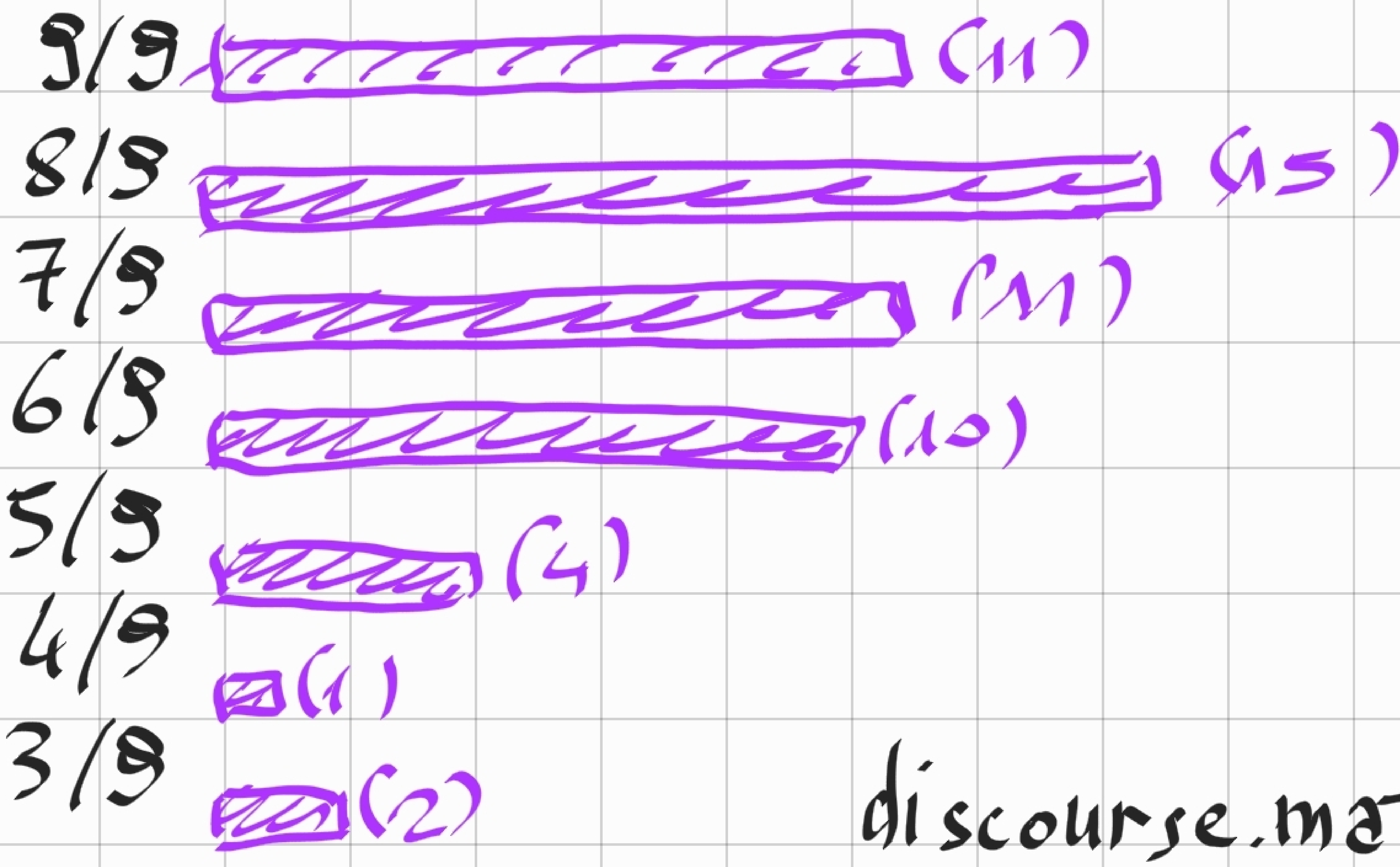
ANALISI MATEMATICA B
LEZIONE 20 - 9.11.2020

