INTEGRALE delle FUNZIONI RAZIONALI

$$\int \frac{P(x)}{Q(x)}\,dx \qquad P, Q \text{ polinomi}$$

– i polinomi li sappiamo integrare

Es

$$\int \frac{2x^2-3x+7}{x-5}\,dx$$

$$\begin{array}{r} 2x^2-3x+7 \\ 2x^2-10x \\ \hline 7x+7 \end{array} \qquad 2x(x-5) = 2x^2-10x$$

$$P(x) = 2x^2-3x+7 = 2x(x-5) + 7x+7$$

$$7x+7 \quad 7(x-5) = 7x-35$$

$$\begin{array}{r} 7x+7 \\ 7x-35 \\ \hline 42 \end{array} \qquad 7x+7 = 7\cdot(x-5)+42$$

$$= 2x(x-5)+7(x-5)+42$$

$P(x) \qquad Q(x)$

$$
\begin{array}{r|l}
\cancel{2x^2} - 3x + 7 & x - 5 \\
\cancel{2x^2} - 10x & \overline{2x + 7} \\
\hline
\cancel{7x} + 7 & \\
\cancel{7x} - 35 & \\
\hline
42 &
\end{array}
$$

$2x + 7 \leftarrow$ Quociente

$42 \leftarrow$ Resto

$$\int \frac{2x^2 - 3x + 7}{x - 5}\,dx = \int \left[2x + 7 + \frac{42}{x-5}\right] dx$$

$$= x^2 + 7x + \int \frac{42}{x-5}\,dx$$

$$= x^2 + 7x + 42 \cdot \ln|x-5|$$

$\frac{1}{x}$

$$\int \frac{3x-4}{x^2-6x+8}\,dx$$

$$x^2-6x+8=(x-4)(x-2)$$

$\Delta>0$

$P(x)$

$$\frac{3x-4}{(x-4)(x-2)} \overset{?}{=} \frac{A}{x-4}+\frac{B}{x-2}$$

$\frac{1}{x-4}$ $\frac{1}{x-2}$

$$= \frac{A(x-2)+B(x-4)}{(x-4)(x-2)}$$

Se $(x-4)$ e $(x-2)$ sono polinomi indipendenti

allora $\operatorname{span}\{x-4,\,x-2\}$ = tutti i polinomi di grado 1.

$$\frac{(A+B)x+(-2A-4B)}{\cancel{\%}} = \frac{3x-4}{\cancel{\%}}$$

$$\begin{cases} A+B=3 \\ -2A-4B=-4 \end{cases}$$

$x,\ 1$

$$\begin{cases} B = 3 - A \\ 2A + 4(3-A) = 4 \end{cases}$$

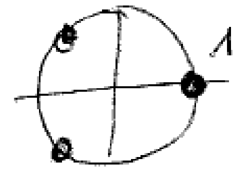
$$\begin{cases} B = 3 - A \\ -2A + 12 = 4 \end{cases} \quad \begin{cases} B = -1 \\ A = 4 \end{cases}$$

$$\int \frac{3x-4}{x^2-6x+8}\,dx = \int \left[\frac{4}{x-4} - \frac{1}{x-2} \right] dx$$

$$= 4 \ln|x-4| - \ln|x-2|$$

$$= \ln \frac{(x-4)^4}{|x-2|}$$

$$\int \frac{3x}{x^3-1}\,dx \qquad z^3 = 1$$



$$Q(x) = x^3 - 1 = (x-1)\underbrace{(x^2+x+1)}_{\Delta<0}$$

$$\frac{3x}{x^3-1} \overset{?}{=} \frac{A}{x-1} + \frac{B+Cx}{x^2+x+1}$$

$$= \frac{A(x^2+x+1) + B\cancel{x}(x-1) + Cx(x-1)}{(x-1)(x^2+x+1)}$$

(La teoria mi garantisce che le frazioni razionali $\frac{1}{x-1}$, $\frac{1}{x^2+x+1}$, $\frac{x}{x^2+x+1}$ sono linearmente indipendenti)

$$= \frac{(A+C)x^2 + (A+B-C)x + A - B}{x^3-1}$$

$$\begin{cases} A+C = 0 \\ A+B-C = 3 \\ A-B = 0 \end{cases} \qquad \begin{cases} A = 1 \\ B = 1 \\ C = -1 \end{cases}$$

$$\int \frac{3x}{x^3-1}\,dx = \int \left[\frac{1}{x-1} + \frac{1-x}{x^2+x+1}\right] dx$$

