ANISOTROPIC ISOPERIMETRIC DOUBLE TILINGS OF THE PLANE

FRANCESCO NOBILI, MATTEO NOVAGA, AND EMANUELE PAOLINI

Abstract. We study periodic partitions of the plane into two distinct cells minimizing the anisotropic ℓ_1-perimeter. For a rectangular lattice G, we compute explicitly the (G, ℓ_1)-isoperimetric profile and classify all minimizers. When one cell has small area, the optimal tiling is generated by a square and a chipped rectangle, while in the remaining regime the tiling is generated by two adjacent rectangles. Further minimizing over all possible planar lattices, we show that the ℓ_1-isoperimetric profile is attained by the Pythagorean double tiling of two axis-aligned squares sharing a vertex. This configuration is unique unless the two cells are assigned the same area. Finally, we prove that the limiting (non periodic) partitions obtained by sending one volume to infinity are locally ℓ_1-isoperimetric.

Contents

1. Introduction

The study of configurations with minimal interface energy is a longstanding research program in the calculus of variations and geometric measure theory, with applications ranging from material science, crystallography, and physics. Depending on the formulation and on the concept of interface energy, the natural targets of these theoretical investigations are isoperimetric-type inequalities with geometric and physical flavour. An important goal is typically the understanding and characterization of minimal-energy configurations.

A relevant example from discrete geometry is the so-called Kelvin problem ([Tho87]), which consists of finding a partition of the Euclidean space $\mathbb{R}^d$, for $d \geq 2$, into cells of equal volume so that the interface area is minimal. Except for the planar case $d = 2$, where the problem takes the name of the Honeycomb conjecture and was finally solved in [Hal01] (see also [FT63, FT64b, FT53, FT64a, MFG98, Mor99]), little is currently known for $d > 2$ (cf. [WP96]).

Another source of examples from geometric measure theory is the characterization of minimal clusters ([Alm76]). The central objects are multi-isoperimetric inequalities and the characterization of minimal clusters, a problem going under the name of the k-bubble problem. A k-bubble in $\mathbb{R}^d$ with $k \leq d + 1$ is a cluster with k distinguished chambers with prescribed volumes that is conjectured to

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have minimal interface area ([SM96]). Currently, several advances have been obtained in the series of works [FAB+93, CLM17, MW02, DLSW09, HHS95, HS00, HMRR02, Rei08, Wic04, PT18a, PT18b, PT20, MN22, MN24], depending on the dimension d, the number of chambers k, and the prescribed volumes. We refer to [Mor09] and the survey [Mil25] for more insights and references.

For our goals, we recall that these geometric problems have also been investigated with anisotropic surface energies, namely with a different notion of interface energy that is often motivated by the study of material science and crystallography ([Wul01, Tay78, Her51]). Given a norm φ in $\mathbb{R}^d$, we recall the concept of φ-surface energy, also called anisotropic perimeter, which is defined for each Borel set $E \subset \mathbb{R}^d$ as

$$\mathrm{Per}_\varphi(E) \coloneqq \int_{\partial^* E} \varphi(\nu_E(x))\,\mathrm{d}\mathcal{H}^{d-1}(x). \tag{1.1}$$

We refer to Section 2 for the notations and concepts appearing in the above formula, and to the books [AFP00, Mag12] for a detailed introduction. It is well known that, in this scenario, the regularity theory for perimeter or cluster minimizers is different from the standard one; see [FPS23] for a thorough investigation.

For instance, double bubbles have also been investigated in this anisotropic scenario for the so-called Manhattan norm, namely $\varphi(x) = \|x\|_{\ell_1} = \sum_i |x_i|$. We recall the work of [MFG98], who initiated the study of Wulff planar clusters, and also the recent advances [FGS24, FGS25a, FGS25b] up to three dimensions, and novel geometric settings [FS19].

In this work, we are interested in the study of periodic partitions of the plane with two distinct repeating cells, minimizing the total ℓ_1-anisotropic length. Periodic isoperimetric problems for the standard isotropic area have been commented on in [Mor99] and also numerically tested in [FT, FGT]. Furthermore, these theoretical investigations align with both the Kelvin problem and the study of minimal clusters in terms of methodology and geometric flavour. Finally, as we are going to see in the next set-up paragraph, our goals are motivated by a recent series of works [CN24b, NPST22, NPST23, CN24a, CFN23] aiming at studying isoperimetric periodic partitions of the Euclidean space, and its recent generalization with unequal cells in [NN24, NNP26].

Theoretical set-up. We consider d-dimensional lattices G, namely groups of translations in the Euclidean space $\mathbb{R}^d$. The fundamental domain of a lattice G is a Borel set $D \subset \mathbb{R}^d$ satisfying

$$\left|\mathbb{R}^d \setminus \bigcup_{g\in G}(D+g)\right| = 0 \qquad |(D+g)\cap D| = 0, \quad \forall g \in G\setminus\{0\}\,.$$

Since the volume of the fundamental domain only depends on the group G, we will denote the volume of the lattice G by $|G| \coloneqq |D|$ for any (hence all) fundamental domains D. In the planar case, we will write $\mathrm{Area}(G) = |G|$. Given a lattice G with $\mathrm{Area}(G) > 0$, there are vectors $\{g_1, \dots, g_d\} \subset G$ called generators which are independent vectors and such that, for every $g \in G$, we have $g = \sum_{k=1}^d c_k g_k$ with $c_k \in \mathbb{Z}$. A canonical choice of fundamental domain is the parallelepiped spanned by the g_i's, namely $D = \{\sum_{k=1}^d c_k g_k \colon c_k \in [0,1]\}$, such that $|D| = |\det(g_1, \dots, g_d)|$.

A tiling in $\mathbb{R}^d$ is a collection $\mathbf{T}$ of Borel sets such that $|E \cap F| = 0$ for all $E, F \in \mathbf{T}$, $E \neq F$, and $\left|\mathbb{R}^d \setminus \bigcup_{E\in\mathbf{T}} E\right| = 0$. A periodic N-tiling, for $N \in \mathbb{N}$, is a tiling $\mathbf{T}$ such that there are N Borel sets $E_1, \dots, E_N$ called generators and a lattice G such that we have

$$\mathbf{T} = \{E_i + g : i = 1, \dots, N, g \in G\}.$$

We shall shortly say that $\mathbf{T}$ is an N-tiling relative to G generated by $E_1, \dots, E_N$. Moreover, as customary, we shall consider tilings up to Borel representatives and say that $\mathbf{T}$ is generated by

$\tilde{E}_1, \dots, \tilde{E}_N$ also if $|E_i \triangle \tilde{E}_i| = 0$ for $i = 1, \dots, N$. We call $\mathbf{T}$ non-degenerate provided $|E_i| > 0$ for every $i = 1, \dots, N$. Given a norm $\varphi\colon \mathbb{R}^d \to [0, \infty)$, we define the φ-perimeter of a tiling $\mathbf{T}$ as

$$\mathrm{Per}_\varphi(\mathbf{T}) \coloneqq \frac{1}{2} \sum_{i=1}^N \mathrm{Per}_\varphi(E_i),$$

where $\mathrm{Per}_\varphi(E)$ is the anisotropic φ-perimeter of a Borel set $E \subset \mathbb{R}^d$ (cf. (1.1)). A periodic N-tiling $\mathbf{T} = \{E_i + g\colon i = 1, \dots, N, g \in G\}$ is called φ-isoperimetric relative to a lattice G provided that for any other N-tiling $\mathbf{T}' = \{E_i' + g\colon i = 1, \dots, N, g \in G\}$ with $|E_i| = |E_i'|$, $i = 1, \dots, N$, one has

$$\mathrm{Per}_\varphi(\mathbf{T}) \le \mathrm{Per}_\varphi(\mathbf{T}'). \tag{1.2}$$

In this note, we are interested in classifying φ-isoperimetric double tilings relative to a lattice G of the plane ($N = 2$ and $d = 2$). This can be seen as the extension of the case of $N = 1$ cells, in which case the isoperimetric problem is equivalent to that of isoperimetric fundamental domains, initiated by [Cho89] and subsequently studied in the series of works [MNPR17, CN24b, NPST22, CN24a, CFN23] dealing with different types of local and non-local perimeter functionals.

Main results. We next state our main result on anisotropic isoperimetric double tilings of the plane with the so-called Manhattan anisotropy given by

$$\varphi(x, y) = \|(x, y)\|_{\ell_1} = |x| + |y|\,.$$

We shall also consider rectangular lattices G generated by the vectors $g_1 = (a, 0), g_2 = (0, b)$ for $a, b > 0$ so that $\mathrm{Area}(G) = ab$. By scaling, it will be enough to suppose that $ab = 1$. A key notion for our goals is that of the (G, ℓ_1)-isoperimetric profile function defined by

$$\mathcal{I}_{G,\ell_1}(x) := \inf \{\mathrm{Per}_{\ell_1}(\mathbf{T})\colon \mathbf{T} = \{E_i + g\colon i = 1, 2,\ g \in G\} \text{ with } |E_1| = x,\ |E_2| = 1 - x\}\,,$$

for all $x \in [0, 1]$. Observe that double tilings $\mathbf{T}$ minimizing $\mathcal{I}_{G,\ell_1}(x)$, whose existence is reported for completeness in Theorem 3.1 below, are precisely ℓ_1-isoperimetric double tilings relative to G, according to (1.2).

