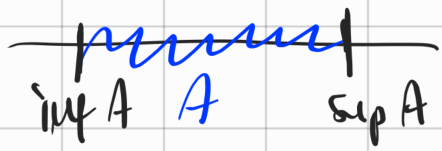
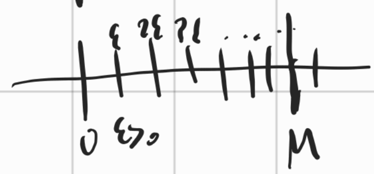
ANALISI MATEMATICA B
LEZIONE 13 – 21.10.2020

Estremo superiore/inferiore   sup e inf.



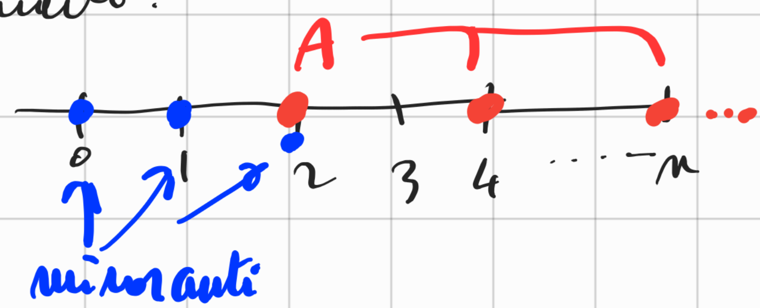
Proprietà Archimedea



Principio del buon ordinamento:

$A \subseteq \mathbb{N}$, $A \neq \emptyset$

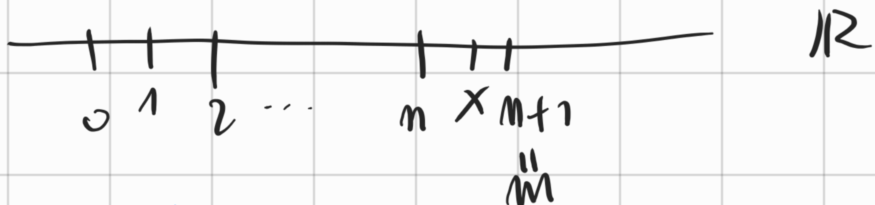
min A esiste!



dim. per assurdo se

n è un minorante ($n \leq A$) anche $n+1 \leq A$

Parte intera



Definiamo la parte intera di $x \in \mathbb{R}$

come $\lfloor x \rfloor \in \mathbb{Z}$ tale che

$$x-1 < \lfloor x \rfloor \leq x$$

e $\lceil x \rceil \in \mathbb{Z}$ tale che $x \leq \lceil x \rceil < x+1$

Cosicché $\lfloor x \rfloor \le x \le \lceil x \rceil$

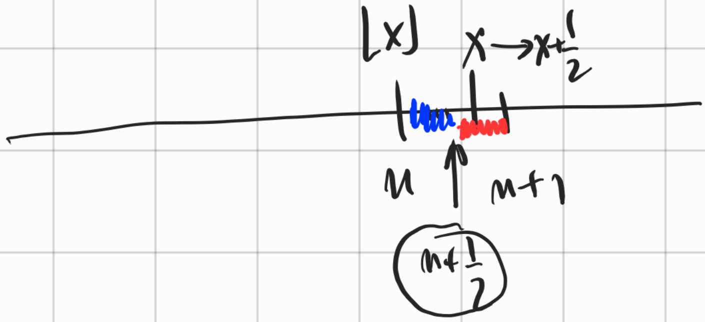
Es $\lfloor 3.14 \rfloor = 3$, $\lceil 3.14 \rceil = 4$, $\lfloor -3.14 \rfloor = -4$

Es $\longrightarrow \dfrac{\lfloor 100\pi \rfloor}{100} = 3.14$ $\dfrac{\lceil 100\pi \rceil}{100} = 3.15$

$\pi = 3.1415926\ldots$

$\dfrac{\lceil 10'000\pi - \frac{1}{2} \rceil}{10'000} = 3.1416$

$\{x\} = x - \lfloor x \rfloor$



NUMERI RAZIONALI $\mathbb{Q}$

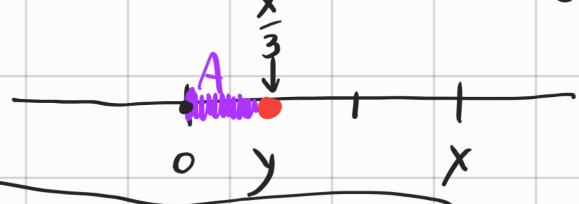
Divisione $y = \dfrac{x}{n}$ se $ny = x$ $x, y \in \mathbb{R}$, $n \in \mathbb{N}$

