, $E \in \mathcal{P}$ bounded, and Proposition 4.3(1) then every $F \in T_h(E)$ is bounded and satisfies (88) with θ depending only on universal constants and $r_0 \leq C\sqrt{h}$.

Therefore Lemma 4.13 implies that there exists a constant $C > 0$ depending only on the isoperimetric constant and on ω_{CC} such that for all $l \leq r_0 \leq C\sqrt{h}$ and for all $F \in T_h(E)$ there holds

$$|F \triangle G| \leq C\left(lP_{\mathbb{H}}(F) + \frac{1}{l}\int_{F\triangle G} d_{CC}(p, \partial F(1))\, dp\right). \tag{89}$$

Proof. We follow [5, Proposition 3.3]. Define

$$V := \left\{ p \in F \triangle G \,:\, d_{CC}(p, \partial F(1)) \leq \frac{l}{2} \right\}, \qquad W = \left\{ p \in F \triangle G \,:\, d_{CC}(p, \partial F(1)) \geq \frac{l}{2} \right\}$$

so that $|F \triangle G| \leq |V| + |W|$. We have

$$\frac{2}{l} \int_{F \triangle G} d_{CC}(p, \partial F(1))\, dp \geq \frac{2}{l} \int_W d_{CC}(p, \partial F(1))\, dp \geq |W|.$$

Next, by the compactness of $\partial F(1)$ and the definition of V, there exists a finite family of disjoint balls $B_{CC}(p_j, l)$ with $p_j \in \partial F(1)$, such that $V \subseteq \cup_{j=1}^m B_{CC}(p_j, 5l)$, due to the covering theorem in [24, Theorem 1.2]). If $l \geq r_0$ recalling the property (88) we get

$$\begin{aligned} |V| &\leq \sum_{j=1}^m |B_{CC}(p_j, 5l)| = \sum_{j=1}^m \omega_{CC} (5l)^4 = \omega_{CC} \frac{(5l)^4}{\theta r_0^3} \sum_{j=1}^m \theta r_0^3 \\ &\leq \omega_{CC} \frac{(5l)^4}{\theta r_0^3} \sum_{j=1}^m P_{\mathbb{H}}(F, B_{CC}(p_j, r_0)) \leq \omega_{CC} \frac{(5l)^4}{\theta r_0^3} P_{\mathbb{H}}\big(F, \cup_{j=1}^m B_{CC}(p_j, r_0)\big) \leq \omega_{CC} \frac{(5l)^4}{\theta r_0^3} P_{\mathbb{H}}(F) \end{aligned}$$

where we have used the fact that the balls $B_{CC}(p_j, r_0) \subset B_{CC}(p_j, l)$ are disjoint. If $l \leq r_0$ we obtain similarly

$$\begin{aligned} |V| &\leq \sum_{j=1}^m |B_{CC}(p_j, 5l)| = \omega_{CC} \frac{5^4 l}{\theta} \sum_{j=1}^m \theta l^3 \\ &\leq \omega_{CC} \frac{5^4 l}{\theta} \sum_{j=1}^m P_{\mathbb{H}}(F, B_{CC}(p_j, l)) \leq \omega_{CC} \frac{5^4 l}{\theta} P_{\mathbb{H}}\big(F, \cup_{j=1}^m B_{CC}(p_j, l)\big) \leq \omega_{CC} \frac{5^4 l}{\theta} P_{\mathbb{H}}(F) \end{aligned}$$

and the claim follows. □

4.4 The flat flow

We define the following discrete time scheme. Take a bounded set $E \in \mathcal{P}$, and consider any $T_h(E)$ minimizer of (64). Then we construct the following piecewise constant flows of sets

$$t \in [0, T] \longrightarrow E_h(t) \qquad E_h(t) = \begin{cases} E & t \in [0, h) \\ T_h(E) & t \in [h, 2h) \\ T_h(T_h(E)) =: T_h^2(E) & t \in [2h, 3h) \\ \dots & \dots \end{cases}$$

In order to have a unique selection, we choose each $T^k(E)$ to be $(T^k(E))^+$ (equivalently we could choose $T^k(E)$ to be $(T^k(E))^-$). Note that by Proposition 4.3, item 4. the map $t \to P_{\mathbb{H}}(E_h(t))$ is monotone nonincreasing, that is

$$P_{\mathbb{H}}(E_h(t)) \leq P_{\mathbb{H}}(E_h(s)) \qquad 0 \leq s \leq t. \tag{90}$$

Moreover $E_h(t)$ is bounded for $t > 0$ by Proposition 4.3(1) independently of h.