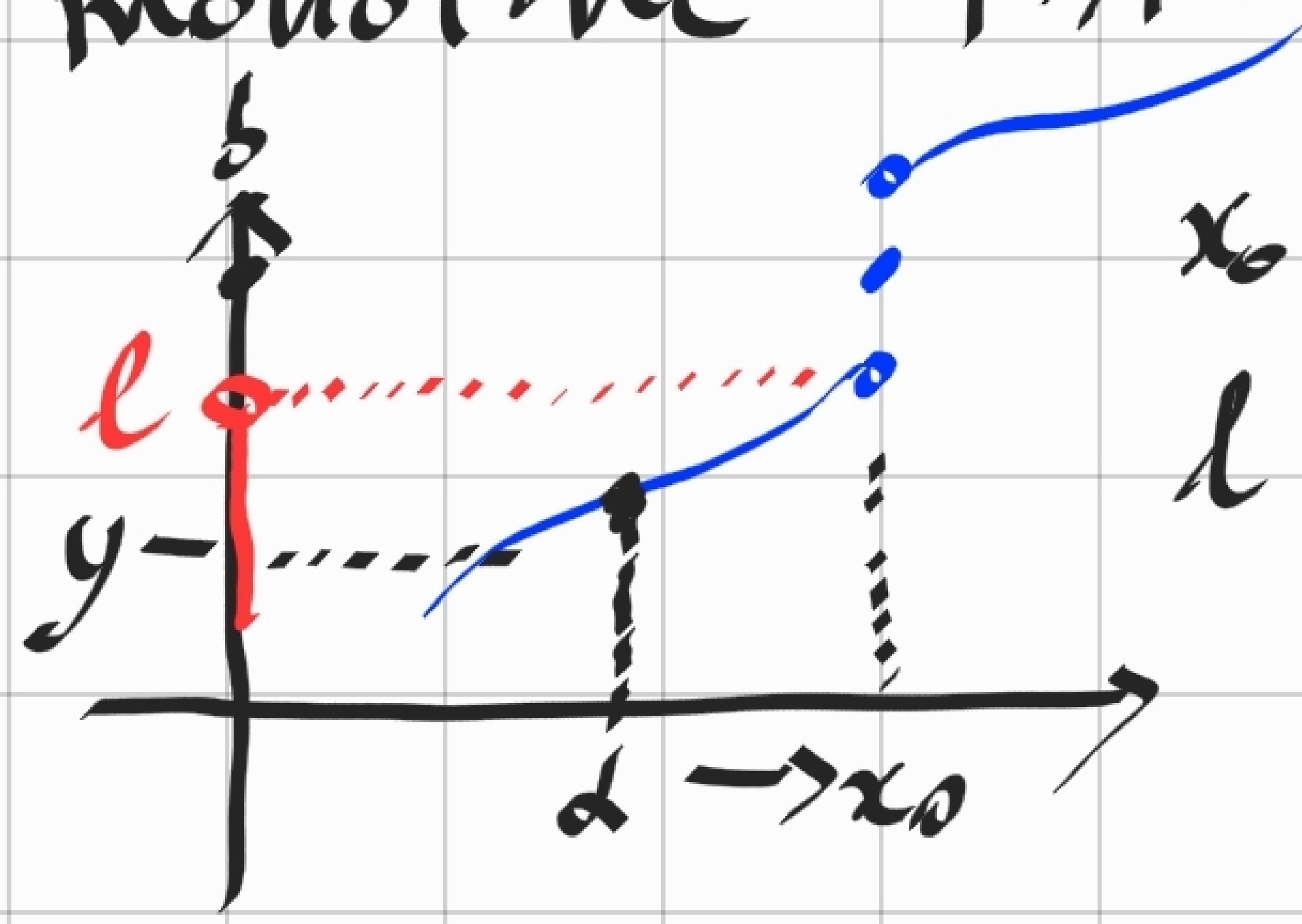
TEST SETTIMANALE (54)



discourse.matb.it

Limite delle fn. monotone $f: A \subseteq \mathbb{R} \to \mathbb{R}$

1 f crescente



$$\boxed{\lim_{x \to x_0^-} f(x) = \sup_{x < x_0} f(x)}$$

$x_0 \in [-\infty, +\infty]$

$l \in [-\infty, +\infty]$

$$= \sup \{ f(x) : x < x_0 \}$$

x_0^- pto di accumulazione A

(l'insieme $\{ \dots \}$ non è vuoto.)

<u>dim</u> $l = \sup_{x < x_0} f(x)$

l minimo dei maggioranti di $\{ f(x) : x < x_0 \}$

→ (1) $l \geq f(x) \quad \forall x < x_0$

(2) $\forall y < l : \underbrace{\exists \alpha < x_0 : f(\alpha) > y}_{y \text{ non è un maggiorante.}}$

Voglio mostrare:

$$\forall U \in \mathcal{B}_l : \exists V \in \mathcal{B}_{x_0^-} : \forall x \in A \setminus \{x_0\} : x \in V \Rightarrow f(x) \in U$$

$U = (y, l]$ con $y < l$

Dato $y < l$ $\exists \alpha < x_0$ t.c. $x > \alpha \Rightarrow f(x) \overset{f \text{ crescente}}{\geq} f(\alpha) \overset{(2)}{>} y$

(1) $\forall x < x_0 \quad f(x) \leq l$

$\forall x \in \underbrace{(\alpha, x_0)}_{} \cap \mathbb{R} : \boxed{y < f(x) \leq \ell}$