Our first main result is the explicit characterization of the isoperimetric profile together with the classification of its minimizers.

Theorem 1.1. *Let G be the rectangular lattice with $\mathrm{Area}(G) = 1$. Let us denote by $(a, 0), (0, b)$ the generators of G so that $ab = 1$, and let us assume that $a \ge b$. Then, it holds*

$$\mathcal{I}_{G,\ell_1}(x) = \begin{cases} 2\sqrt{x} + a + b & \text{if } x \le \frac{b^2}{4}, \\ a + 2b & \text{if } \frac{b^2}{4} < x \le \frac{1}{2}, \\ \mathcal{I}_{G,\ell_1}(1 - x), & \text{otherwise.} \end{cases}$$

In particular, $[0, 1] \ni x \mapsto (\mathcal{I}_{G,\ell_1}(x) - \mathcal{I}_{G,\ell_1}(0))^2$ is concave. Furthermore, it holds:

- o) *if $x = 0$, then $\mathcal{I}_{G,\ell_1}(0)$ is minimized by the 1-tiling generated by $[0, a] \times [0, b]$;*
- i) *if $0 < x \le \frac{b^2}{4}$, then $\mathcal{I}_{G,\ell_1}(x)$ is minimized by all the double tilings generated by an axis-aligned square and a chipped rectangle (see Figure 2 and Figure 5);*
- ii) *if $\frac{b^2}{4} < x \le \frac{1}{2}$, then $\mathcal{I}_{G,\ell_1}(x)$ is minimized by the double tilings generated by two adjacent rectangles (see Figure 3).*

Finally, the minimizers described above are the only possible minimizers, up to translations.

The above statement gives an explicit description of the (G, ℓ_1)-isoperimetric profile function; see Figure 1 for the plot of $x \mapsto \mathcal{I}_{G,\ell_1}(x)$ and its minimizers. First, we note that when the double tiling is degenerate, namely when $x \in \{0, 1\}$, then the resulting (G, ℓ_1)-isoperimetric 1-tiling relative to G is precisely the fundamental domain given by the parallelepiped spanned by the vectors $(a, 0), (0, b)$.

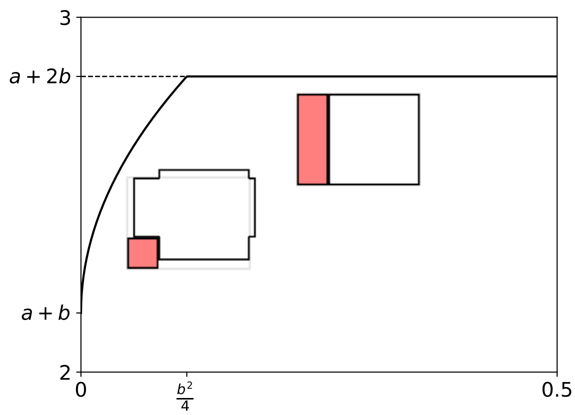

FIGURE 1. Isoperimetric profile $\mathcal{I}_{G,\ell_1}(x)$.

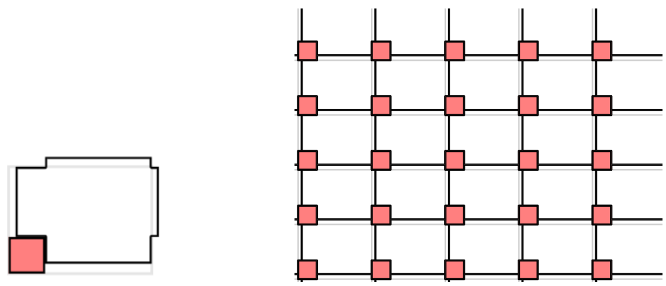

FIGURE 2. The ℓ_1-isoperimetric double tiling generated by a square and a chipped rectangle.

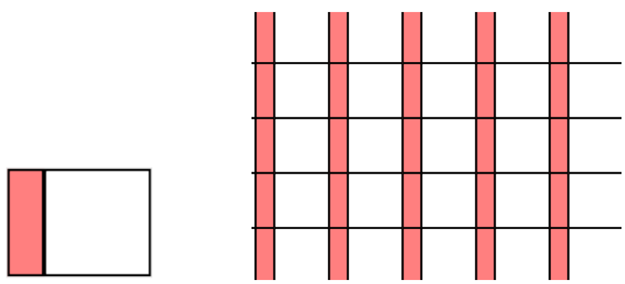

FIGURE 3. The ℓ_1-isoperimetric double tiling generated by two adjacent rectangles.

Instead, when $x \in (0,1)$, the minimizers are also completely classified, and the minimizing models are depicted in Figure 2 and Figure 3, attaining precisely the cost $\mathcal{I}_{G,\ell_1}(x)$ as $x \in (0,1)$. Finally, these models are unique up to translations, and we recall that in Case i) there are several energetically equivalent possibilities to allocate the small axis-aligned square as described in Figure 4 (see some concrete examples in Figure 5). This uniqueness conclusion is rather different from that obtained in [NNP26, Theorem 3.5] up to isometries. This is not surprising, as the ℓ_1-perimeter is not invariant under arbitrary rotations and allows extra flexibility of minimal shapes, see e.g. [MFG98].

A natural isoperimetric profile function can be obtained by further minimizing among *all* possible lattices G of the plane (not just rectangular ones). In particular, one can consider the ℓ_1-isoperimetric

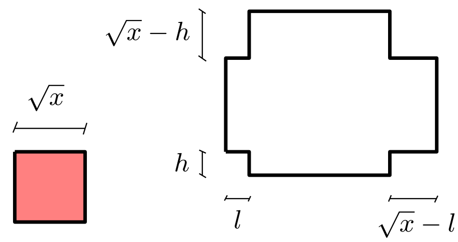

FIGURE 4. Given $x \le \frac{b^2}{4}$, any choice of E_1 (red) and E_2 (white) with $h, l \in [0, \sqrt{x}]$ generates a double tiling with perimeter equal to $\mathcal{I}_{G,\ell_1}(x)$.

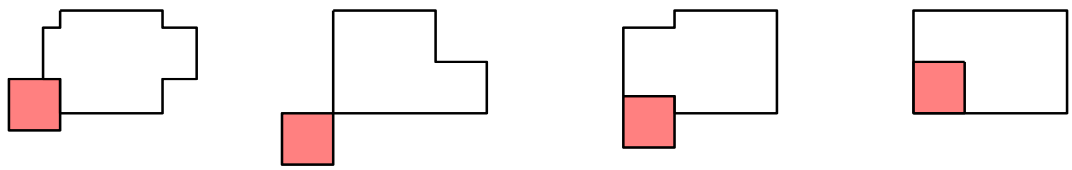

FIGURE 5. Example of generators with same total ℓ_1-perimeter.

profile function defined by

$$[0,1] \ni x \mapsto \mathcal{I}_{\ell_1}(x) := \inf\{\mathcal{I}_{G,\ell_1}(x) \colon G \text{ lattice with } \mathrm{Area}(G) = 1\}.$$

For the standard perimeter (i.e. for the choice of anisotropic norm given by $\varphi(x) = \|x\|_{\ell_2}$), the above was considered by the authors in [NNP26], and a surprisingly explicit description was achieved with three different minimizing models (cf. [NNP26, Theorem 3.5]). In the current ℓ_1-anisotropic setting, it turns out that the very same analysis is much more rigid.

Theorem 1.2. *It holds*

$$\mathcal{I}_{\ell_1}(x) = 2\sqrt{x} + 2\sqrt{1-x}, \qquad \forall x \in [0,1].$$

Furthermore, for all $x \in [0,1]$, $x \neq \frac{1}{2}$, the Pythagorean double tilings generated by two adjacent axis-aligned squares sharing one of their vertices (see Figure 6) are the only minimizers. In particular, the squares tile the plane with a lattice G generated by two orthogonal unit vectors.

Finally, if $x = \frac{1}{2}$ the only possible minimizers are composed by alternating strips parallel to the coordinate axes (see Figure 8) filled by squares with the same side length $\frac{1}{\sqrt{2}}$ that tile the plane with a lattice G generated by the two vectors $g_1 = (\frac{1}{\sqrt{2}}, 0)$, $g_2 = (h, \sqrt{2})$ or the symmetric ones $g_1' = (0, \frac{1}{\sqrt{2}})$, $g_2' = (\sqrt{2}, h)$ for any $h \in [0, \frac{1}{\sqrt{2}}]$.

The above result is obtained by a direct application of the Wulff isoperimetric inequality for the ℓ_1-perimeter. Indeed, see Theorem 2.1 and (2.2), it always holds that $\mathcal{I}_{\ell_1}(x) \ge 2\sqrt{x} + 2\sqrt{1-x}$. The uniqueness part is instead obtained by a careful use of the rigidity of the Wulff shape in the ℓ_1-isoperimetric inequality in the plane. We also refer to [GS87, MMS98] for more insights around planar tilings generated by squares.