Remark 4.15. Recalling Remark 4.2 we infer that

$$P_{\mathbb{H}}(T_h(E)) + \frac{1}{h} \int_{E \triangle (T_h(E))} dist(p, \partial E) dp \leq P_{\mathbb{H}}(E). \tag{91}$$

By Lemma 4.13 and in particular by (89) in Remark 4.14, , we obtain for $l \leq C\sqrt{h}$, where C is a constant depending only on the isoperimetric constant, the following holds for any $k \in \mathbb{N}$:

$$\begin{aligned} |T_h^k(E) \triangle T_h^{k+1}(E)| &\leq C l P_{\mathbb{H}}(T^k(E)) + \frac{C}{l} \int_{T^k(E) \triangle T^{k+1}(E)} \mathrm{dist}(p, \partial T_h^k(E))\, dp \\ &\leq C l P_{\mathbb{H}}(T_h^k(E)) + \frac{C}{l} h \Big(P_{\mathbb{H}}(T_h^k(E)) - P_{\mathbb{H}}(T^{k+1}(E)) \Big). \end{aligned} \tag{92}$$

Remark 4.16. For sets A, B, C there holds: $A \triangle C \subset (A \triangle B) \cup (B \triangle C)$.

We provide the existence of the flat flow, as the time step $h \to 0$, starting from E bounded.

Theorem 4.17. *Let $E \in \mathcal{P}$ be a bounded set. There exists a sequence $h_i \to 0$, and a constant $C > 0$ only depending on isoperimetric constant and on $p_0 = P_{\mathbb{H}}(E)$, such that the iteration scheme $t \to E_{h_i}(t)$ converges to a map $t \to E(t) \in \mathcal{P}$, $E(t)$ bounded and such that*

$$|E(t) \triangle E(t')| \leq C|t - t'|^{1/2} \qquad \forall\, t, t' \in (0, \infty), \quad |t - t'| < 1. \tag{93}$$

In addition, if $|\partial E| = 0$, then (93) holds also for any $t, t' \in [0, \infty), |t - t'| < 1$.

Proof. Let $0 < t < t'$ such that $0 < t' - t < 1$ and let us fix $0 < h \leq t' - t$ with $\lfloor \frac{t}{h} \rfloor \geq 1$, where for $r \geq 0$ we denote $\lfloor r \rfloor := \max\{n \in \mathbb{N}, n \leq r\}$. Let us introduce $k, m \in \mathbb{N}$, such that $k = \lfloor \frac{t}{h} \rfloor \geq 1$ and $k + m = \lfloor \frac{t'}{h} \rfloor \geq 1$. We introduce the following notation

$$[t]_h := kh, \qquad [t']_h := (k+m)h, \qquad h \leq mh = [t']_h - [t]_h \leq |t - t'| + h \leq 2|t - t'|.$$

Note that by definition $E_h(t) = E_h([t]_h) = T_h^k(E)$, and $E_h(t') = E_h([t']_h) = T_h^{k+m}(E)$.

We choose $l = \frac{Ch}{2|t-t'|^{1/2}} < C\sqrt{h}$, where C is the constant in (92), so that $ml \leq C|t - t'|^{1/2}$. Now, we use estimate (92), Remark 4.16, and the fact that $P_{\mathbb{H}}(T^k(E)) \leq P_{\mathbb{H}}(E) = p_0$ (see (90), to get

$$\begin{aligned} |E_h(t) \triangle E_h(t')| = |T_h^k(E) \triangle T_h^{k+m}(E)| &\leq \sum_{j=1}^{m} |T_h^{k+j}(E) \triangle T_h^{k+j-1}(E)| \\ &\leq Clp_0 m + \frac{C}{l} h \Big(P_{\mathbb{H}}(T^k(E)) - P_{\mathbb{H}}(T^{k+m}(E)) \Big) \\ &\leq C^2 p_0 |t - t'|^{1/2} + 2p_0 |t - t'|^{1/2} = (C^2 p_0 + 2p_0)|t - t'|^{1/2}. \end{aligned} \tag{94}$$