$f(x) \in (y, \ell]$

$V = (\alpha, x_0]$

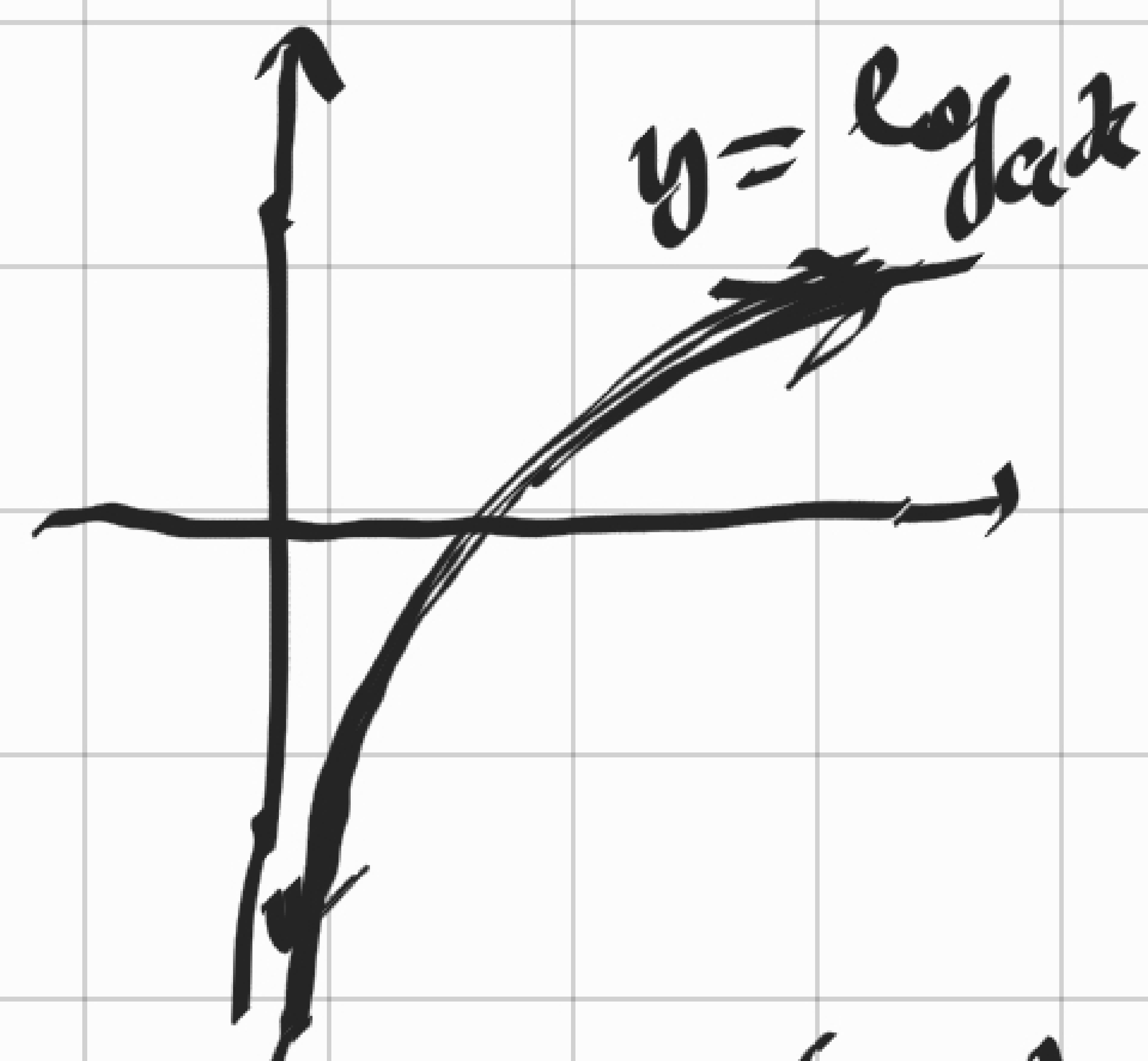
$\forall x \in (A \setminus \{x_0\}) \cap V : f(x) \in U$ $\square$