Teorema (divisibilità dei numeri reali)

Se $x \in \mathbb{R}$, $n \in \mathbb{N}$, $n \neq 0$ $\exists!\, y \in \mathbb{R}$ t.c. $ny = x$.
(scriviamo $y = \frac{x}{n}$)

dim

x fissato. $x > 0$



$$\boxed{A = \{ a \in \mathbb{R} : na \le x \}}$$

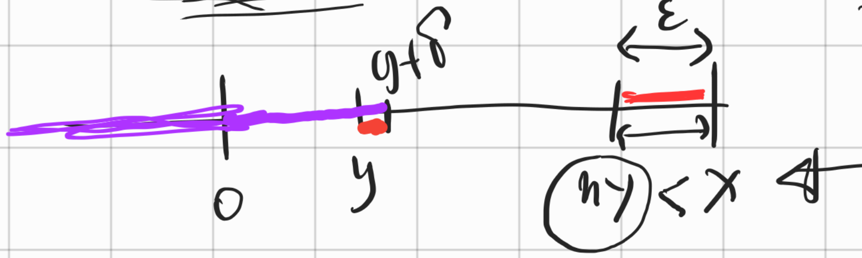
$y = \sup A$.

→ $A \neq \emptyset$ perché $0 \in A$

→ A è superiormente limitato? $x \geq A$ | $y = \sup A$ esiste.

→ $\boxed{ny = x}$??

Per assurdo $ny < x$ (poi faremo $ny > x$)



$\varepsilon = x - ny$, $\varepsilon > 0$. Voglio trovare $\delta > 0$ tale che $y+\delta \in A$ (sarebbe assurdo)

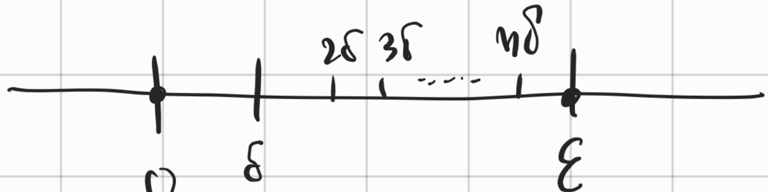
$n(y+\delta) \leq x$ ($y < y+\delta \in A$ (y non è maggiorante

$\|$

$ny + n\delta \leq x$ cioè $n\delta \leq \varepsilon$.

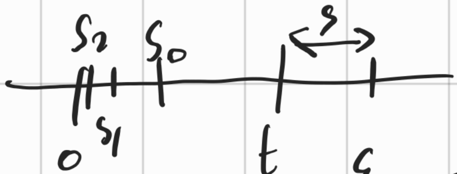
(*) Devo dimostrare che esiste $\delta > 0$ t.c. $n\delta \leq \varepsilon$.

↑ devo usare la densità.



Per densità

$\exists\, t$: t.c.



$s = t$

$0 < t < \varepsilon$ o $2 \cdot t \leq \varepsilon$ o $2t > \varepsilon$

se $2t > \varepsilon$ prendo $s = \varepsilon - t$

$$2s = 2\varepsilon - 2t < 2\varepsilon - \varepsilon = \varepsilon$$

$\boxed{2s_0 < \varepsilon}$

$s_0 = s$ $\quad s_1$ t.c. $2s_1 \le s_0$

s_2 t.c. $2s_2 \le s_1$

$4s_2 \le s_0$

$8s_2 \le \varepsilon$

$s_n \cdot 2^{n+1} \le \varepsilon$

$$\underline{n \cdot s_n} \le 2^{n+1} \cdot s_n \le \varepsilon. \quad \square$$

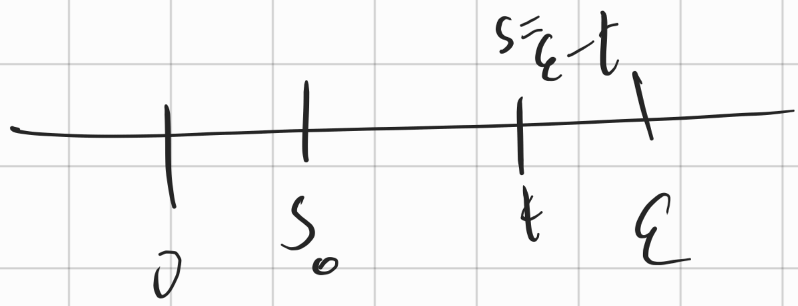
Esercizio

$n \le 2^{n+1}$! $\forall n$

(a posteriori $y = \max A$).