By the a priori bounds given by (90), and by the fact that $E_h(t)$ is bounded uniformly in h, we may apply Theorem 3.22. to get that for all $t > 0$, there exists a subsequence (depending on t) $h_t \to 0$ such that $E_h(t)$ converges in L^1 to some $E(t)$. Using a simple diagonalization argument we can find a sequence $h_i \to 0$ such that

$$\lim_{i \to \infty} E_{h_i}(t) =: E(t) \qquad \forall t \in \mathbb{Q} \cap (0, \infty).$$

From (94) we infer that

$$|E(t) \triangle E(t')| \leq (C^2 p_0 + 2p_0)|t - t'|^{1/2} \qquad \forall t, t' \in \mathbb{Q} \cap (0, \infty), |t - t'| < 1. \tag{95}$$

Using again Theorem 3.22 and (95) we extend univocally $E(t)$ to $(0, \infty)$ such that

$$|E(t) \triangle E(t')| \leq (C^2 p_0 + 2p_0)|t - t'|^{1/2} \qquad \forall t, t' \in (0, \infty), |t - t'| < 1$$

holds, yielding (93).

Finally let us consider the continuity of $E(t)$ at the initial time $t = 0$. Obviously $E_h(0) = E$ for any $h > 0$. Notice that by (91) we have

$$\int_{E_{h_i}(h_i) \triangle E} dist(x, \partial E) dp \leq h_i P_{\mathbb{H}}(E) \to 0 \text{ for } h_i \to 0.$$

On the other hand by Theorem 3.22 for a subsequence of h_i (still denoted as h_i for simplicity) we have that $\chi_{E_{h_i}(h_i)} \to \chi_B$ in $L^1(\mathbb{H})$ for some $B \in \mathcal{P}$. By Fatou's lemma we get

$$\int_{E \Delta B} dist(p, \partial E) dp \leq \liminf_i \int_{E \Delta E_{h_i}(h_i)} dist(p, \partial E) dp \leq \liminf_i h_i P_{\mathbb{H}}(E) = 0.$$

This implies that $dist(p, \partial E) = 0$ for a.e. $p \in B \setminus E$ and $p \in E \setminus B$, which implies that $B \subseteq \overline{E}$. Moreover Since $|\partial E| = 0$, it follows that $\chi_E = \chi_B$ that is $\chi_{E \setminus E_{h_i}(h_i)} \to 0$ in $L^1(\mathbb{H})$. Finally given $t \in (0, 1)$ we compute

$$|E_{h_i}(t) \triangle E| \leq |E_{h_i}(t) \triangle E_{h_i}(h_i)| + |E_{h_i}(h_i) \triangle E| \leq C|t - h_i|^{1/2} + |E \Delta E_{h_i}(h_i)|$$

and passing to the limit we obtain $|E(t) \triangle E| \leq C|t|^{1/2}$ as claimed. □

4.5 Remarks on the consistency of the scheme

Following [7, 18, 4] a family of surfaces $t \in [0,T) \mapsto \Sigma_t = \partial E_t$, where Σ_t is a surface in $\mathbb{H}$, given by the boundary of a C^2 set $E_t \subseteq \mathbb{H}$ for every $t \in [0,T)$, evolves by horizontal mean curvature flow if all its non-singular points $p(t) \in \Sigma_t \setminus (\Sigma_t)_0$ move according to the velocity

$$\partial_t p(t) = -H_{\mathbb{H}}(p(t), E_t)\nu_{\mathbb{H}}(p(t)) \tag{96}$$

where $\nu_{\mathbb{H}}(p(t))$ is the horizontal exterior normal to E_t at the point $p(t)$. If Σ_t can be described as the 0-level set of a function u, that is $\Sigma_t = \{(t,p) \in [0,T) \times \mathbb{H} \,:\, u(t,p) = 0\}$, differentiation with respect to time of the equation $0 = u(t,p(t))$ yields (recalling (16) and (23))

$$\begin{aligned}
0 &= u_t(t,p(t)) + \nabla u(t,p(t)) \cdot \partial_t p(t) \\
&= u_t(t,p(t)) - \nabla u(t,p(t)) \cdot H_{\mathbb{H}}(p(t),E_t)\frac{\nabla_{\mathbb{H}} u(p(t))}{|\nabla_{\mathbb{H}} u(p(t))|_g} \\
&= u_t - H_{\mathbb{H}} \nabla u \cdot \frac{Xu\,X + Yu\,Y}{|\nabla_{\mathbb{H}} u|_g} = u_t - H_{\mathbb{H}} \frac{(Xu)^2 + (Yu)^2}{|\nabla_{\mathbb{H}} u|_g} \\
&= u_t(t,p(t)) - H_{\mathbb{H}}(p(t),E_t)|\nabla_{\mathbb{H}} u(p(t))|_g.
\end{aligned}$$