$\Downarrow$

$\alpha < x < x_0$

Es $\lim_{x \to 0^+} \log_a x = -\infty$ $(a>1)$

$\uparrow$



$\lim_{x \to 0^+} \log_a x = \inf_{x>0} \log_a x = \inf \log_a(\mathbb{R}_+)$

$= \inf \mathbb{R} = -\infty$

$\lim_{x \to +\infty} \log_a x = +\infty$

$= \sup_{x < +\infty} \log_a x = \sup \mathbb{R} = +\infty$

$\lim_{x \to x_0} \log_a x = \log_a x_0 \leftarrow$ continuità

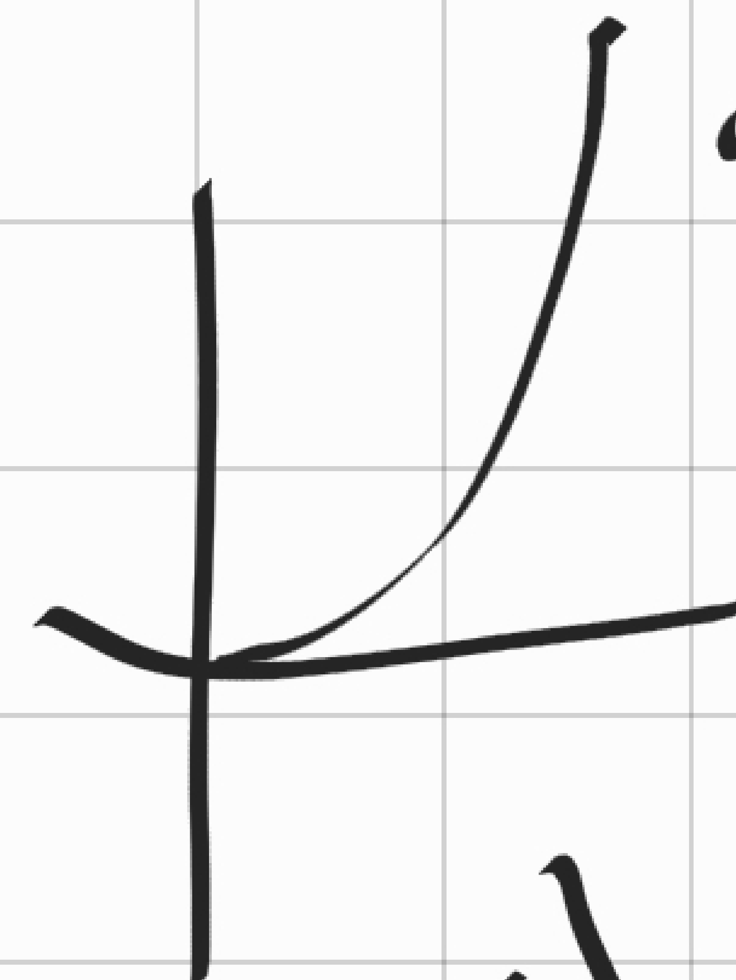
$0 < x_0 < +\infty$

$$\lim_{x \to 0} \sqrt{x} = 0$$

$\uparrow$

$$\lim_{x \to +\infty} \sqrt{x} = +\infty$$

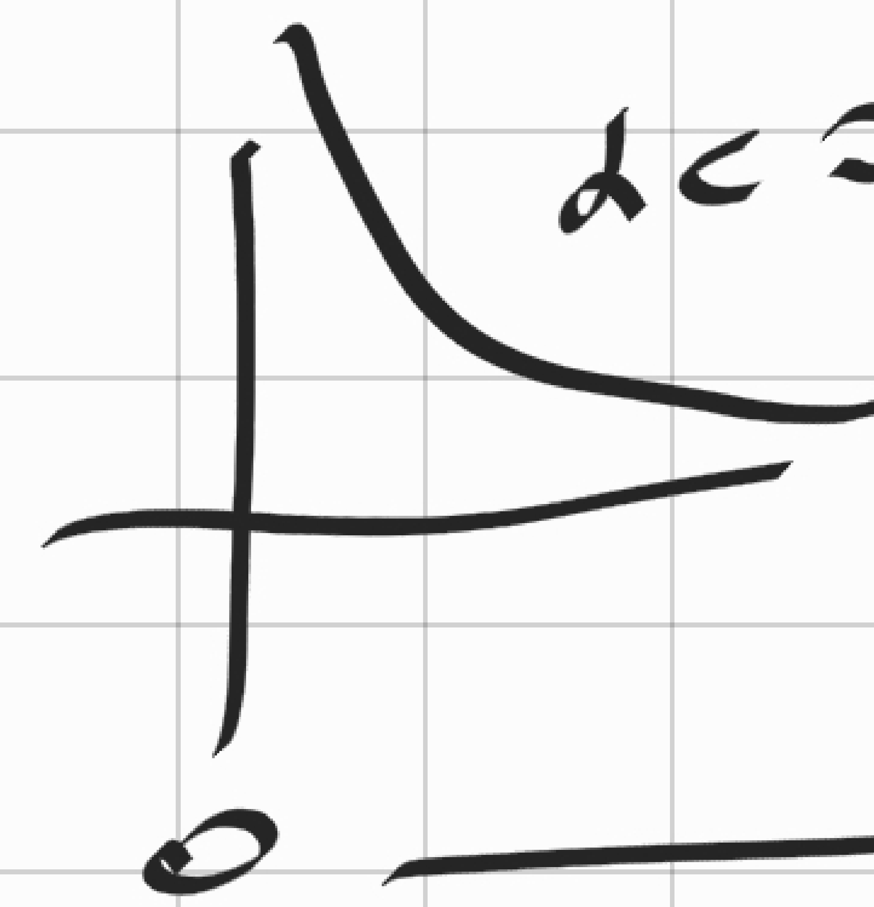
$\alpha > 0$



$$x^{\alpha}$$

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$\alpha < 0$



Operazioni con i limiti:

Es

$$\lim_{x \to 3} \frac{\sqrt{1+x^2}}{\log_2(x+2^x)} = \frac{\sqrt{1+3^2}}{\log_2(3+2^3)}$$

$$= \frac{\sqrt{10}}{\log_2 11}$$

Se per $x \to x_0$

$$f(x) \to l_1$$
$$g(x) \to l_2$$

Allora $\boxed{f(x)+g(x) \to l_1+l_2}$

$-$ $f(x) \cdot g(x) \to l_1 \cdot l_2$

$-x$ $-$ $f(x)-g(x) \to l_1-l_2$

$\frac{1}{x}$ $-$ $\dfrac{f(x)}{g(x)} \to \dfrac{l_1}{l_2}$

Sempre che le operazioni siano definite!

