Analisi Matematica B
Lezione 21 - 11.11.2020

Criteri di confronto

<u>Teorema</u> della permanenza del segno

$$\left.\begin{array}{l} f(x)\to \ell>0 \\ \text{per } x\to x_0 \end{array}\right| \begin{array}{l} \exists V \text{ intorno di } x_0 \\ \forall x\in V,\ x\neq x_0,\ f(x)>0 \end{array}$$

<u>Notazione</u>: proprietà <u>definitive</u> e <u>frequenti</u>

Per $x\to x_0$ $\quad P(x)$ predicato.

- diremo che $P(x)$ vale **definitivamente** se esiste V intorno di x_0 tale che
$$\forall x\in V\setminus\{x_0\} : P(x).$$
- diremo che $P(x)$ vale **frequentemente** se per ogni V intorno di x_0
$$\exists x\in V\setminus\{x_0\} : P(x).$$

Sono proprietà "complementari" $\quad \begin{array}{|l} \neg\,\text{freq}\,\neg = \text{def} \\ \neg\,\text{def}\,\neg = \text{freq} \end{array}$

$\neg$ frequentemente $P(x)$
è equivalente a
definitivamente $\neg P(x)$.

$(\text{def } P) \wedge (\text{def } Q)$
$= \text{def } (P \wedge Q)$

$\text{freq } P \vee \text{freq } Q$
$= \text{freq } P \vee Q$

Esempi $A = \mathbb{N}$ $n \to +\infty$

def. $n > 10'000$

def. $2^n \geq n^2 \iff \neg \text{ freq } 2^n < n^2$

freq. n è pari $\iff \neg$ def. n dispari

Teorema permanenza del segno:

Se $f(x) \to l$ per $x \to x_0$

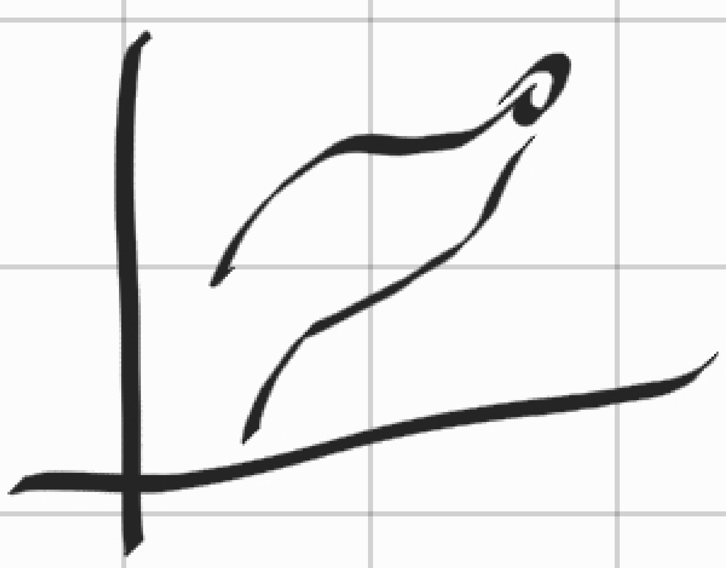
$l > 0$ allora $f(x) > 0$
definitivamente.

Confronti: $x \to x_0$

$$(\le)$$

(1) $\left.\begin{array}{l} f(x) \geq g(x) \\ f(x) \to l_1 \\ g(x) \to l_2 \end{array}\right\}$ allora $\quad \overset{(\le)}{l_1 \geq l_2}$

dim Se fosse $l_1 < l_2$



$$g(x) - f(x) \to l_2 - l_1 > 0$$

$g(x) - f(x) > 0$ definitivamente

$g(x) > f(x)$ assurdo

$f(x) \geq g(x)$ □

(2) $\left.\begin{array}{l} \text{Se } g(x) \to +\infty \\ \text{e } f(x) \geq g(x) \end{array}\right\}$ allora $f(x) \to +\infty$

dim $\forall\, U = (\alpha, +\infty]$: def: $g(x) \in U$

$g(x) > \alpha$

Se $f(x) \geq g(x) > \alpha$

def: $f(x) \in U$ □

(3) (teorema dei due carabinieri)

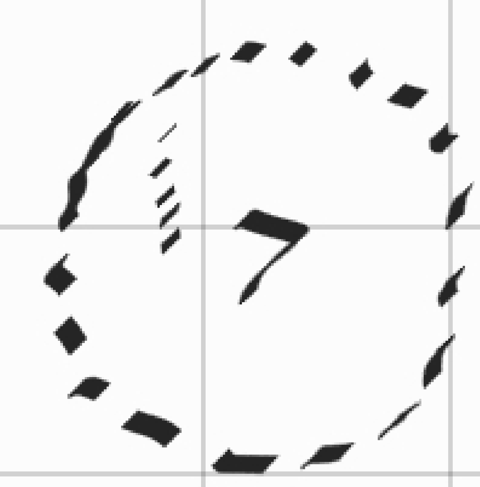
$$\left.\begin{array}{l} f(x) \le g(x) \le h(x) \\ f(x) \to l \\ h(x) \to l \end{array}\right\} \text{allora } g(x) \to l$$

dim $\forall U$ intorno di l def. $\begin{cases} f(x) \in U \\ h(x) \in U \end{cases}$

è un intervallo $\Rightarrow$ def $g(x) \in U$.

