ANALISI MATEMATICA (B)

LEZIONE n.69

(CdS FISICA)

INIZIO ORE 9:10

INTEGRALI IMPROPRI

Proprietà

monotonia $\quad f \leq g \Rightarrow \int_a^b f \leq \int_a^b g$

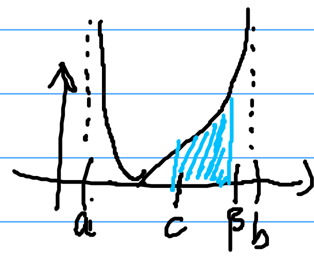
additività $\quad \int_a^b f = \int_a^c f + \int_c^b f$

linearità $\quad \int_a^b \lambda f + \mu g = \lambda \int_a^b f + \mu \int_a^b g$

le funzioni integrabili in senso improprio formano uno spazio vettoriale.

Teorema (integrabilità delle funzioni positive)
Sia $f:(a,b)\to\mathbb{R}$ localmente R-integrabile, se $f\geq 0$

Allora esiste $\int_a^b f(x)\,dx \geq 0$



dim $\quad c\in(a,b) \quad \int_a^b = \int_a^c + \int_c^b$

$$\int_c^b f(x)\,dx = \lim_{\beta\to b^-} \int_c^\beta f(x)\,dx$$

$$F(x) = \int_c^x f(t)\,dt \qquad \text{se } f\geq 0$$

F è crescente

$$x_1 \leq x_2 \Rightarrow F(x_2)-F(x_1) = \int_{x_1}^{x_2} f(t)\,dt \geq 0$$

□

Teorema (criteri di confronto).
$f,g : [a,b) \to \mathbb{R}$, localmente R-integrabile
$(a,b]$
$f,g \geq 0$ (allora $\int_a^b f$, $\int_a^b g$ esistono)

1. (confronto puntuale) se $f\leq g \Rightarrow \int_a^b f \leq \int_a^b g$
(se g è integrabile $\Rightarrow f$ è integrabile)

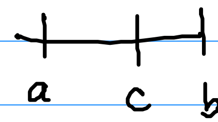
2. (confronto asintotico) se $f \ll g$ per $x\to b^-$ ($x\to a^+$)
(ovvero $\lim_{x\to b^-} \frac{f(x)}{g(x)} = 0$, ovvero $f\in o(g)$)
allora se $\int_a^b g(x)\,dx < +\infty$ $\Rightarrow$ $\int_a^b f(x)\,dx < +\infty$
(g integrabile) $\quad$ (f integrabile)

3. (confronto asintotico) se $f \sim g$ per $x \to b^-$ $\quad x \to a^+$

$\left(\text{ovvero } \lim_{x \to b^- \; a^+} \frac{f(x)}{g(x)} = 1 \right)$

allora $\int_a^b f(x)\,dx$ e $\int_a^b g(x)\,dx$ hanno lo stesso carattere.

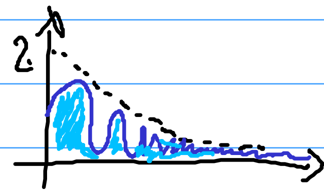
Oss Se $f : (a,b) \to \mathbb{R}$



$$\int_a^b f = \int_a^c f + \int_c^b f.$$

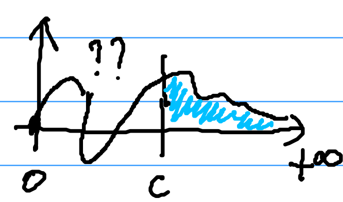
Morale In ogni "punto cattivo" devo studiare l'andamento asintotico da destra e da sinistra

Es. $\int_0^{+\infty} e^{-x}\left(1+\sin(x^2)\right) dx$



$$\leq \int_0^{+\infty} 2e^{-x}\,dx = 2\left[-e^{-x}\right]_0^{+\infty} = 2$$

Es $\int_0^{+\infty} \frac{x^2+x^3-x+x^5}{x^6+x^4+1}\,dx = +\infty$



$\underset{f(x)}{\overset{\shortparallel}{}} \overset{?}{\geq} 0 \qquad \lim_{x \to +\infty} f(x) = 0$

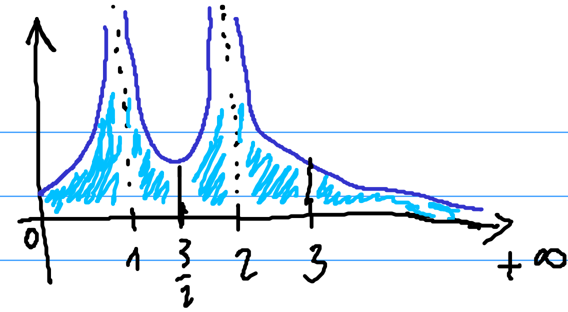
$\lim_{x \to +\infty} x^2+x^3-x+x^5 = +\infty \Rightarrow \exists c :$ $f(x) > 0$ su $[c,+\infty)$