Avendo anche posto: $l \in \mathbb{R}$

$$\begin{cases} (+\infty)+l = +\infty \\ (-\infty)+l = -\infty \end{cases}$$

$$\left|\begin{array}{l}(+\infty)+(+\infty)=+\infty \\ (-\infty)+(-\infty)=-\infty\end{array}\right|$$

Rimane <u>non definito</u> $\boxed{\begin{array}{c}(+\infty)+(-\infty) \\ (-\infty)+(+\infty)\end{array}}$

<u>forma indeterminata</u>

<u>Dim</u> (cenni)

Se $\underbrace{f(x)\to l_1}$, $g(x)\to l_2$ per $x\to x_0$

$l_1, l_2 \in \mathbb{R}$ (finiti)

se $x_0\in\mathbb{R}$ l'abbiamo già dimostrato

$f(x)\to\tilde{f}(x_0)=l_1$ $\tilde{f}$ estensione continua

$g(x)\to\tilde{g}(x_0)=l_2$

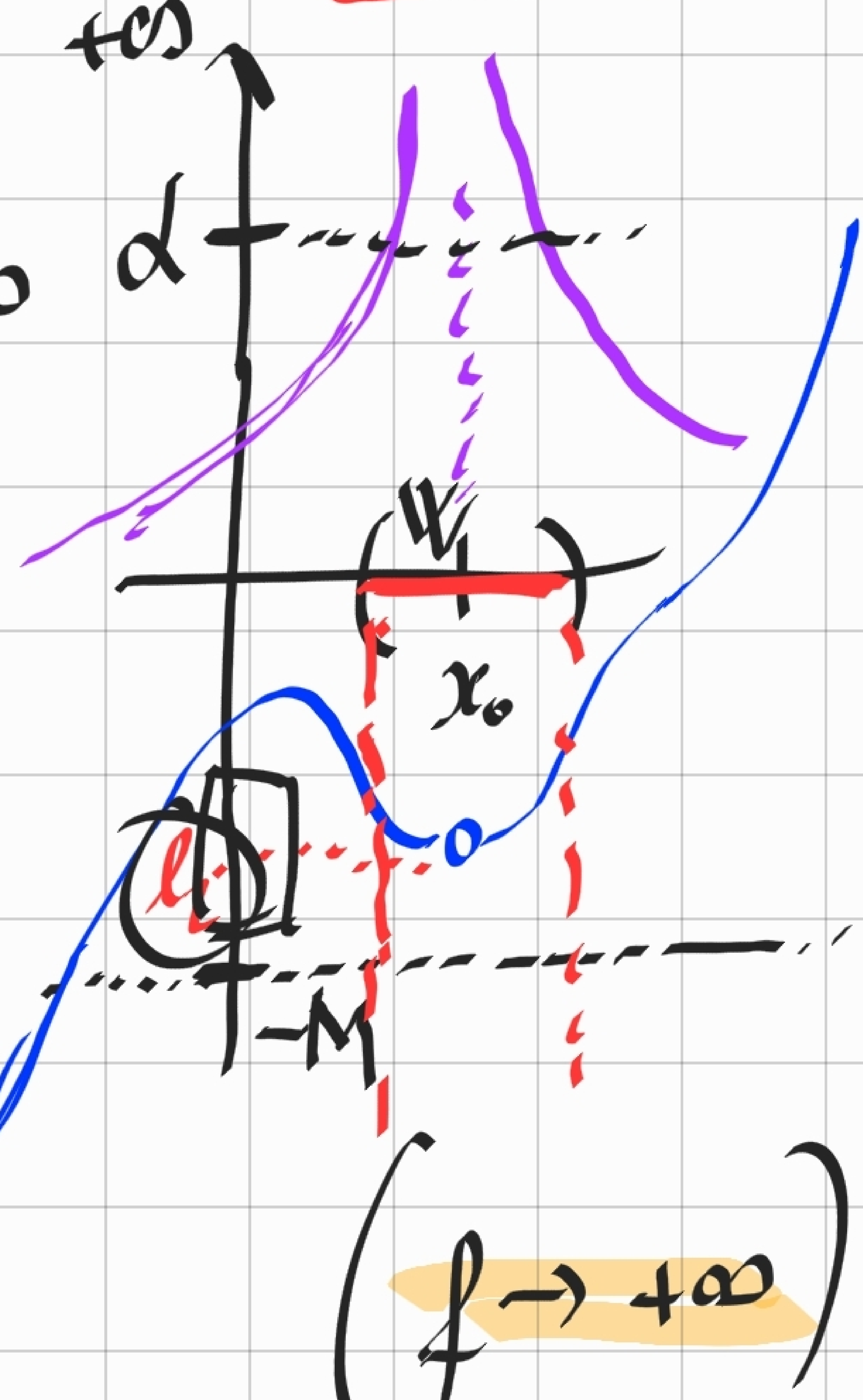
$f(x)+g(x)$ è continua

$$f(x)+g(x) \to f(x_0)+g(x_0) = \ell_1+\ell_2$$

Altro caso $\boxed{\ell_1 = +\infty}$, $\ell_2 > -\infty$

$f(x) \to +\infty$

$g(x) \to \ell_2 > -\infty$



Allora in un intorno W di x_0