$g(x)$ è un punto intermedio tra $f(x)$ e $h(x)$ $\square$

Esempio $\lim_{x \to +\infty} \frac{1}{x^2} = 0$



$\frac{1}{x^2} > 0 \to 0$

dim (con gli intorni) per $x \to x_0$

$f(x) \to l$

$h(x) \to l$

$\forall U \in \mathcal{B}_\ell \; \exists V_1 \in \mathcal{B}_{x_0} : x \neq x_0, x \in V_1 \Rightarrow f(x) \in U$

$\forall U \in \mathcal{B}_\ell \; \exists V_2 \in \mathcal{B}_{x_0} : x \neq x_0, x \in V_2 \Rightarrow h(x) \in U$

$\exists V = V_1 \cap V_2 : x \neq x_0, x \in V_1 \cap V_2 \Rightarrow \begin{cases} f(x) \in U \\ h(x) \in U \end{cases} \Rightarrow g(x) \in U$

$U \in \mathcal{B}_\ell$, U è un intervallo

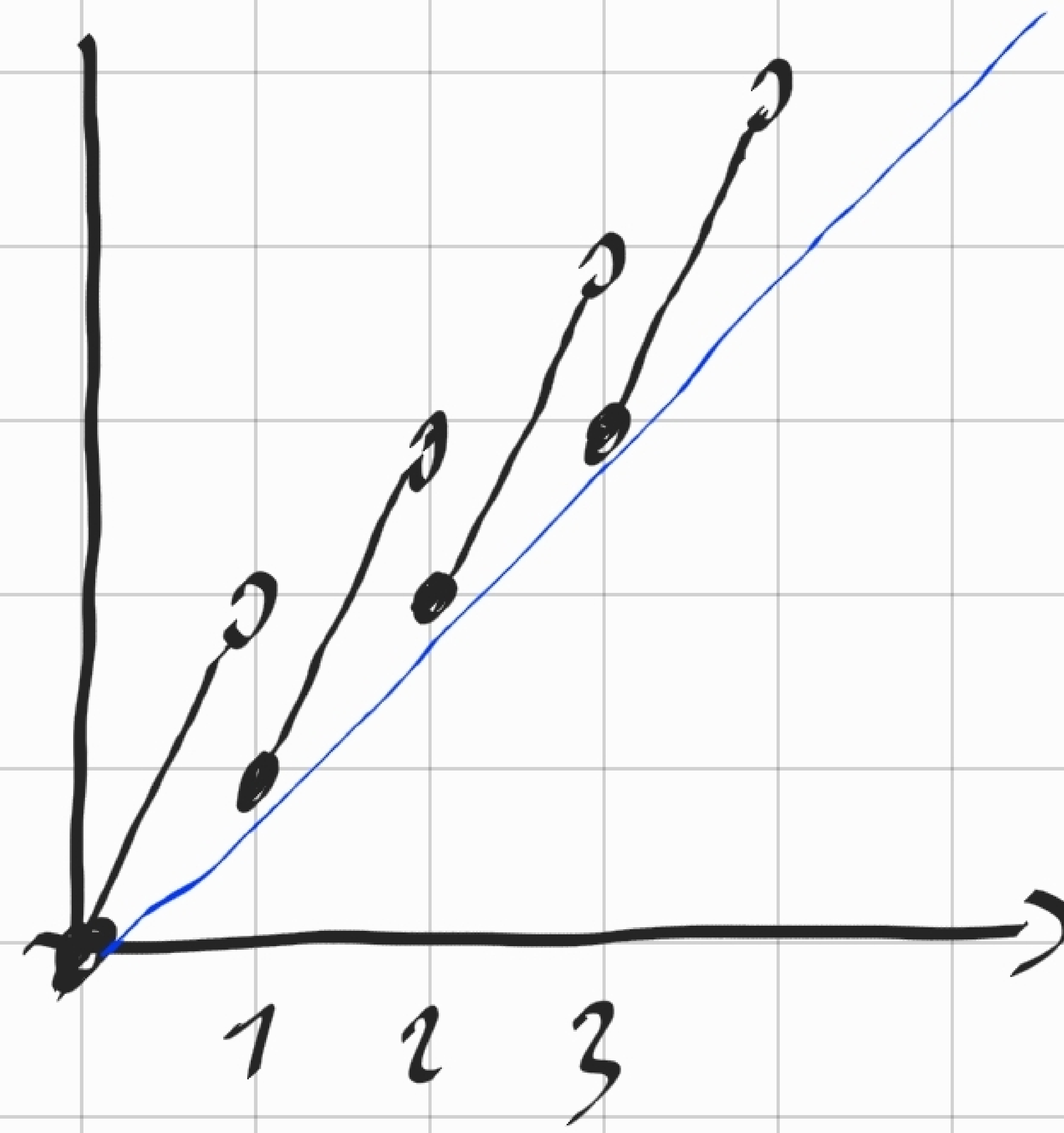
$(\ell \in \mathbb{R},\quad U = (\ell - \varepsilon, \ell + \varepsilon)$