$2t < \varepsilon$

o $2(\varepsilon - t) \le \varepsilon$



$2 \cdot s_0 \le \varepsilon$

$2 \cdot s_1 \le s_0 \qquad 4 \cdot s_1 \le 2 \cdot s_0 \le \varepsilon$

$2 \cdot s_2 \le s_1 \qquad 8 \cdot s_2 \le 4s_1 \le \varepsilon$

$\vdots$

$2 \cdot s_{n+1} \le s_n \qquad 2^{n+1} \cdot s_n \le \varepsilon$

$\delta = s_n \qquad \boxed{n \cdot \delta} \le 2^{n+1} \delta \le \varepsilon.$

$n \le 2^{n+1}$

Razionali:

$$\mathbb{Q} = \mathbb{Z} \Big/ _{\mathbb{N} \setminus \{0\}} = \left\{ \frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{N},\ q \neq 0 \right\}$$

$\cap$

$\mathbb{R}$

$$\frac{p}{q} \cdot x = \frac{p \cdot x}{q} = p \cdot \frac{x}{q}$$

Decimali:

$$3.14 = \frac{314}{10^2}$$

$$314 = (3 \cdot (9+1) + 1) \cdot (9+1) + 4 \qquad 10 = 9+1$$

$$\pi \underset{\uparrow}{\approx} 3.14 \quad \text{se} \quad |\pi - 3.14| \leq 0{,}01$$

approssimativamente

Decimali periodici:

$$x = \underline{0{,}1\overline{6}} \qquad (0{,}16666\ldots)$$

$$= \underline{0{,}16\overline{6}}$$

$10(x-0{,}1)$

$y = 10x - 1$

$y = 0.\bar{6}$

$\boxed{10y - 6 = y}$

$6{,}\bar{6} -$
6
$0{,}\bar{6}$

$10(10x-1) - 6 = 10x - 1$

$100x - 10 - 6 = 10x - 1$

$90x = 16 - 1$

$x = \dfrac{16-1}{90}$

$0{,}\bar{9} = 1$

$0{,}9999\ldots\ldots$
$1{,}0000\ldots\ldots$

$0{,}\bar{9}$ $\mathbb{R}$

1

$x = 0{,}\bar{9}$

$10x - 9 = x$

$9x = 9$

$x = 1.$

$9{,}\bar{9} -$
9
$0{,}\bar{9}$

$0{,}999\ldots$

$1{,}0000$

ε

??

$1 - 0{,}9999\ldots$

sarebbe infinitesimo -

ISOMORFISMI DEI GRUPPI ORDINATI (DENSI, CONTINUI)

Dati, $R \longrightarrow S$ $\quad 1, 2, 7, \frac{3}{2} \longleftrightarrow 1, 2, 7, \frac{3\pi}{2}$

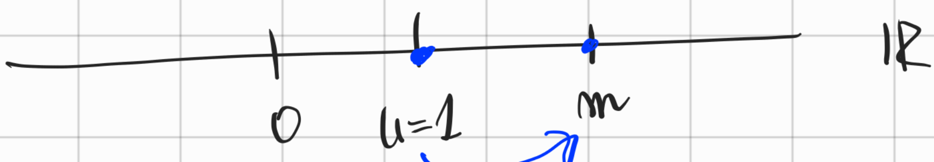
Esiste l'unica $f : R \to S$ biunivoca tale che

$f(0)=0 \longrightarrow \boxed{f(x+y) = f(x) + f(y)}$ (omomorfismo)

(monotona crescente) $\boxed{x \leq y \Leftrightarrow f(x) \leq f(y)}$

Teorema Siano R, S due gruppi additivi, totalmente ordinati, densi e continui. Scelto $u \in R$, $u \neq 0$, scelto $m \in S$ esiste una unica $f : R \to S$ che è un OMOMORFISMO MONOTONO tale che $f(u) = m$.

Es. $R = \mathbb{R}$, $S = \mathbb{R}$.



$f(x) = m x$.

$$f(x+y) = m(x+y)$$

$$\Vert$$

$$f(x)+f(y) = mx + my$$

Se $m > 0$

$$x \geq y \Rightarrow mx \geq my$$