$g(x) > -M \in \mathbb{R}$

$\forall \alpha \in \mathbb{R}$ $\quad U = (\alpha, +\infty]$ $\quad (f \to +\infty)$

$\exists V \in \mathcal{B}_{x_0}$ t.c. $x \in V \setminus \{x_0\}$

$\Rightarrow f(x) > \alpha$

$x \in V \cap W = V'$ $\quad \Rightarrow \boxed{f(x)+g(x) > \alpha - M}$

$\forall \beta \in \mathbb{R}$ $\exists \alpha$: $\alpha - M > \beta$ $\qquad \alpha = \beta + M$.

$$\exists V' \in \mathcal{B}_{x_0} : \quad x \in V' \setminus \{x_0\} \Rightarrow f(x)+g(x) > \beta$$

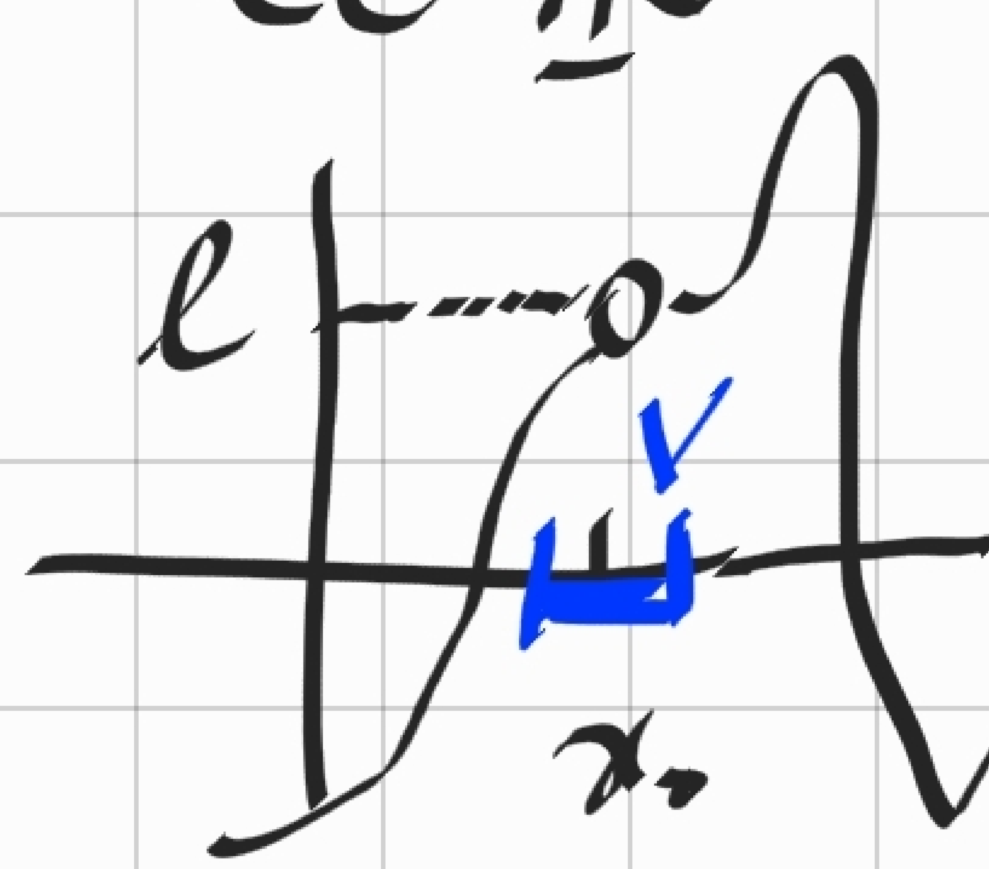
$$f(x)+g(x) \to +\infty \quad \text{per } x \to x_0$$

"Se f molto grande g non molto negativo allora $f+g$ è molto grande".

Teorema della permanenza del segno

Se $\lim_{x \to x_0} f(x) = l > 0$ $\quad l \in \overline{\mathbb{R}}$

Allora esiste un intorno V di x_0 tale che $f(x) > 0 \quad \forall x \in V$.