The main results obtained in Theorem 1.1 and Theorem 1.2 are the starting point to further investigate the local minimality of the partitions induced by the ℓ_1-isoperimetric double tilings. The natural concept, see Definition 6.1, is that of a locally φ-isoperimetric partition.

Remark 1.3. We remark that in the case when G is different from the lattice of the Pythagorean tiling and $x \neq \frac{1}{2}$, the isoperimetric tilings relative to a rectangular lattice G cannot induce a locally ℓ_1-isoperimetric partition in the sense of Definition 6.1. In fact, denote by (E_n) the partition of $\mathbb{R}^2$

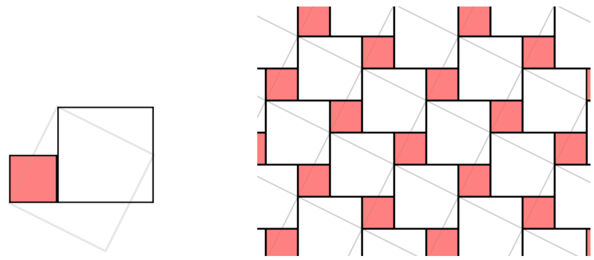

FIGURE 6. The Pythagorean double tiling.

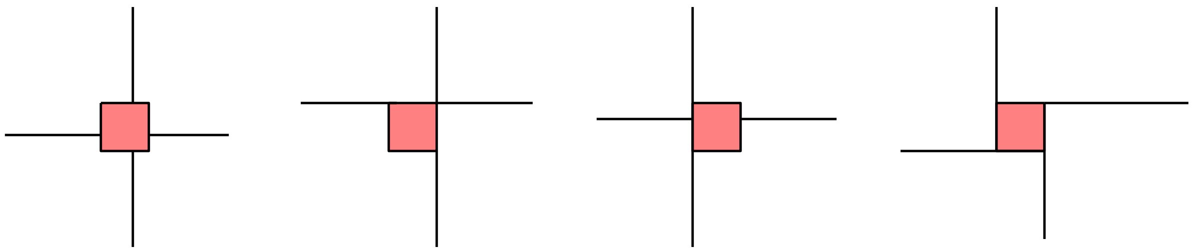

FIGURE 7. Possible partitions obtained by sending $|E_2| \uparrow \infty$ while keeping $|E_1| = 1$ on the isoperimetric double tilings: the first three partitions are obtained from Case i) in Theorem 1.1, the last one (the *vortex*) is obtained from Theorem 1.2.

induced by any of the ℓ_1-isoperimetric double tilings relative to a rectangular lattice G, given by Theorem 1.1. Given a square $Q_r = [-r, r]^2$ with $r > 0$, consider the partition (F_n^r) with $|F_n^r| = |E_n|$ so that $E_n \triangle F_n^r \Subset Q_{r+1}$ and F_n^r is a square for all $n \in \mathbb{N}$ such that $E_n \subset Q_r$ and fill the annulus $Q_{r+1} \setminus Q_r$ in a suitable way. Then, we have

$$\lim_{r\to\infty} \frac{\frac{1}{2}\sum_{n\in\mathbb{N}} \mathrm{Per}_{\ell_1}(F_n^r, Q_{r+1})}{4r^2} = 2\sqrt{x} + 2\sqrt{1-x} = \mathcal{I}_{\ell_1}(x).$$

Since, by assumption, we have $\mathcal{I}_{\ell_1}(x) < \mathcal{I}_{\ell_1,G}(x)$ we deduce that (E_n) cannot be a locally ℓ_1-isoperimetric partition.

On the other hand, the Pythagorean double tiling induces a locally ℓ_1-isoperimetric partition, because every tile must satisfy the isoperimetric inequality. ■

In light of the above remark, a possible variant is to sent one area to infinity ($|E_2| \uparrow \infty$, hence also $\mathrm{Area}(G) \uparrow \infty$) while the remaining one is kept fixed ($|E_1| = 1$).

Theorem 1.4. *Let $h, l \in [-1, 1]$, $E_1 = (h, l) + [-1, 1]^2 \subset \mathbb{R}^2$, and let E_2, E_3, E_4 be the four connected components of $\mathbb{R}^2 \setminus (E_1 \cup \{x = 0\} \cup \{y = 0\})$. Then the partition of the plane given by $\mathbf{E} = (E_1, \dots, E_5)$ is locally ℓ_1-isoperimetric (cf. Definition 6.1).*

The same property holds true if we consider the vortex *partition $\mathbf{E}$ given by $E_1 = [0, 1]^2$, and the connected components E_2, E_3, E_4, E_5 of $\mathbb{R}^2 \setminus (E_1 \cup \{y = 1, x \geq 1\} \cup \{y = 0, x \leq 0\} \cup \{x = 0, y \geq 1\} \cup \{x = 1, y \leq 0\})$ (see Figure 7).*

The result above proves that certain 5-partitions of the plane with four regions with infinite area are local minimizers of the anisotropic perimeter. This is indeed obtained by exploiting Theorem 1.1 and Theorem 1.2, and by studying their local minimality properties. Finally, we recall the recent series of works studying minimality properties of clusters with chambers of possibly infinite assigned volume, [ABV25, NPT25a, BNNS25, NPT25b, NNS26] for the isotropic perimeter, and [Ben26] in the anisotropic case.

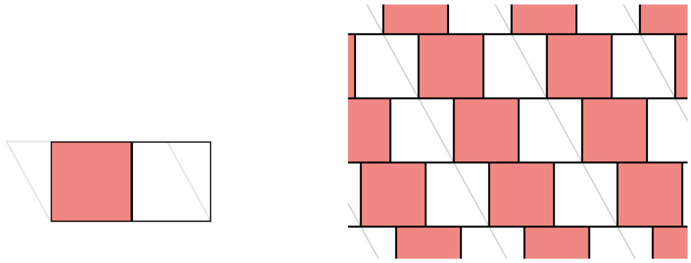

FIGURE 8. A possible ℓ_1-isoperimetric double tiling composed of two cells of equal area and with the same perimeter of the Pythagorean double tiling.

2. Preliminaries

In this section, we recall some useful facts around the classical and anisotropic perimeters.

2.1. **Sets of finite perimeter.** We shall list some useful facts around sets of finite perimeter, referring to [AFP00] for a detailed introduction. For every $E \subseteq \mathbb{R}^d$ Borel and $\Omega \subseteq \mathbb{R}^d$, we recall that E is said a set of finite perimeter in Ω if

$$\mathrm{Per}(E,\Omega) \coloneqq \sup\left\{\int_E \mathrm{div}(X)\,\mathrm{d}\mathcal{L}^d \colon X \in C_c^\infty(\Omega,\mathbb{R}^d), \|X\|_{L^\infty} \leq 1\right\} < +\infty.$$

The value $\mathrm{Per}(E,\Omega)$ is called the relative perimeter of E in Ω, and it extends to a locally finite nonnegative measure $\mathrm{Per}(E,\cdot)$ on the whole Borel σ-algebra. This will be called perimeter measure of E and, when $\Omega = \mathbb{R}^d$, we shall simply write $\mathrm{Per}(E) \coloneqq \mathrm{Per}(E,\mathbb{R}^d)$. We then say that E is a set of locally finite perimeter, provided $\mathrm{Per}(E,\Omega) < \infty$ is finite for all $\Omega \subset \mathbb{R}^d$ bounded open sets. In this case, there is a vector valued measure, denoted $D\chi_E$, that satisfies $|D\chi_E| = \mathrm{Per}(E,\cdot)$ and

$$D\chi_E = \nu_E \mathcal{H}^{d-1} \llcorner \partial^* E,$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, and $\partial^* E$ is the reduced boundary of E, defined by $\partial^* E \coloneqq \{x \in \mathbb{R}^d \colon m(E,x) > 0, m(E^c,x) > 0\}$, with $m(E,x) \coloneqq \lim_{r\to 0} \frac{\mathcal{L}^d(B_r(x)\cap E)}{\mathcal{L}^d(B_r(x))}$, and $\nu_E(x)$ is the outer unit normal of the set E, which is $\mathcal{H}^{d-1}$-a.e. defined on $\partial^* E$.

2.2. **Anisotropic surface energy.** In this note, we shall consider anisotropic variants of the standard Euclidean perimeter. We recall first some basic facts around anisotropic norms and the Wulff problem, referring to [Mag12, Chapter 20] for a detailed discussion.