Therefore away from the singular points u is a solution to the PDE

$$\frac{u_t}{|\nabla_{\mathbb{H}} u|_g} = H_{\mathbb{H}}(u). \tag{97}$$

There are very few explicit examples of smooth evolutions by horizontal mean curvature flow. A first trivial observation is the following. Every family of axially symmetric cylinders of the form $C(r)_t = \{p = (x,y,z), \sqrt{x^2+y^2} = \sqrt{r^2-2t}\}$ is an evolution by horizontal mean curvature flow for every $t \in [0, \frac{r^2}{2})$. Indeed in every point of $C(r)_t = \partial E(r)_t = \partial\{p = (x,y,z), \sqrt{x^2+y^2} < \sqrt{r^2-2t}\}$ the horizontal mean curvature coincides with the euclidean mean curvature.

A less trivial example is the evolution of the Korányi sphere, obtained in [18, Section 3.1]. Recalling the definition given in Remark 3.11, the Korányi sphere is

$$\partial B_K(0,R) = \{(x,y,z),\ (x^2+y^2)^2 + 16z^2 = R^4\}$$

and its evolution by horizontal mean curvature is

$$\partial B_K^t(0,R) = \{(x,y,z),\ (x^2+y^2)^2 + 16z^2 + 12t(x^2+y^2) + 12t^2 = R^4\}.$$

for $t \in \left[0, \frac{R^2}{2\sqrt{3}}\right)$. It is interesting to observe that despite the fact that $B_K(0,R) \in \mathcal{B}$ (see Definition 3.27), the sets $B_K^t(0,R)$ for $t \in \left(0, \frac{R^2}{2\sqrt{3}}\right)$ do not belong to $\mathcal{B}$ since the horizontal mean curvature

$$H_{\mathbb{H}}(p, (B_K)_t(0,1)) = \frac{3(x^2+y^2)+6t}{\sqrt{x^2+y^2}\sqrt{1+24t^2}}$$

is positive but unbounded and $\delta_\lambda B_K^t(0,R) = B_K^{\lambda^2 t}(0,\lambda R)$ holds.

In general we cannot expect to have smooth solutions to the horizontal mean curvature flow, so weak notions of the evolutions have been introduced, mainly in the setting of level set formulation and viscosity solutions [7, 18]. We recall also some results providing optimal control interpretation of the evolution by horizontal mean curvature flow, [16, 15, 23]. In particular in [18] existence and uniqueness of the viscosity solution to the level set equation associated to the horizontal mean curvature flow has been provided in the class of axially symmetric sets. More recently [4] establishes a general uniqueness theorem and comparison principle for viscosity solutions to the horizontal mean curvature flow, overcoming limitations near characteristic points by proving that continuous viscosity solutions can be derived as uniform limits of approximating Riemannian flows.

In this paper we provide the construction of the flat flow, Theorem 4.17, which can be interpreted as another weak formulation of the horizontal mean curvature flow.

A natural question that we leave open is the consistency of the scheme we propose, that is the flat flow, with the smooth evolution by horizontal mean curvature flow and with other weak formulation of the flow, in particular with level set viscosity solutions.

The first problem would be the consistency with respect to smooth evolutions. First of all, we observe that due to Theorem 3.5 we may define the evolution by horizontal mean curvature flow for smooth sets as follows.

Definition 4.18. *We say that a family of surfaces $t \mapsto \Sigma_t = \partial E_t$, where $E(t)$ are bounded C^3 sets is a smooth horizontal mean curvature flow if*

$$\partial_t d_{E(t)}(p) = \mathrm{div}_{\mathbb{H}} \nabla_{\mathbb{H}} d_{E(t)}(p) \qquad \forall p \in \Sigma(t) \setminus \Sigma_0(t),$$

where $\Sigma_0(t)$ is the set of singular points in $\Sigma(t)$ and d_E is the signed distance function from E as defined in (10) *(with respect to the Carnot–Carathéodory distance d_{CC}).*