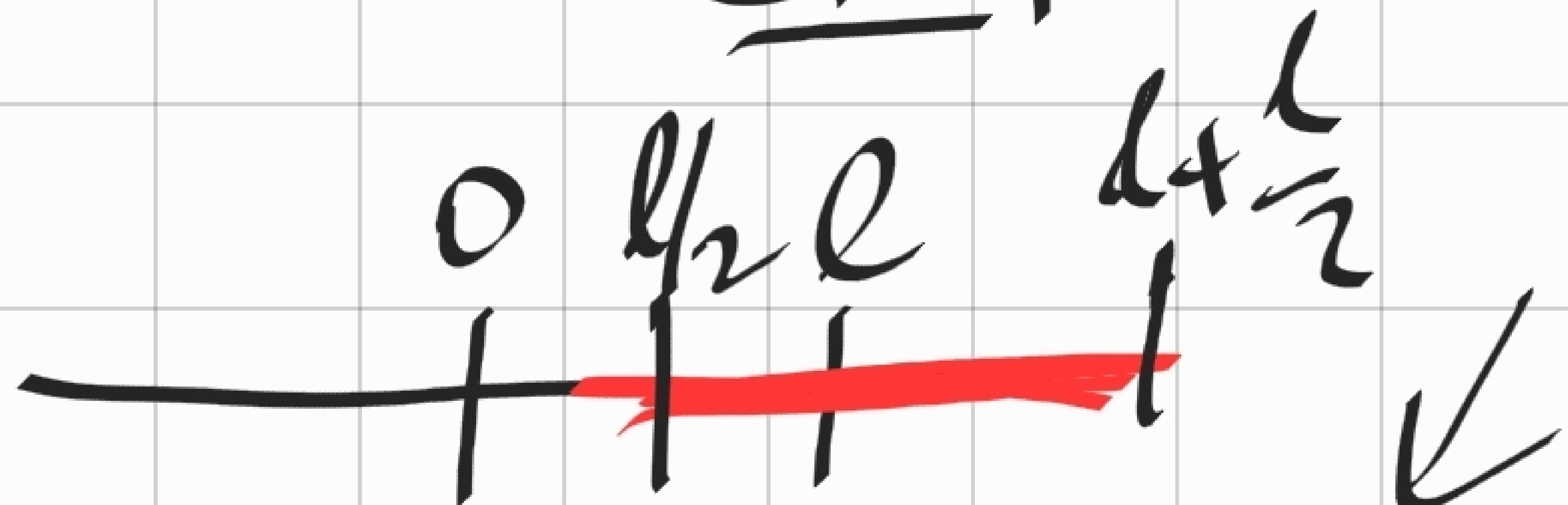
dim (più generale)

$f(x) \to \ell$ per $x \to x_0$

$\forall U \in B_\ell : \exists V \in B_{x_0} : \forall x \neq x_0$

$x \in V \Rightarrow \boxed{f(x)} \in U$

Se $\ell > 0$



$\exists U \in B_\ell : U \subseteq (0, +\infty]$

$U = \left(\frac{\ell}{2}, \ell + \frac{\ell}{2}\right)$ se $\ell \in \mathbb{R}$

$U = (1, +\infty)$ se $\ell = +\infty$

$\square$

Differenza:

$f(x) - g(x) = f(x) + (-g(x))$

$= f(x) + h(g(x))$

$$h(y) = -y$$

$$g(x) \to \ell_2 \qquad -g(x) \to -\ell_2$$

se ℓ_2 finito è la continuità di h.

Se $\ell_2 = +\infty$ | $g(x) \to +\infty$, $-g(x) \to -\infty$

$$\boxed{-(+\infty) = -\infty}$$

$$\boxed{-(-\infty) = +\infty}$$

FORME INDETERMINATE: $\boxed{\begin{array}{c}(+\infty)-(+\infty)\\(-\infty)+(-\infty)\end{array}}$

Prodotto

$x \to x_0$

$$\left.\begin{array}{l} f(x) \to l_1 \\ g(x) \to l_2 \end{array}\right\} \Rightarrow f(x) \cdot g(x) \to l_1 \cdot l_2$$

Se $l_1, l_2 > 0$
in un intorno di x_0

$$f(x) > 0 \quad e \quad g(x) > 0$$

Passando ai logaritmi: $\boxed{a > 1}$

$$\log f(x) \cdot g(x) = \log f(x) + \log g(x)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\log(l_1 \cdot l_2) = \log l_1 + \log l_2$$

$$\uparrow \qquad \text{se } \boxed{l_1, l_2 \in \mathbb{R}} \qquad \uparrow$$

$$f(x) \cdot g(x) \to l_1 \cdot l_2$$

Se $l_1 = +\infty$ $\quad \boxed{l_2 > 0}$

$f(x) \to \underline{+\infty}$ $\quad \log f(x) \to \underline{+\infty}$ $l_1 \in \mathbb{R}$

$g(x) \to l_2$ $\quad \begin{cases} \log g(x) \to \log(l_2) \\ \log g(x) \to \underline{+\infty} \end{cases}$

$l_1 = +\infty$

$\log f(x) + \log g(x) \longrightarrow +\infty$

$f(x) \cdot g(x) \to +\infty$.