$\int_0^c f(x)\,dx = A \in \mathbb{R}$ $\qquad \int_c^{+\infty} f(x)\,dx \qquad f(x) \sim \frac{1}{x}$ per $x \to +\infty$

$\hookrightarrow$ ha lo stesso carattere di $\int_c^{+\infty} \frac{1}{x}\,dx = +\infty$

$f \geq 0$

Es $\int_0^{+\infty} \frac{1}{(x-1)^2 (x-2)^2} \, dx = +\infty$



$= \int_0^1 f + \int_1^2 f + \int_2^{+\infty} f$

$= \underbrace{\int_0^{1} f}_{A} + \underbrace{\int_{1}^{3/2} f}_{B} + \underbrace{\int_{\frac{3}{2}}^{2} f}_{C} + \underbrace{\int_{2}^{3} f}_{D} + \underbrace{\int_3^{+\infty} f}_{E}$

Basta che uno di questi 5 integrali sia $+\infty$ affinché tutto l'integrale sia $+\infty$. (non può essere $-\infty$ perché $f \geq 0$)

E) $E = \int_3^{+\infty} f(x) \, dx$ per $x \to +\infty$ $f(x) \sim \frac{1}{x^4}$

$$\int_3^{+\infty} \frac{1}{x^4} \, dx = \left[\frac{x^{-3}}{-3} \right]_3^{+\infty} = \frac{1}{3^4} < +\infty$$

Quindi $E < +\infty$

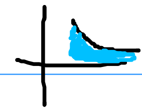
A) $A = \int_0^{1} \frac{1}{(x-1)^2 (x-2)^2} \, dx = +\infty$

$$f(x) = \frac{1}{(x-1)^2} \frac{1}{(x-2)^2} \sim \frac{1}{(x-1)^2}$$

$$\int_0^1 \frac{1}{(x-1)^2} \, dx = \left[-\frac{1}{x-1} \right]_0^1 = +\infty$$

Ricorda: $\displaystyle\sum_{k=1}^{+\infty} \frac{1}{k^p}$

Esempi di base



$$\int_1^{+\infty} \frac{1}{x^p} = \int_1^{+\infty} x^{-p}\,dx = \frac{1}{1-p}\left[x^{1-p}\right]_1^{+\infty} \begin{cases} p<1 & = +\infty \\ p>1 & = \dfrac{1}{p-1} < +\infty \end{cases} \qquad p \neq 1$$

$$p=1 \qquad = \int_1^{+\infty} \frac{1}{x}\,dx = \left[\ln x\right]_1^{+\infty} = +\infty$$

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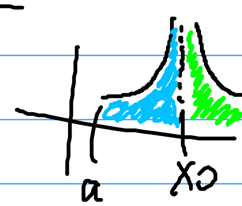
$a < x_0$



$$\int_a^{x_0} \frac{1}{(x_0-x)^p}\,dx = \frac{-1}{1-p}\left[(x_0-x)^{1-p}\right]_a^{x_0} = \begin{cases} p<1 & = 0 + \dfrac{1}{1-p}(x_0-a)^{1-p} < +\infty \\ p>1 & = +\infty \end{cases} \qquad p \neq 1$$

$$p=1 \qquad = -\left[\ln(x_0-x)\right]_a^{x_0} = +\infty$$

$\boxed{\displaystyle\int_{x_0}^{b} \frac{1}{(x-x_0)^p}\,dx \text{ analogo.}}$

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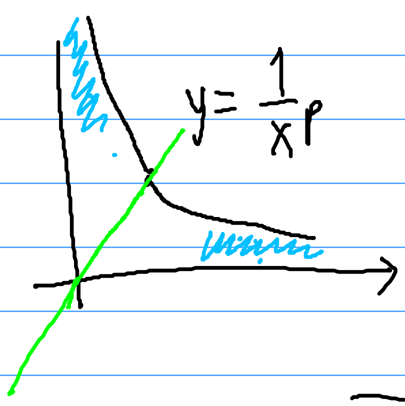

$x^p y = 1 \qquad x = \left(\frac{1}{y}\right)^{\frac{1}{p}} = \frac{1}{y^{1/p}}$

$y = \frac{1}{x^2} \qquad x = \frac{1}{\sqrt{y}}$

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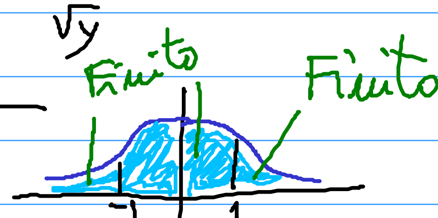
Es $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}\,dx < +\infty$



$f(x) = \frac{1}{e^{x^2}} \ll \frac{1}{x^p} \quad \forall p. \qquad \int_1^{+\infty} \frac{1}{x^p} < +\infty$ se $p > 1$.

Es Trovare $f:[0,+\infty)$, $f\geq 0$, continua tale che

$$\int_0^{+\infty} f(x)\,dx < +\infty$$

ma non $\lim_{x\to+\infty} f(x) = 0$.

(l'integrale può convergere anche se la funzione non è infinitesima)