$\ell - \varepsilon < f(x) \leq g(x) \leq h(x) < \ell + \varepsilon \quad \square$

Esempio $f(x) = 2x - \lfloor x \rfloor$

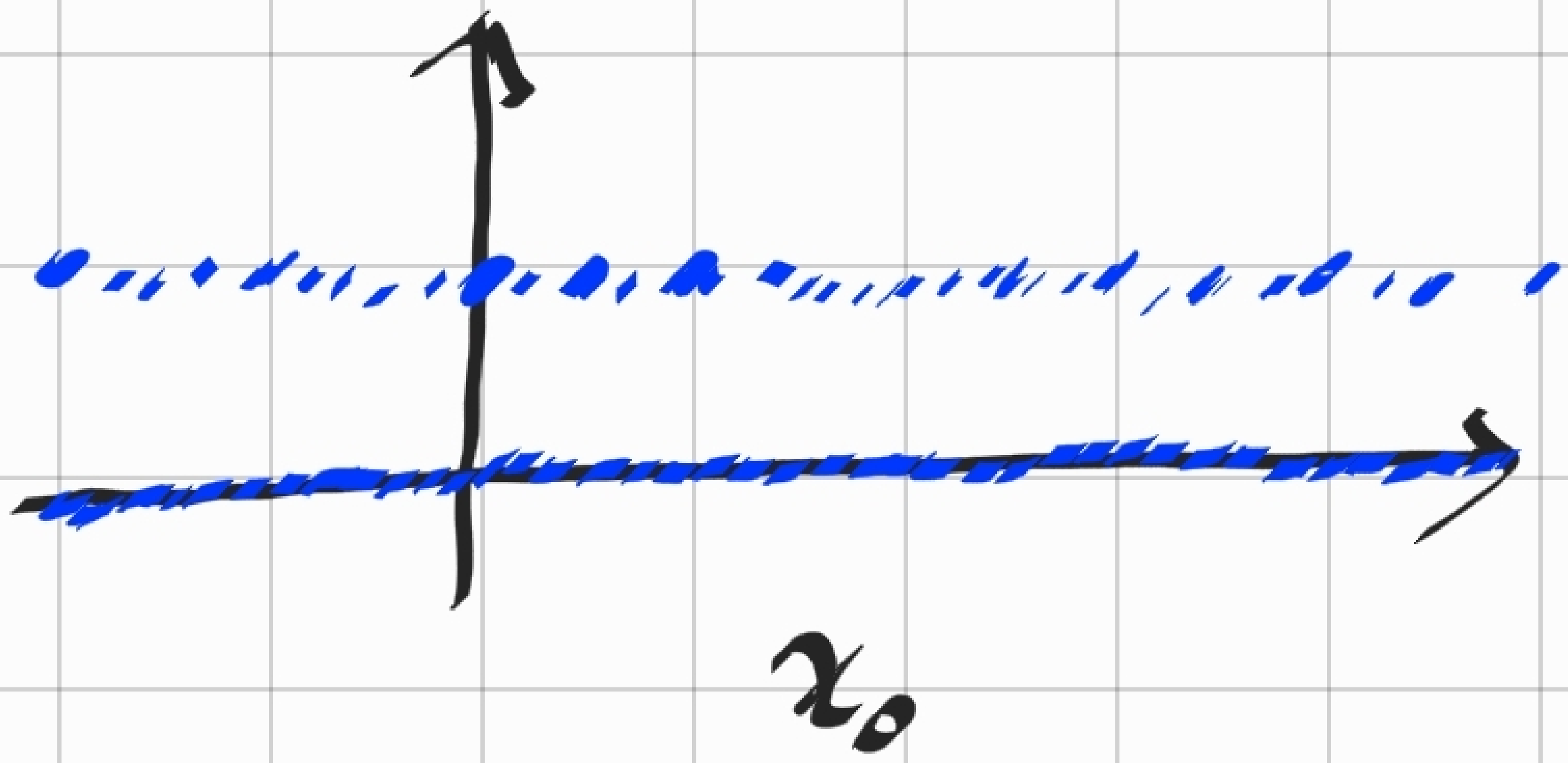
$\lim_{x \to +\infty} f(x) \stackrel{?}{=} +\infty$