$l_1 = +\infty \qquad l_2 \in (0, +\infty]$

$$\boxed{(+\infty) \cdot l_2 = +\infty}$$

$$\boxed{(+\infty) \cdot (+\infty) = +\infty}$$

$$\boxed{+\infty \cdot 0}$$ FORMA

$\{\log$ INDET.

$\boxed{+\infty + (-\infty)}$

$$\boxed{\begin{aligned}(-\infty)\cdot l_2 &= -\infty \\ (-\infty)\cdot(+\infty) &= -\infty\end{aligned}}$$ se $l_2 > 0$

$$\boxed{\begin{aligned}+\infty\cdot(-l_2) &= -\infty \\ +\infty\cdot(-\infty) &= -\infty \\ (-\infty)\cdot(-l_2) &= +\infty \\ (-\infty)\cdot(-\infty) &= +\infty\end{aligned}}$$

$\boxed{\infty \cdot 0}$

Rapporto

$$\left.\begin{array}{l} f(x) \to l_1 \\ g(x) \to l_2 \end{array}\right\} \quad \frac{f(x)}{g(x)} \to \frac{l_1}{l_2}$$

$$\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$$

$$= f(x) \cdot r(g(x))$$

$$r(y) = \frac{1}{y} \leftarrow \quad \left(\frac{1}{+\infty} = 0\right)$$

$$\frac{0}{+\infty} = 0 \qquad 0 \cdot 0 = 0$$

$$\frac{0}{-\infty} = 0 \qquad \text{"} \left|\frac{+\infty}{0}\right| = +\infty \text{"}$$

$$\frac{+\infty}{+\infty} \quad \frac{+\infty}{-\infty} \quad \frac{-\infty}{+\infty} \quad \frac{-\infty}{-\infty}$$

FORME INDET.

$$\frac{0}{0} \quad \frac{+\infty}{0} \quad \frac{-\infty}{0} \quad \frac{l_1}{0}$$

[1

$$\frac{l_1}{0^+} = \begin{cases} +\infty & \text{se } l_1 > 0 \\ -\infty & \text{se } l_1 < 0 \end{cases}$$

]

$$\frac{+\infty}{0^+} = +\infty \qquad \frac{+\infty}{0^-} = -\infty$$

"

.....

$$\Big[\quad \lg(+\infty \cdot 0) = (\lg +\infty) + \lg 0$$

$$= \underset{\nearrow}{+\infty} + (\underset{\nearrow}{-\infty}) = ?? \Big]$$

Es $\lim_{x \to +\infty} \dfrac{x + x^2}{2 - \frac{1}{x}}$

per $x \to +\infty$

$x \to +\infty$

$x^2 = x \cdot x \to [+\infty \cdot +\infty] = +\infty$

$x + x^2 \to +\infty + (+\infty) = +\infty$

$\frac{1}{x} \to 0 \qquad -\frac{1}{x} \to -0 = 0$

$2 - \frac{1}{x} \to 2 - 0 = 2$

$$\frac{x+x^2}{2-\frac{1}{x}} \to \frac{+\infty}{2} = +\infty \cdot \frac{1}{2} = +\infty$$

$$\lim_{x\to+\infty} \frac{x+x^2}{2-\frac{1}{x}} = +\infty \qquad \square$$

ERRORE!

Es (sbagliato)

NON GIUSTIFICATO

$$\lim_{x\to+\infty} \frac{x+x^2}{2-\frac{1}{x}} = \lim_{x\to+\infty} \frac{x+x^2}{2-0}$$

Non è consentito porre al limite solo in una sottoespressione

$= +\infty$ è giusto

$$x^3 = x \cdot x \cdot x \quad << x$$

Es $\lim\limits_{x\to0^+} \dfrac{\cancel{x}+x^3}{x^2}$

NO $= \lim\limits_{x\to0^+} \dfrac{x^3}{x^2} = \lim\limits_{x\to0^+} x = 0$ NO

INGIUSTIFICATO

$$\frac{x+x^3}{x^2} \overset{\forall x}{=} \frac{x\cdot(1+x^2)}{x^2}$$

per $x\to0^+$

$$= \frac{1+x^2}{x} = \frac{1}{x} + x \to +\infty$$

$$\to +\infty + 0 = +\infty$$

$\frac{1}{x} \to +\infty$ se $x \to 0^+$

$$\lim_{x\to 0^+} \frac{x+x^3}{x^2} = \lim_{x\to 0^+} \frac{x(1+x^2)}{x^2}$$

$$\ldots = \lim_{x\to 0^+} \left(\frac{1}{x}+x\right) = +\infty$$

OK MA SCONSIGLIATO.

$$\left[\begin{array}{l} \lim_{x\to x_0} f(x) = \ell \\ \\ \quad f(x) \longrightarrow \ell \quad \text{per } x \to x_0 \end{array}\right]$$

$$\frac{x+x^3}{x^2} = \ldots = \frac{1}{x}+x \overset{\text{NO}}{=} +\infty$$

$$\frac{x+x^3}{x^2} \overset{\text{NO}}{=} \lim \frac{1}{x}+x$$