Given a norm $\varphi\colon \mathbb{R}^d \to [0,\infty)$, we define the anisotropic surface φ-energy of a set $E \subset \mathbb{R}^d$ relative to an open set $\Omega \subset \mathbb{R}^d$ by

$$\mathrm{Per}_\varphi(E,\Omega) \coloneqq \int_{\partial^* E\cap\Omega} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1}(x) \in [0,\infty],$$

where the above is canonically set to $+\infty$ if E is not a set of locally finite perimeter. As before, we simply write $\mathrm{Per}_\varphi(E) \coloneqq \mathrm{Per}_\varphi(E,\mathbb{R}^d)$. The Wulff problem is the following variant of the isoperimetric problem

$$\inf\{\mathrm{Per}_\varphi(E)\colon E \subset \mathbb{R}^d, |E| = m\}, \qquad \forall m > 0. \tag{2.1}$$

and the minimizers are called Wulff shape and are completely characterized by the next theorem (see, e.g. [Mag12, Theorem 20.8 and Eq. (20.14)]). Below, φ^* is the dual norm defined by $\varphi^*(v) \coloneqq \sup\{v\cdot x\colon \varphi(x) < 1\}$ for all $v \in \mathbb{R}^d$.

Theorem 2.1. *For all $m > 0$ there is $r > 0$ such that $m = r^d|W_\varphi|$, where $W_\varphi \subset \mathbb{R}^d$ is the Wulff shape given by*

$$W_\varphi := \{v \in \mathbb{R}^d \colon \varphi^*(v) < 1\}.$$

Then, for every $x_0 \in \mathbb{R}^d$, the set $x_0 + rW_\varphi$ is a minimizer of (2.1)*. In particular, for every $E \subset \mathbb{R}^d$ Borel, the Wulff inequality holds*

$$\mathrm{Per}_\varphi(E) \geq n|W_\varphi|^{\frac{1}{d}}\,|E|^{\frac{d-1}{d}}.$$

We also note, see [BM94, FMP10], that the Wulff shape is also the unique minimizer up to translation (see also [FMP10, FET05, Neu16] for quantitative improvements). The above result, in the planar case with the Manhattan anisotropy $\varphi(x) = \|x\|_{\ell_1}, \varphi^*(v) = \|v\|_{\ell_\infty}$ reads as follows. Since $W_{\ell_1} \subset \mathbb{R}^2$ is a square with area $|W_\varphi| = 4$, Theorem 2.1 gives

$$\mathrm{Per}_{\ell_1}(E) \geq 4\sqrt{|E|}, \qquad \forall E \subset \mathbb{R}^2, \tag{2.2}$$

and equality occurs if and only if E is equivalent to a square of edge-length $\sqrt{|E|}$ and aligned with the coordinate axes, up to translations.

3. Existence and regularity

In this section, we shall discuss basic existence and topological regularity properties of anisotropic isoperimetric tilings, with special attention to the planar setting.

3.1. **Existence of anisotropic isoperimetric tilings.** The following existence result was obtained in [NN24], but a short proof is included for completeness with emphasis on a certain construction in the planar setting. We recall that an N- tiling $\mathbf{T} = \{E_i + g \colon i = 1, \dots, N,\ g \in G\}$ is said locally finite if for all $i = 1, \dots, N$ and for all compact set $C \subset \mathbb{R}^d$ it holds

$$N_i := \#G_i < \infty, \qquad \text{where } G_i := \{g \in G \colon |(E_i + g) \cap C| > 0\} \tag{3.1}$$

Let us also define the packing and covering radius of G, respectively given by

$$\rho_G := \sup\{r > 0 \colon B_r(g_1) \cap B_r(g_2) = \emptyset \text{ for all } g_1 \neq g_2 \in G\};$$
$$r_G := \inf\{r > 0 \colon \cup_g \left(B_r(0) + g\right) = \mathbb{R}^d\}.$$

Theorem 3.1. *Let $N \in \mathbb{N}, d \geq 2$ and let G be a lattice in $\mathbb{R}^d$. Let $v = (v_1, \dots, v_N)$ be so that $\sum_{i=1}^N v_i = |G|$. Let $\varphi \colon \mathbb{R}^d \to [0, \infty)$ be a norm. Then, there exists a minimizer of*

$$\inf\{\mathrm{Per}_\varphi(\mathbf{T}) \colon \mathbf{T} = \{E_i + g \colon i = 1, \dots, N,\ g \in G\} \textit{ with } |E_i| = v_i\}.$$

In particular, the minimizer is a φ-isoperimetric N-Tiling relative to G. Finally, if $d = 2$, there is a minimizing tiling such that each generator is essentially bounded (in particular, it is locally finite).

Proof. In [NN24, Theorem 3.2] (with the choice $\mu = 0$), it was shown that there exists a minimizer of the following variational problem:

$$\inf_D \inf \left\{ \frac{1}{2}\sum_{i=1}^N \mathrm{Per}(E_i) \colon \begin{array}{ll} E_i \subseteq D \text{ Borel}, & |E_i| = v_i, \quad i = 1, ..., N \\ |E_i \cap E_l| = 0, & \left|D \setminus \cup_i E_i\right| = 0, \end{array} \right\}, \tag{3.2}$$

where the first infimum is taken among all fundamental domains D of G, and the second infimum is taken among all Borel partitions (E_i) of D with assigned areas $v_1, .., v_N$. Evidently, $D, (E_i)$ minimizes the above if and only if $\mathbf{T} := \{E_i + g \colon i = 1, \dots, N, g \in G\}$ is an ℓ_1-isoperimetric N-tiling relative to G with prescribed volumes $v_1, \dots, v_N$.

Finally, if $d = 2$, the last conclusion is implied by [NN24, Proposition 5.1]. Since we shall later need the precise construction, we report it briefly here. Let us consider the fundamental domain $D := \cup_i E_i$

and its decomposition into indecomposable sets (D_l) satisfying $|D| = \sum_l |D_l|, \mathrm{Per}_\varphi(D) = \sum_l \mathrm{Per}_\varphi(D_l)$ ([ACMM01]). For all $l \in \mathbb{N}$, consider $g_l \in G$ so that $(D_l - g_l) \cap B_{r_G}(0) \neq \emptyset$ (here $r_G > 0$ is the covering radius of the lattice G) and define the sets

$$E_i' := \cup_l \big((E_i \cap D_l) - g_l\big), \qquad \tilde{D}_l := D_l - g_l, \quad \forall i, l. \tag{3.3}$$

By construction, $D' := \cup_l \tilde{D}_l$ is also a fundamental domain for G and (E_i') is a partition up to negligible sets of D'. By construction, it also holds

$$\sum_i \mathrm{Per}_\varphi(E_i') = \sum_i \mathrm{Per}_\varphi(E_i).$$

and in particular $\mathbf{T}' := \{E_i' + g \colon i = 1, \dots, N,\ g \in G\}$ is a φ-isoperimetric N-tiling relative to G satisfying $|E_i'| = v_i$ for all $i = 1, \dots, N$. Moreover, we have

$$\mathrm{diam}(E_i') \leq \mathrm{diam}(\tilde{D}) \leq 2r_G + \sum_l \mathrm{diam}(D_l) \leq 2r_G + c \sum_l \mathrm{Per}_\varphi(D_l) \leq 2r_G + c' \, \mathrm{Per}_\varphi(\mathbf{T}) < \infty,$$

for some dimensional constants c, c', having used the diameter to perimeter estimates for indecomposable planar sets of finite perimeter (cf. [DMNP22, Lemma 2.13]). Therefore, the partition $(E_i' + g)_{i, g \in G}$ is locally finite, and the generators E_i' are essentially bounded, by construction. □

3.2. **Topological regularity of anisotropic tilings.** In this part, we establish basic density estimates and topological regularity properties of anisotropic tilings.

We first start with a definition of local minimality for partitions, and then show that it is satisfied by an isoperimetric tiling.

Definition 3.2. *Let φ be a norm in $\mathbb{R}^d$, let (E_n) be a Borel partition of an open set $\Omega \subset \mathbb{R}^d$. We say that (E_n) is a volume constrained minimizer of the φ-perimeter relative to Ω, provided for every B open bounded subset of Ω and every partition (F_n) such that*

$$|E_n \cap \Omega| = |F_n \cap \Omega|, \qquad E_n \triangle F_n \Subset B, \qquad \forall n \in \mathbb{N},$$

it holds

$$\sum_{n \in \mathbb{N}} \mathrm{Per}_\varphi(E_n, B) \leq \sum_{n \in \mathbb{N}} \mathrm{Per}_\varphi(F_n, B).$$

Lemma 3.3. *Let φ be a norm in $\mathbb{R}^d$, let G be a lattice, and let $\mathbf{T} := \{E_i + g \colon g \in G, i = 1, \dots, N\}$ be a φ-isoperimetric N-tiling relative to a lattice G. Let $\Omega \subset \mathbb{R}^d$ be an open set such that $\overline{\Omega} \cap (\overline{\Omega} + g) = \emptyset$ for all $g \in G \setminus \{0\}$ (for instance, $\Omega = B_\rho(x)$ for all $x \in \mathbb{R}^d, \rho < \rho_G$). Then, the partition $(E_i + g)_{i,g}$ of $\mathbb{R}^d$ is a volume constrained local minimizer of the φ-perimeter relative to Ω.*

Proof. Consider a fundamental domain $D \supset \Omega$ with the property $\mathrm{Per}_\varphi(E_i, \partial D) = 0$ for all $i = 1, \dots, N$. This is possible by assumptions on Ω, and up to an arbitrary small translation of D. Let (g_l) for $l \in \mathbb{N}$ be an enumeration of G and let us denote for brevity $E_{i,l} := E_i + g_l$ for every $l \in \mathbb{N}, i$.