We may define also smooth strict sub-and superflow of evolutions: a family $\Sigma(t) = \partial E(t)$, where $E(t)$ denotes a C^3 bounded set in $\mathbb{H}$, is a strict superflow (resp. subflow) of the smooth horizontal mean curvature flow in $[t_0, t_1]$ if there exists $A \supseteq \cup_{t\in[t_0,t_1]}\Sigma(t)$, such that $d_{E(t)}$ is $C^3(A \setminus \cup_{t\in[t_0,t_1]}\Sigma_0(t))$, and $\hat{\delta} > 0$ for which

$$\partial_t d_{E(t)} \geq \mathrm{div}_{\mathbb{H}} \nabla_{\mathbb{H}} d_{E(t)} + \hat{\delta} \qquad (\text{resp. } \partial_t d_{E(t)} \leq \mathrm{div}_{\mathbb{H}} \nabla_{\mathbb{H}} d_{E(t)} - \hat{\delta})$$

for $p \in A \setminus \cup_{t\in[t_0,t_1]}\Sigma_0(t)$, $t \in [t_0, t_1]$.

We expect that if $\Sigma(t) = \partial E(t)$ is a smooth superflow, (resp. subflow) then there exists h_0 such that for all $h \leq h_0$, $t \in [t_0, t_1 - h]$, it holds

$$E(t + h) \subseteq T_h(E(t)) \qquad (\text{resp. } E(t + h) \supseteq T_h(E(t))).$$

Such a result could be obtained in the case of evolving smooth surfaces without characteristic points, by adapting the calibration argument used in Proposition 3.28. Nonetheless the validity of a general result is at the moment unclear to us. We recall also that in [12] a new approximation and regularization of horizontal mean curvature flow, similar to the Bence-Merriman-Osher algorithm, has been proposed, and a consistency result in that case was obtained for smooth non-characteristic evolutions.

References

[1] F. Almgren, J. E. Taylor, L. Wang, Curvature-Driven Flows: A Variational Approach, SIAM J. Control Optim. 31, (1993), no. 2, 387–438.

[2] N. Arcozzi, F. Ferrari, Metric normal and distance function in the Heisenberg group, Math. Z. 256, (2007), no. 3, 661–684.

[3] N. Arcozzi, F. Ferrari, The Hessian of the distance from a surface in the Heisenberg group, Ann. Acad. Sci. Fenn. Math. 33, (2008), no. 1, pp 35–63.

[4] E. Baspinar, G. Citti, Uniqueness of viscosity mean curvature flow solution in two sub-Riemannian structures. SIAM J. Math. Anal. 51 (2019), no. 3, 2633–2659.

[5] G. Bellettini, A. Chambolle, S. Kholmatov, Minimizing movements for forced anisotropic mean curvature flow of partitions with mobilities. Proc. Roy. Soc. Edinburgh Sect. A 151, (2021), no.4, 1135–1170.

[6] G. Bombardieri, M. Fogagnolo, V. Franceschi, Mean-convex horizontal mean curvature flow in the Heisenberg group, In preparation.

[7] L. Capogna, G. Citti, Generalized mean curvature flow in Carnot groups. Comm. Partial Differential Equations 34 (2009), no. 7-9, 937–956.

[8] L. Capogna, D. Danielli, S.D. Pauls, J. T. Tyson, **An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem.** Progress in Mathematics, 259. Birkhäuser Verlag, Basel, 2007.

[9] A. Cesaroni, M. Novaga, K-mean convex and K-outward minimizing sets. Interfaces Free Bound. 24 (2022), no. 1, 35–61.

[10] A. Chambolle, D. De Gennaro, M. Morini, Minimizing movements for anisotropic and inhomogeneous mean curvature flows. Advances in Calculus of Variations, vol. 17, no. 4, 2024, pp. 1095-1129.

[11] A. Chambolle, M. Morini, M. Ponsiglione, Nonlocal curvature flows. Arch. Ration. Mech. Anal. 218 (2015), no. 3, 1263–1329.

[12] G. Citti, N. Dirr, F. Dragoni, R. Grande, Horizontal mean curvature flow as a scaling limit of a mean field equation in the Heisenberg group. Multiscale Modeling & Simulation 24 (2026), no. 2, 720–741.

[13] D. Danielli, N. Garofalo, D.M. Nhieu, S. D. Pauls, Instability of graphical strips and a positive answer to the Bernstein problem in the Heisenberg group $\mathbb{H}^1$. J. Differential Geom. 81 (2009), no. 2, 251–295.