$\ln|x-1|$

$$\int \frac{1-x}{x^2+x+1}\,dx = -\frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \int \frac{1+\frac{1}{2}}{x^2+x+1}\,dx$$

$-\frac{1}{2}\ln\left(x^2+x+1\right)$

$$\frac{3}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$\int \frac{1}{1+y^2}\,dy = \operatorname{arctg} y$$

COMPLETAMENTO del QUADRATO

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{4} = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$= \frac{3}{4}\left[\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}} + 1\right]$$

$$= \frac{3}{4}\left[\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2 + 1\right]$$

$$\frac{3}{2}\int \frac{1}{x^2+x+1}\,dx = \frac{3}{2}\int \frac{1}{\frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right]}\,dx$$

$$= \int \frac{2}{\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right]}\,dx \qquad \left\{\begin{array}{l} y=\frac{2x+1}{\sqrt{3}} \\ dy=\frac{2}{\sqrt{3}}\,dx \end{array}\right\}$$

$$= \sqrt{3}\,\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$\int \frac{3x}{x^3-1}\,dx = \ln|x-1| - \frac{1}{2}\ln\left(x^2+x+1\right) + \sqrt{3}\,\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$= \ln\frac{|x-1|}{\sqrt{x^2+x+1}} + \sqrt{3}\,\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$\int \frac{9x+8}{x^3+2x^2+x+2}\,dx$$

-2 è rodiφ

$$x^3+2x^2+x+2 = (x+2)(x^2+1)$$

$$\frac{9x+8}{x^3+2x^2+x+2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$$\cancel{\left[\ln \frac{x^2+1}{x^2-1} + 5 \operatorname{arctg} x\right]}$$

$$\left[\ln \frac{x^2+1}{(x+2)^2} + 5 \operatorname{arctg} x\right]$$

$$\int \frac{x^5 - x + 1}{x^4 + x^2}\, dx = \int x\, dx - \int \frac{x^3 + x - 1}{x^4 + x^2}\, dx$$

$$\frac{x^2}{2}$$

$$\begin{array}{l|l} x^5 - x + 1 & x^4 + x^2 \\ \hline x^5 + x^3 & x \\ \hline -x^3 - x + 1 & \end{array}$$

$$x^4 + x^2 = x^2 (x^2 + 1)$$

$$\int \frac{x^3 + x - 1}{x^4 + x^2}\, dx$$

$$\frac{x^3 + x - 1}{x^4 + x^2} = \frac{A + Dx}{x^2} + \frac{Bx + C}{x^2 + 1}$$

$$= \frac{A}{x^2} + \frac{D}{x} + \frac{Bx + C}{x^2 + 1}$$

Sono indipendenti: $\frac{1}{x^2}, \frac{1}{x}, \frac{1}{x^2+1}, \frac{x}{x^2+1}$

$$x^3 + x - 1 = (A + Dx)(x^2 + 1) + (Bx + C)x^2$$

$$x^3+x-1 = (D+B)x^3+(A+C)x^2+Dx+A$$

$$\begin{cases} D+B=1 \\ A+C=0 \\ D=1 \\ A=-1 \end{cases} \qquad \begin{cases} B=0 \\ A=-1 \\ C=1 \\ D=1 \end{cases}$$

$$\int \frac{x^3+x-1}{x^4+x^2}\,dx = \boxed{\int \frac{x-1}{x^2}}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \int \frac{1}{x}\,dx - \int \frac{1}{x^2}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \ln|x| + \frac{1}{x} + \operatorname{arctg} x$$