Let $(F_{i,l})$ be another partition of $\mathbb{R}^d$ with the property that $E_{i,l} \triangle F_{i,l} \Subset B$ and $|E_{i,l} \cap \Omega| = |F_{i,l} \cap \Omega|$ for all i, l. Set

$$F_i := \left(E_i \setminus \bigcup_l (\Omega - g_l) \right) \cup \left(\bigcup_l (F_{i,l} - g_l) \right).$$

We note that $\tilde{D} := \cup_i F_i$ is a fundamental domain for G, using that D is a fundamental domain and that $(D + g_l) \triangle (\cup_i F_{i,l}) \Subset \Omega$ by assumptions. Define then the N-tiling

$$\mathbf{T}' := \{F_i + g \colon g \in G, i = 1, \dots, N\},$$

that is periodic relative to G and satisfies by construction $|F_i| = |E_i|$ for all $i = 1, \dots, N$. Using at first that $\mathbf{T}$ is φ-isoperimetric relative to G, we deduce

$$\begin{aligned} 0 \le \mathrm{Per}_\varphi(\mathbf{T}') - \mathrm{Per}_\varphi(\mathbf{T}) &= \frac{1}{2}\sum_i \mathrm{Per}(F_i) - \frac{1}{2}\sum_i \mathrm{Per}(E_i) \\ &\le \sum_i \mathrm{Per}_\varphi\left(F_i, \cup_l(\Omega - g_l)\right) - \sum_i \mathrm{Per}_\varphi\left(E_i, \cup_l(\Omega - g_l)\right) \\ &\le \sum_{i,l} \mathrm{Per}_\varphi(F_{i,l}, \Omega) - \sum_{i,l} \mathrm{Per}_\varphi(E_{i,l}, \Omega) \end{aligned}$$

having used the subadditivity of the perimeter on disjoint sets, its translation invariance, and the choice of D. This concludes the proof. □

Regularity theory is by now well understood in the case of sets and clusters, for the classical isotropic perimeter. Many basic regularity results, however, also hold true for the φ-anisotropic perimeter for any norm φ (without any regularity assumptions). We refer to the general references [Alm76, MFG98, Mag12, FPS23] for further details on the regularity of isoperimetric clusters. The following result establishes a basic topological regularity result for partitions minimizing the anisotropic perimeter, with an omitted proof as it would not bring any novelty. The idea is the same in all cases: first, a volume fixing result must be established (Almgren's lemma for partitions), this will trade the volume fixing constraint with a quasi-minimality property (see [BCT25, Appendix A] for a similar setting), then the co-area formula will produce a differential inequality to obtain a suitable decay of the densities on small balls. The constants might depend on the partition, and on a prescribed distance from the boundary of the open domain, but are uniform on any such point.

Theorem 3.4 (density estimates for partitions)**.** *Let $\varphi\colon \mathbb{R}^d \to [0, +\infty)$ be a norm. Let $\Omega \subset \mathbb{R}^d$ be an open and connected set. Let $(E_1, \dots, E_M)$ be a volume constrained minimizer of the φ-perimeter relative to Ω. Then, for every $\varepsilon > 0$ there exists $c > 0$, $\rho_0 > 0$ such that if $x \in \Omega$, $\mathrm{dist}(x, \Omega^c) > \varepsilon$ it holds*

$$|E_n \cap B_\rho(x)| \ge c\rho^d, \qquad \mathrm{Per}_\varphi(E_n, B_\rho(x)) \ge c\rho^{d-1}, \tag{3.4}$$

for all $\rho < \rho_0$ and $n = 1, \dots, M$.

Leveraging on the density estimates for partitions minimizing the φ-perimeter, we will establish a crucial regularity property for anisotropic isoperimetric tilings that are locally finite (cf. (3.1)).

Theorem 3.5 (regularity of isoperimetric tilings)**.** *Let $N \in \mathbb{N}$, let φ be a norm in $\mathbb{R}^d$, let G be a lattice, and let $\mathbf{T} \coloneqq \{E_i + g \colon g \in G, i = 1, \dots, N\}$ be a φ-isoperimetric N-tiling relative to the lattice G. Assume that $(E_i + g)_{i,g}$ is a locally finite partition. Then there are $c, \rho_0 > 0$ such that for all $x \in \mathbb{R}^d$ and $i = 1, \dots, N$, it holds*

$$|E_i \cap B_\rho(x)| > c\rho^d, \qquad \mathrm{Per}_\varphi(E_i, B_\rho(x)) > c\rho^{d-1}, \qquad \forall \rho \le \rho_0.$$

In particular, there exists a representative $\tilde{E}_i$ with $|E_i \triangle \tilde{E}_i| = 0$ and $\mathcal{H}^{d-1}(\partial \tilde{E}_i \setminus \partial^* \tilde{E}_i) = 0$.

Proof. Let $D \subset \mathbb{R}^d$ be bounded fundamental domain for the lattice G, take $r < \rho_G$ and consider $x_1, \dots, x_n \in \mathbb{R}^d$ such that the family of balls $(B_{r/2}(x_k))$ is an open covering for the compact set $\bar{D}$. For $k = 1, \dots, n$, we apply Lemma 3.3 and Theorem 3.4 (with $\varepsilon = r/4$) with the partition obtained by restricting $(E_i + g)_{i, g \in G}$ to the ball $\Omega = B_r(x_k)$. Observe that, since the N-tiling $\mathbf{T}$ is assumed to induce a locally finite partition, this restriction yields each time a *finite* partition of Ω.

Consider the constants $c > 0$, $\rho_0 > 0$ given by each application of Theorem 3.4 as $k = 1, \dots, n$. By selecting the worst possible constants, we can assume that c and ρ_0 do not depend on k. For any

$x \in \mathbb{R}^d$, there is $g \in G$ such that $x + g \in D$ and there is k such that $x + g \in B_{r/2}(x_k)$. In particular, for every $\rho < r/4$, it holds that $B_\rho(x + g) \subset B_r(x_k)$ and $\operatorname{dist}\big(x + g, \mathbb{R}^d \setminus B_r(x_k)\big) > \varepsilon$. Hence the conclusion of Theorem 3.4 yields the desired measure and perimeter density estimates for each E_i in $B_\rho(x + g)$. The same estimates therefore hold at the point x, by G translation invariance. Finally, the last conclusion holds as a byproduct of the density estimates. □

As a byproduct of the above, we can obtain two crucial results for planar anisotropic isoperimetric tilings. These will be essential to prove our main result.

Corollary 3.6. *Let $N \in \mathbb{N}$ and let G be a lattice in the plane $\mathbb{R}^2$. Let $v = (v_1, \ldots, v_N)$ be so that $\sum_{i=1}^N v_i = |G|$. Let $\varphi\colon \mathbb{R}^d \to [0, \infty)$ be a norm. Then, there exists a φ-isoperimetric N-tiling $\mathbf{T} \coloneqq \{E_i + g\colon g \in G, i = 1, \ldots, N\}$ relative to a lattice G satisfying $|E_i| = v_i$, and each generator E_i admits an open and bounded representative $\tilde{E}_i$ with $|E_i \triangle \tilde{E}_i| = 0$ and $\mathcal{H}^1(\partial \tilde{E}_i \setminus \partial^* \tilde{E}_i) = 0$.*

Proof. By the last conclusion of Theorem 3.1, we know that there is a minimizing tiling $\mathbf{T} \coloneqq \{E_i + g\colon g \in G, i = 1, \ldots, N\}$ inducing a locally finite partition and with E_i essentially bounded. Therefore, the regularity result in Theorem 3.5 applies, giving in turn the existence of open and bounded representatives $\tilde{E}_i$ satisfying the desired properties. □

Lemma 3.7. *Let φ be a norm in the plane $\mathbb{R}^2$ and let $\mathbf{T} \coloneqq \{E_i + g\colon g \in G, i = 1, 2\}$ be a φ-isoperimetric double tiling relative to a lattice G. Assume that E_1, E_2 are open and bounded generators of $\mathbf{T}$ such that $\mathcal{H}^1(\partial E_i \setminus \partial^* E_i) = 0$, $i = 1, 2$. If $\mathbb{R}^d \setminus (\bar{E}_1 + G)$ is connected, then E_2 is connected.*

Proof. Suppose, by contradiction, that E_2 can be decomposed as the union of two open disjoint sets E_2', E_2''. Consider a point $x' \in E_2'$ and a point $x'' \in E_2''$. By the assumptions on E_1 we know that there exists a curve contained in $U = \mathbb{R}^d \setminus (\bar{E}_1 + G)$ connecting x' to x''. By continuity there is a point x, along the curve, such that $x \in (\partial E_2' \cap \partial(E_2'' + g)) \setminus \bar{E}_1$ for some $g \in G$. By perturbing the curve, we can also suppose that $x \in (\partial^* E_2' \cap \partial^*(E_2'' + g)) \setminus \bar{E}_1$. Consider $\rho > 0$ such that $B_\rho(x) \cap \bar{E}_1 = \emptyset$. By Theorem 3.5 we know that $\mathcal{H}^1(\partial E_2' \cap \partial(E_2'' + g) \cap B_\rho(x)) > 0$.