[14] G. De Philippis and T. Laux Implicit time discretization for the mean curvature flow of sets Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 21 (2020), 911–930.

[15] N. Dirr, F. Dragoni, R. Grande, Asymptotics for optimal controls for horizontal mean curvature flow. Nonlinear Anal. 202 (2021), Paper No. 112076, 17 pp.

[16] N. Dirr, F. Dragoni and M. von Renesse, Evolution by mean curvature flow in sub-Riemannian geometries: A stochastic approach. Commun. Pure Appl. Anal. 9(2) (2010), 307–326.

[17] F. Dragoni, D. Filali. Starshaped and Convex Sets in Carnot Groups and in the Geometries of Vector Fields. Journal of Convex Analysis 26 (2019), No. 4, 1349–1372.

[18] F. Ferrari, Q. Liu, J.J. Manfredi, On the horizontal mean curvature flow for axisymmetric surfaces in the Heisenberg group. Commun. Contemp. Math. 15 (2013), Paper No. 1350027.

[19] M. Fogagnolo, A. Pinamonti, Strict starshapedness of solutions to the horizontal p-Laplacian in the Heisenberg group. Math. Eng. 3 (2021), no. 6, Paper No. 046, 15 pp.

[20] V. Franceschi, F. Montefalcone, R. Monti, CMC spheres in the Heisenberg group. Anal. Geom. Metr. Spaces 7 (2019), no. 1, 109–129.

[21] V. Franceschi, G.P. Leonardi, R. Monti, Quantitative isoperimetric inequalities in H^n. Calc. Var. Partial Differential Equations 54 (2015), no. 3, 3229–3239.

[22] B. Franchi, R. Serapioni, F. Serra Cassano, Rectifiability and perimeter in the Heisenberg group. Math Ann 321, 479–531 (2001).

[23] R. Grande, A stochastic representation for the solution of approximated mean curvature flow. NoDEA Nonlinear Differential Equations Appl. 29 (2022), no. 1, Paper No. 9, 21 pp.

[24] J. Heinonen, **Lectures on analysis on metric spaces**. Universitext, Springer Verlag 2001.

[25] G.P. Leonardi, S. Rigot, Isoperimetric sets on Carnot groups. Houston J. Math. 29(3), 609–637 (2003).

[26] S. Luckhaus, T. Sturzenhecker, Implicit time discretization for the mean curvature flow equation. Calc. Var. Partial Differential Equations 3 (1995), 253–271.

[27] F. Maggi, **Sets of Finite Perimeter and Geometric Variational problems. An introduction to geometric measure theory.** Cambridge studies in advanced mathematics 135, 2012.

[28] R. Montgomery, **A Tour of sub-Riemannian Geometries, their Geodesics and Applications**. Math. Surv. and Monographs 91 AMS, 2000.

[29] R. Monti, **Isoperimetric problem and minimal surfaces in the Heisenberg group** Geometric Measure Theory and Real Analysis, in: L. Ambrosio (Ed.), CRM Series, vol. 17, 2015, pp. 57–130.

[30] R. Monti, Heisenberg isoperimetric problem. The axial case. Adv. Calc. Var. 1(1), 93–121 (2008).

[31] R. Monti, M. Rickly, Convex isoperimetric sets in the Heisenberg group. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8(2), 391–415 (2009). MR2548252

[32] L. Mugnai, C. Seis, E. Spadaro, Global solutions to the volume-preserving mean-curvature flow. Calc. Var. Partial Differential Equations 55 (2016).

[33] P. Pansu, Une inégalité isopérimétrique sur le groupe de Heisenberg. C. R. Acad. Sci. Paris Sér. I Math. 295(2), 127–130 (1982).

[34] J. Pozuelo, M. Ritorè, Pansu-Wulff shapes in H^1. Adv. Calc. Var. (2021).

[35] M. Ritorè, A proof by calibration of an isoperimetric inequality in the Heisenberg group H^n. Calc. Var. Partial Differ. Equ. 44(1–2), 47–60 (2012).

[36] M. Ritorè, C. Rosales, Area-stationary surfaces in the Heisenberg group H^1. Adv. Math. 219(2), 633–671 (2008).

[37] M. Ritorè, C. Rosales, Rotationally invariant hypersurfaces with constant mean curvature in the Heisenberg group H^n. J. Geom. Anal. 16, no. 4, 703–720,(2006).