We can hence define $F_1 \coloneqq E_1, F_2 \coloneqq E_2' \cup (E_2'' + g)$ and notice that $\mathbf{T}' \coloneqq \{F_i + g\colon g \in G, i = 1, 2\}$ is another double tiling relative to G with the same prescribed areas. However we have

$$\operatorname{Per}_\varphi(F_2) \leq \operatorname{Per}_\varphi(E_2) - \mathcal{H}^1(\partial^* E_2' \cap \partial^*(E_2'' + g)) < \operatorname{Per}_\varphi(E_2),$$

contradicting the minimality of $\mathbf{T}$. This concludes the proof. □

4. Estimates of the ℓ_1-perimeter of double tilings

In this section, we develop key properties for the ℓ_1-perimeter of double tilings.

4.1. Preliminary lemmas.

Since, in this work, we are interested in planar sets, it will be convenient to decompose the ℓ_1-perimeter of a set $E \subset \mathbb{R}^2$ of finite perimeter into a horizontal and vertical contribution. To this aim, we set

$$P_V(E) \coloneqq \int_{\partial^* E} |\nu_E \cdot e_1| \, d\mathcal{H}^1, \qquad P_H(E) \coloneqq \int_{\partial^* E} |\nu_E \cdot e_2| \, d\mathcal{H}^1,$$

so that we have can rewrite $\operatorname{Per}_{\ell_1}(E) = P_H(E) + P_V(E)$. A first important observation is the following simple lemma, where we denote by π_1, π_2 the projections respectively on the x, y axes.

Lemma 4.1. *Let $E \subset \mathbb{R}^2$ be a set of finite perimeter. Then we have*

$$P_H(E) \geq 2\mathcal{H}^1(\pi_1(\partial^* E)), \qquad P_V(E) \geq 2\mathcal{H}^1(\pi_2(\partial^* E)). \tag{4.1}$$

Finally, it also holds

$$|E| \le \frac{1}{4} P_H(E) \cdot P_V(E).$$

Proof. We subdivide the proof into different steps.
STEP 1. Let us consider $(E_i)_{i\in I}$ the family of indecomposable components of E, namely the collection of Borel subsets $E_i \subset E$ satisfying

$$|E| = \sum_{i\in I} |E_i|, \qquad \mathrm{Per}_\varphi(E) = \sum_{i\in I} \mathrm{Per}_\varphi(E_i).$$

See [ACMM01] for the existence of such a decomposition. By the perimeter-diameter estimates for planar indecomposable sets of finite perimeter (see, e.g. [DMNP22, Lemma 2.13]), we also deduce that E_i is (essentially) bounded and, up to change of representative, we can assume that E_i's are bounded sets. For all $i \in I$, we can define $\tilde{E}_i = \mathrm{sat}(E_i)$, where the saturation $\mathrm{sat}(E)$ of a set E is the union of all the bounded indecomposable components of E^c (i.e. the union of E and its *holes*). Since E_i is indecomposable, then $\tilde{E}_i$ is simple. Hence, by [ACMM01], there exists a Jordan curve γ_i such that $\partial^* \tilde{E}_i = \Gamma_i := \mathrm{Im}(\gamma_i)$ (modulo $\mathcal{H}^1$-a.e. equality).
STEP 2. We show the first in (4.1), the second being analogous. By the previous discussion, we can estimate

$$P_H(E) = \sum_{i\in I} P_H(E_i) \ge \sum_{i\in I} P_H(\tilde{E}_i).$$

Since γ_i is a Lipschitz loop spanning $\partial \tilde{E}_i$ modulo $\mathcal{H}^1$-a.e., we deduce

$$P_H(\tilde{E}_i) \ge 2\mathcal{H}^1(\pi_1(\Gamma_i)), \qquad \forall i \in I.$$

The sought estimate follows, by sub-additivity and noticing

$$\sum_{i\in I} \mathcal{H}^1(\pi_1(\Gamma_i)) \ge \mathcal{H}^1(\cup_{i\in I}\pi_1(\Gamma_i)) = \mathcal{H}^1(\pi_1(\cup_{i\in I}\Gamma_i)) = \mathcal{H}^1(\pi_1(\cup_i \partial^* \tilde{E}_i)) = \mathcal{H}^1(\pi_1(\partial^* E)),$$

having used, lastly, that the *holes* in each E_i do not contribute to the total length of the projection.
STEP 3. For all $i \in I$, we can then denote R_i the smallest rectangle containing E_i aligned with the coordinate axes, and we infer, by the previous step, that

$$|E_i| \le |R_i|, \qquad P_H(E_i) \ge P_H(R_i), \qquad P_V(E_i) \ge P_V(R_i).$$

Indeed, the first is obvious while the claimed estimates on the perimeter are a direct consequence of (4.1). Therefore, we conclude the proof observing

$$\begin{aligned} |E| = \sum_{i\in I} |E_i| \le \sum_{i\in I} |R_i| &= \frac{1}{4}\sum_{i\in I} P_H(R_i)P_V(R_i) \le \frac{1}{4}\sum_{i\in I} P_H(E_i)P_V(E_i) \\ &\le \frac{1}{4}\sum_{i\in I} P_H(E_i)\sum_{i\in I} P_V(E_i) = \frac{1}{4} P_H(E) \cdot P_V(E). \end{aligned}$$

□

We next show a result aiming at characterizing rectangular sets. This will be needed to deduce the uniqueness part of our main results.

Lemma 4.2. *Let $h, b > 0$ and suppose that $E \subset \mathbb{R}^2$ is an open set of finite perimeter with $\mathcal{H}^1(\partial E \setminus \partial^* E) = 0$, $|E| = hb$, $P_H(E) = 2h$, $P_V(E) = 2b$. If $\bar{E} \supset \{0\} \times [0, b]$ then a horizontal translation of $\bar{E}$ is $[0, h] \times [0, b]$.*

Proof. Consider the Borel function $u(t) := \mathcal{H}^1(E \cap \{y = t\})$. By (4.1), we have

$$2u(t) = 2\mathcal{H}^1(\pi_1(E \cap \{y = t\})) \leq 2\mathcal{H}^1(\pi_1(E)) \leq P_H(E) = 2h.$$

On the other hand, we have

$$hb = |E| = \int_{-\infty}^{+\infty} u(y)\, dy = \int_0^b u(y)\, dy,$$

where, the last equality follows from $2b = P_V(E) = 2\mathcal{H}^1(\pi_2(E)) \geq 2\mathcal{H}^1(\pi_2(\{0\} \times [0, b])) = 2b$, which guarantees that $u(y) = 0$ when $y \notin [0, b]$. Putting together we deduce $u(y) = h$ for a.e. $y \in [0, b]$. Since $\mathcal{H}^1(\pi_1(E)) = h$ while $\mathcal{H}^1(E \cap \{y = t\}) = h$ for all $t \in [0, b]$ we conclude that $\bar{E}$ is actually a rectangle. □

4.2. **Estimates for the ℓ_1-perimeter.** In this part, we produce effective estimates on 2-tilings that are periodic with respect to rectangular lattices. We shall always assume that $\mathbf{T}$ is a tiling generated by open sets E satisfying $\mathcal{H}^1(\partial E_i \setminus \partial^* E_i) = 0$. As the following lemmas will be invoked on the ℓ_1-isoperimetric tilings given by Corollary 3.6, this topological regularity will be satisfied.

We first show a useful lower bound on the vertical and the horizontal anisotropic perimeter of the generators.

Lemma 4.3. *Let $\mathbf{T}$ be a 2-tiling relative to a rectangular lattice G in $\mathbb{R}^2$ generated by two vectors $(a, 0)$, $(0, b)$, $a, b > 0$. Let E_1, E_2 be two generators of $\mathbf{T}$ and suppose they are open set with $\mathcal{H}^1(\partial E_i \setminus \partial^* E_i) = 0$. Then, it holds*

$$P_H(E_1) + P_H(E_2) \geq 2a, \qquad P_V(E_1) + P_V(E_2) \geq 2b.$$

Proof. Since a line cannot be contained in the interior of $(E_1 \cup E_2) + G$ (the union of the translations of the interiors of the tiles), every line must touch $\partial\mathbf{T} := \cup_{i=1}^N \partial^* E_i$. This means that given any vertical line v there exists $g \in G$ such that $(v + g) \cap (\partial E_1 \cup \partial E_2) \neq \emptyset$. By taking the lines $\{x = t\}$ with $t \in [0, a]$ we notice that the projection of $(\partial E_1 \cup \partial E_2) + a\mathbb{Z}$ on the x-axis, covers the whole segment $[0, a]$. Hence, by (4.1), we have

$$P_H(E_1) + P_H(E_2) \geq 2(\mathcal{H}^1(\pi_1(\partial E_1)) + \mathcal{H}^1(\pi_2(\partial E_2))) \geq 2a.$$

The second conclusion can be obtained similarly. □

Next, we show a lower bound on the ℓ_1-perimeter of a double tiling when one generator is touched by every horizontal and vertical line.

Proposition 4.4. *Let G be the rectangular lattice in $\mathbb{R}^2$ generated by two vectors $(a, 0)$, $(0, b)$, $a, b > 0$. Suppose that $\mathbf{T}$ is a 2-tiling relative to G generated by two open sets E_1, E_2 with $\mathcal{H}^1(\partial E_i \setminus \partial^* E_i) = 0$ for $i = 1, 2$. Suppose that every horizontal line and every vertical line touches $\bar{E}_2 + G$. Then*

$$\mathrm{Per}_{\ell_1}(\mathbf{T}) \geq a + b + 2\sqrt{|E_1|}. \tag{4.2}$$

Moreover, if equality holds in (4.2) *then $\sqrt{|E_1|} < \min\{a, b\}$ and* T *is a double tiling generated by a square of area $|E_1|$ and chipped rectangle obtained from a translation of $[0, a] \times [0, b]$.*

Proof. By the assumptions on E_2, we have $\mathcal{H}^1(\pi_1(\partial^* E_2)) = a$, $\mathcal{H}^1(\pi_2(\partial^* E_2)) = b$. Therefore, (4.1) yields

$$\int_{\partial^* E} |\nu_E \cdot e_2|\, \mathrm{d}\mathcal{H}^1 \geq a, \qquad \int_{\partial^* E} |\nu_E \cdot e_1|\, \mathrm{d}\mathcal{H}^1 \geq b, \tag{4.3}$$

whence

$$\mathrm{Per}_{\ell_1}(E_2) \geq 2a + 2b.$$

On the other hand, by the ℓ_1-isoperimetric inequality (2.2), we have $\mathrm{Per}_\varphi(E_1) \geq 4\sqrt{|E_1|}$, with equality holding if and only if E_1 is equivalent to the Wulff shape, namely a square with edges of length $\sqrt{|E_1|}$ and parallel to the coordinate axes. The first conclusion is therefore proven.

Suppose now that equality holds in (4.2). Let $I = \pi_1(E_2)$, $J = \pi_2(E_2)$ so that $\mathcal{H}^1(I) = a$, $\mathcal{H}^1(J) = b$ while $\mathrm{Per}_\varphi(E_2) = 2a + 2b$. Since E_2 is connected (by Lemma 3.7) both I and J must be connected too. Hence E_2 is contained in $F := (I \times J) \setminus (E_1 + G)$ which is a set of area $ab - |E_1|$. Since $ab - |E_1| = |E_2|$ we conclude that $E_2 = F$ (up to a negligible set).

Without loss of generality suppose now that $E_1 = (0, \ell)^2$ and let $I = (\alpha, \alpha + a)$, $J = (\beta, \beta + b)$, with $0 \leq \alpha < a$ and $0 \leq \beta < b$, $\ell := \sqrt{|E_1|}$. We now claim that $\alpha \leq \ell$ and $\beta \leq \ell$. In fact if $\alpha > \ell$ we would have $\mathrm{Per}_\varphi(E_2) \geq 2a + 2b + 3\ell$ because at least three sides of the square E_1 would be *inside* E_2. An analogous argument shows that $\beta > \ell$ cannot occur either, concluding thus the proof. □

Finally, we show a last bound on the ℓ_1-perimeter of a double tiling in the situation in which the two generators are touched by two (possibly different) vertical lines.

Lemma 4.5. *Let G be the rectangular lattice in $\mathbb{R}^2$ generated by two vectors $(a, 0)$, $(0, b)$, $a, b > 0$. Suppose that $\mathbf{T}$ is an isoperimetric 2-tiling relative to G generated by two open sets E_1, E_2 with $\mathcal{H}^1(\partial E_i \setminus \partial^* E_i) = 0$ for $i = 1, 2$. Suppose there exists a vertical line contained in $\bar{E}_1 + G$ and another vertical line contained in $\bar{E}_2 + G$. Then*

$$\mathrm{Per}_{\ell_1}(\mathbf{T}) \geq a + 2b. \tag{4.4}$$

Moreover, if equality holds in (4.4)*, then the double tiling $\mathbf{T}$ is generated by two adjacent rectangles of area respectively $|E_1|, ab - |E_2|$ whose union has closure equal to $[0, a] \times [0, b]$.*

Proof. By Lemma 4.3 we have

$$P_H(E_1) + P_H(E_2) \geq 2a.$$

On the other hand, since every horizontal line crosses both vertical lines and hence crosses both sets E_1 and E_2, we have, using (4.1),

$$P_V(E_1) \geq 2b, \qquad P_V(E_2) \geq 2b.$$

Hence

$$\mathrm{Per}_\varphi(\mathbf{T}) = \frac{P_H(E_1) + P_H(E_2) + P_V(E_1) + P_V(E_2)}{2} \geq a + 2b.$$

Suppose now that equality holds. This means that

$$P_V(E_1) = P_V(E_2) = 2b$$

and

$$P_H(E_1) = 2h, \qquad P_H(E_2) = 2(a - h)$$

for some $h \in [0, a]$. We also note, by Lemma 4.1, that

$$ab = |E| = |E_1| + |E_2| \leq \frac{1}{4}(2h)(2b) + \frac{1}{4}(2a - 2h)(2b) = ab,$$

hence, equality must occur in the above inequality, and thus $|E_1| = hb, |E_2| = (a - h)b$. Finally, we can invoke Lemma 4.2 twice to deduce that E_1, E_2 are rectangles up to translations. This concludes the proof of the equality case. □

5. Isoperimetric profiles

In this section, we prove our main results around the isoperimetric profile functions.

5.1. **The case $\mathcal{I}_{G,\ell_1}(x)$.** Here we show the first main result, namely Theorem 1.1. To this aim, we carry out, in the next theorem, the main analysis.

Theorem 5.1. *Let G be the rectangular lattice generated by two vectors $(a, 0)$, $(0, b)$, $a, b > 0$, and let us denote $D = [0, a] \times [0, b]$. Suppose that $\mathbf{T}$ is a ℓ_1-isoperimetric double tiling relative to G generated by E_1, E_2. Then, we have*

$$\mathrm{Per}_\varphi(\mathbf{T}) \geq \min\left\{a + 2b, b + 2a, a + b + 2\sqrt{|E_1|}, a + b + 2\sqrt{|E_2|}\right\}. \tag{5.1}$$

Finally, suppose that equality holds for some $\mathbf{T}$. Then, up to translations, one of the following occurs:

- o) $\mathbf{T}$ *is degenerate (either $|E_1| = 0$ or $|E_2| = 0$) and generated by D;*
- i) $\mathbf{T}$ *is generated by a square and the chipped rectangle D (see Figure 2);*
- ii) $\mathbf{T}$ *is generated by two adjacent rectangles whose union has closure equal to D (see Figure 3).*

Proof. To show (5.1), we can equivalently assume that $\mathbf{T}$ is the ℓ_1-isoperimetric double tiling given by Corollary 3.6 with same prescrived areas (being its perimeter the same), and in particular we can assume that each generator E_i of $\mathbf{T}$ has an open and bounded representative satisfying $\mathcal{H}^1(\partial E_i \setminus \partial^* E_i) = 0$ for $i = 1, 2$. We subdivide the proof into multiple sub-cases, and we write $P := \mathrm{Per}_\varphi(\mathbf{T})$ for brevity.

Case 1: suppose every horizontal line and every vertical line touches $\bar{E}_2 + G$. Then Proposition 4.4 applies and we obtain $P \geq a + b + 2\sqrt{|E_1|}$.

Case 2: Suppose every horizontal line and every vertical line touches $\bar{E}_1 + G$. Then exchanging E_1 and E_2 we reduce to Case 1. Hence we obtain $P \geq a + b + 2\sqrt{|E_2|}$.

Case 3: Suppose Case 1 does not hold. We have two subcases: 3.1 and 3.2.

Case 3.1: Suppose there exists a vertical line v_1 not touching E_2. This means that $v_1 \subset \bar{E}_1 + G$.

Case 3.1.1: Suppose every vertical line touches $\bar{E}_1 + G$. Every horizontal line touches v_1 (from Case 3.1) and hence touches $\bar{E}_1 + G$. So Proposition 4.4 applies with E_1 and E_2 swapped, giving $P \geq a + b + 2\sqrt{|E_2|}$.

Case 3.1.2: Suppose 3.1.1 does not apply. Then, there exists a vertical line not touching $\bar{E}_1 + G$, which means that it is entirely contained in $\bar{E}_2 + G$. Then Proposition 4.5 applies giving $P \geq a + 2b$.

Case 3.2: Suppose there exists a horizonal line not touching E_2. The same subcase analysis performed in Case 3.1 (swapping a, b) gives $P \geq \min\left\{2a + b, a + b + 2\sqrt{|E_2|}\right\}$.

Case 4: Suppose Case 2 does not hold. In this case we swap E_1 and E_2 and reduce to Case 3.

Conclusion: We note that, combining all the cases, we have deduced the isoperimetric estimate (5.1). Therefore, to conclude the proof, we only need to characterize the equality case, and show that one of the conclusions o),i),ii) holds. We shall first characterize the equality case for the isoperimetric tiling $\mathbf{T}$ given by Corollary 3.6, so that each of the previous cases applies. Then, we explain how to conclude the proof.

By symmetry, we shall suppose that $a \geq b$, and $|E_1| \leq |E_2|$. Hence either $P = a + 2b$ or $P = a + b + 2\sqrt{|E_1|}$. Suppose we are in Case 1: in this case, the equality conclusion in Proposition 4.4 yields conclusion i), if $|E_1|$ is strictly positive or conclusion o) if $|E_1| = 0$. Case 2 cannot happen because we have assumed $|E_1| \leq |E_2|$ and $P \leq a + b + 2\sqrt{|E_1|}$ by the minimality assumption. Case 3.1.1 cannot happen for the same reason, so suppose that Case 3.1.2 holds. In this case $P = a + 2b$, and the equality conclusion in Proposition 4.5 yields conclusion ii). Finally, Case 3.2 follows by the very same argument, as well as Case 4. All the cases have been analyzed; hence the tiling $\mathbf{T}$ given by Corollary 3.6 satisfies one of the properties o),i),ii).

Finally, if $\mathbf{T}$ is an arbitrary isoperimetric 2-tiling relative to G, then we recall that the proof of Theorem 3.1 builds another isoperimetric 2-tiling $\mathbf{T}'$ with essentially bounded generators by possibly countable many translations (cf. (3.3)), and Corollary 3.6 deduces the regularity results precisely for such tiling $\mathbf{T}'$. As the conclusion applies to $\mathbf{T}'$, one of the properties o),i),ii) must hold. Consequently, the original tiling $\mathbf{T}$ must also satisfy one of o),i),ii) by recalling again the construction of $\mathbf{T}'$ via (3.3). □

We are finally ready to show our first main result.

Proof of Theorem 1.1. The proof directly follows from Theorem 5.1. We observe that $4\sqrt{x}+2a+2b$ is precisely the sum of the ℓ_1-perimeters of an axis-aligned square of area x and of the chipped rectangle as in Figure 2. The sum of the ℓ_1-perimeter of two adjacent rectangles as in Figure 3 is given by $2a+4b$. In particular, if $x<\frac{b^2}{4}$ then $4\sqrt{x}+2a+2b<2a+4b$, while the opposite holds if $\frac{b^2}{4}<x\leq\frac{1}{2}$. Taking into account Theorem 5.1, we thus obtain the explicit value of $\mathcal{I}_{G,\ell_1}(x)$ for all $x\in[0,1]$. Finally, the conclusions o),i),ii) are in turn obtained by that of Theorem 5.1. □

5.2. **The case $\mathcal{I}_{\ell_1}(x)$.** In this part, we show our second main result when we further minimize among all possible lattices.

Proof of Theorem 1.2. Thanks to the Wulff isoperimetric inequality (2.2), we directly deduce

$$\mathcal{I}_{\ell_1}(x)\geq 2\sqrt{x}+2\sqrt{1-x}.$$

However, the Pythagorean double tiling (cf. Figure 6) where the two squares, with edges parallel to the coordinate axes, have respectively area x and $(1-x)$ is a competitor, whence equality must occur.

Let us now prove that in the case $x\neq\frac{1}{2}$, the Pythagorean tilings are the only possible minimizers. Without loss of generality, we suppose $x<\frac{1}{2}$. Suppose that $\mathbf{T}$ is an ℓ_1-isoperimetric double tiling and denote G its lattice. Namely, $\mathbf{T}$ is periodic relative to G and satisfies $\mathrm{Per}_{\ell_1}(\mathbf{T})=2\sqrt{x}+2\sqrt{1-x}$. Then, by the equality case in the Wulff isoperimetric inequality (2.2), we deduce that the two generators E_1,E_2 are respectively equivalent to two squares with edges parallel to the coordinate axes, and matching areas. Clearly each edge of E_2 (the largest square) touches a translation of E_2 itself because if not, E_2 should be surrounded by translations of E_1 which is not possible since $|E_1|$ is smaller than $|E_2|$ in our assumptions. Denote $a=\sqrt{1-x}$ be the side length of E_2. Hence $(a,h),(w,a)\in G$ for some $h\in[0,a)$ and some $w\in[0,a)$. The complement of $G+E_2$ is composed by rectangles of sides w and h, but E_1 is also a square hence $w=h=\sqrt{x}$ and we obtain a Pythagorean double tiling.

The remaining case is when $x=\frac{1}{2}$. In this case E_1 and E_2 are squares with the same side length $a=\frac{1}{\sqrt{2}}$. It is easy to conclude that the squares, having the edges aligned with the coordinate axes, must fill the plane in horizontal or vertical stripes, as claimed in the statement. □

6. Locally ℓ_1-isoperimetric partitions

In this section, we prove the main result Theorem 1.4. We first give the definition of a locally φ-isoperimetric partition for a general anisotropic norm φ.

Definition 6.1 (locally φ-isoperimetric partition). *Let φ be a norm in $\mathbb{R}^d$, and let (E_n) be a Borel partition (i.e. $|E_i\cap E_i|=0$ for $i\neq j$, and $|\mathbb{R}^d\setminus\cup_n E_n|=0$). We say that (E_n) is locally φ-isoperimetric if for all $R>0$ and for all Borel partitions (F_n) satisfying*

$$|E_n\cap Q_R|=|F_n\cap Q_R|,\qquad E_n\triangle F_n\Subset Q_R,$$

where we denoted $Q_R \coloneqq (-R,R)^d$, it holds that

$$\sum_{n\in\mathbb{N}} \mathrm{Per}_\varphi(E_n, Q_R) \le \sum_{n\in\mathbb{N}} \mathrm{Per}_\varphi(F_n, Q_R).$$

Next, we show a useful local minimality property, and we lastly prove our main result.

Lemma 6.2. *Let G be a lattice and let φ be a norm in $\mathbb{R}^d$. Let $\mathbf{T}$ be a φ-isoperimetric 2-tiling relative to G. Let $\Omega \subset \mathbb{R}^2$ be an open set such that $\bar\Omega \cap (\overline{\Omega} + g) = \emptyset$ for every $g \in G \setminus \{0\}$. Suppose that there exists $E_1 \in \mathbf{T}$ such that $E_1 \Subset \Omega$ and let E_2 be a second generator. Let g_n, $n \in \mathbb{N}$ be an enumeration of G Then the partition*

$$(E_1, E_2 + g_1, \dots, E_2 + g_n, \dots)$$

is a volume constrained minimizer of the φ-perimeter relative to Ω (cf. Definition 3.2).

Proof. Consider a fundamental domain $D \supset \Omega$, i.e. $\mathbb{R}^d = \bigcup_g (D+g)$, $(D+g)\cap D = \emptyset$ for all $g \in G\setminus\{0\}$. Suppose also that $\mathrm{Per}_\varphi(E_1, \partial D) = \mathrm{Per}_\varphi(E_2, \partial D) = 0$. Define

$$\tilde E_1 = F, \qquad \tilde E_2 = \bigcup_n (F_n \cap (D - g_n)), \qquad \tilde{\mathbf{T}} = \{\tilde E_1 + g \colon g \in G\} \cup \{\tilde E_2 + g \colon g \in G\}$$

so that $\tilde{\mathbf{T}}$ is a 2-tiling relative to G. We observe that since $|\tilde E_1| = |E_1|$ then

$$|\tilde E_2| = |D \setminus \tilde E_1| = |D \setminus E_1| = |E_2|.$$

Using that $\mathbf{T}$ is φ-isoperimetric relative to G and by similar arguments as already faced in the proof of Lemma 3.3, we get

$$0 \le 2\,\mathrm{Per}_\varphi(\tilde{\mathbf{T}}) - 2\,\mathrm{Per}_\varphi(\mathbf{T}) = \mathrm{Per}_\varphi(F) - \mathrm{Per}_\varphi(E_1) + \sum_n \mathrm{Per}_\varphi(F_n, \Omega) - \sum_n \mathrm{Per}_\varphi(E_2 + g_n, \Omega),$$

which is the desired conclusion. □

Proof of Theorem 1.4. Let $R > 0$ be given such that $E_1 \Subset Q_R = (-R,R)^2$. Consider the group $G = 2R\cdot\mathbb{Z}^2$ generated by $g_1 = (2R, 0)$, $g_2 = (0, 2R)$ and let $\mathbf{T}$ be a double tiling relative to G generated by E_1 and $\tilde E_2 \coloneqq [0,2R]^2 \setminus (E_1 + G)$. By Theorem 1.1, we know that $\mathbf{T}$ is ℓ_1-isoperimetric relative to G up to choosing R big enough. By Lemma 6.2 we also know that the partition $\mathbf{E}$ is a volume constrained minimizer of the ℓ_1-perimeter relative to Q_R. Since $R > 0$ was arbitrary, we conclude that the partition $\mathbf{E}$ is locally ℓ_1-isoperimetric according to Definition 6.1.

The second part of the statement (about the *vortex* partition) can be obtained by the same argument invoking instead Theorem 1.2. □

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