$2x - \lfloor x \rfloor = x + (x - \lfloor x \rfloor) \geq x \to +\infty$



Esempio (limite che non esiste mai)

$$f(x)=\begin{cases}1 & \text{se } x\in\mathbb{Q}\\ 0 & \text{se } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$$

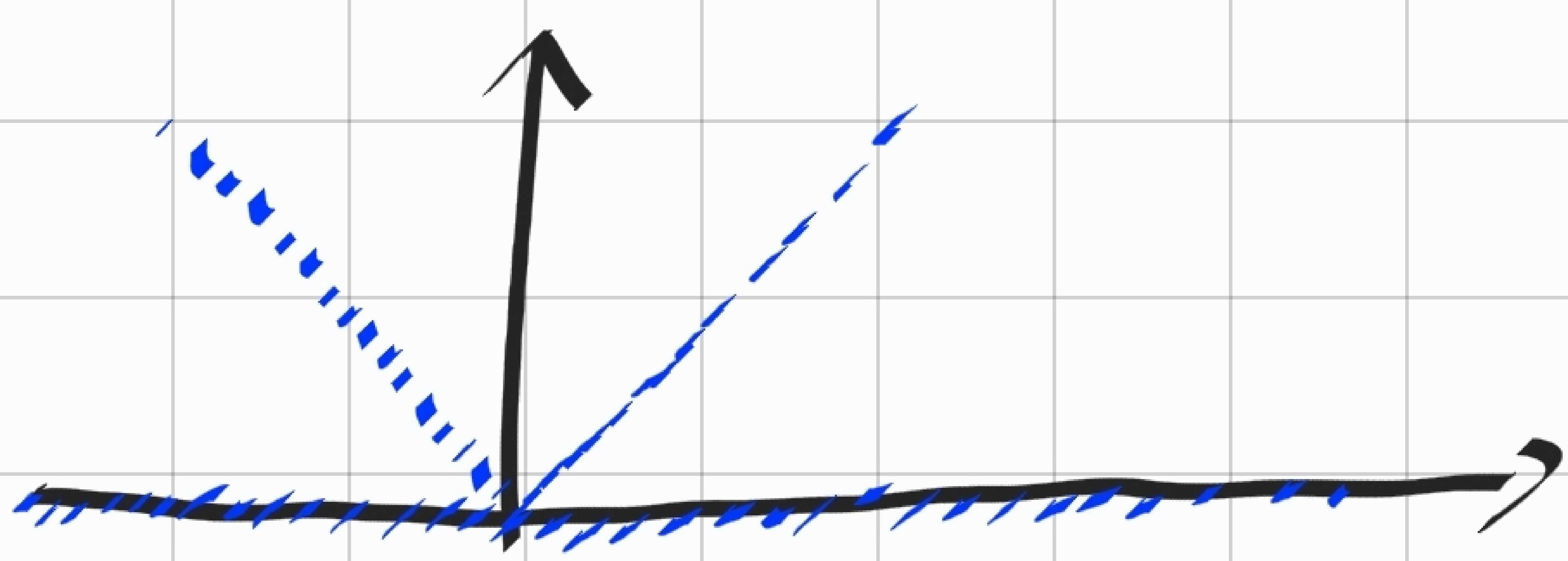


$$\lim_{x\to x_0} f(x) \text{ non esiste}$$

<u>Esempio</u>

$$g(x)=|x|\cdot f(x)$$

$$g(x)=\begin{cases}|x| & \text{se } x\in\mathbb{Q}\\ 0 & \text{se } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$$



$$\lim_{x\to 0} g(x)=0$$

$$\Uparrow$$

$$\begin{array}{ccccc}0 & \le & g(x) & \le & |x|\\ \downarrow & & & & \downarrow\\ 0 & & & & 0\end{array}$$

$\square$

Corollario

f infinitesima

Se $f(x) \to 0$ e $g(x)$ limitata $\Big\}$ allora $f(x) \cdot g(x) \to 0$

dim g limitata: $\exists R > 0$

$$|g(x)| \leq R$$

$$(\text{ovvero } -R \leq g(x) \leq R)$$

$$0 \leq \boxed{|f(x) \cdot g(x)|} = |f(x)| \cdot |g(x)|$$

$$\leq |f(x)| \cdot R$$

$f \to 0$ (f), g, h

$$|f(x)| \cdot R \to 0 \cdot R = 0$$

$$\Downarrow$$

$$|f(x) \cdot g(x)| \to 0$$

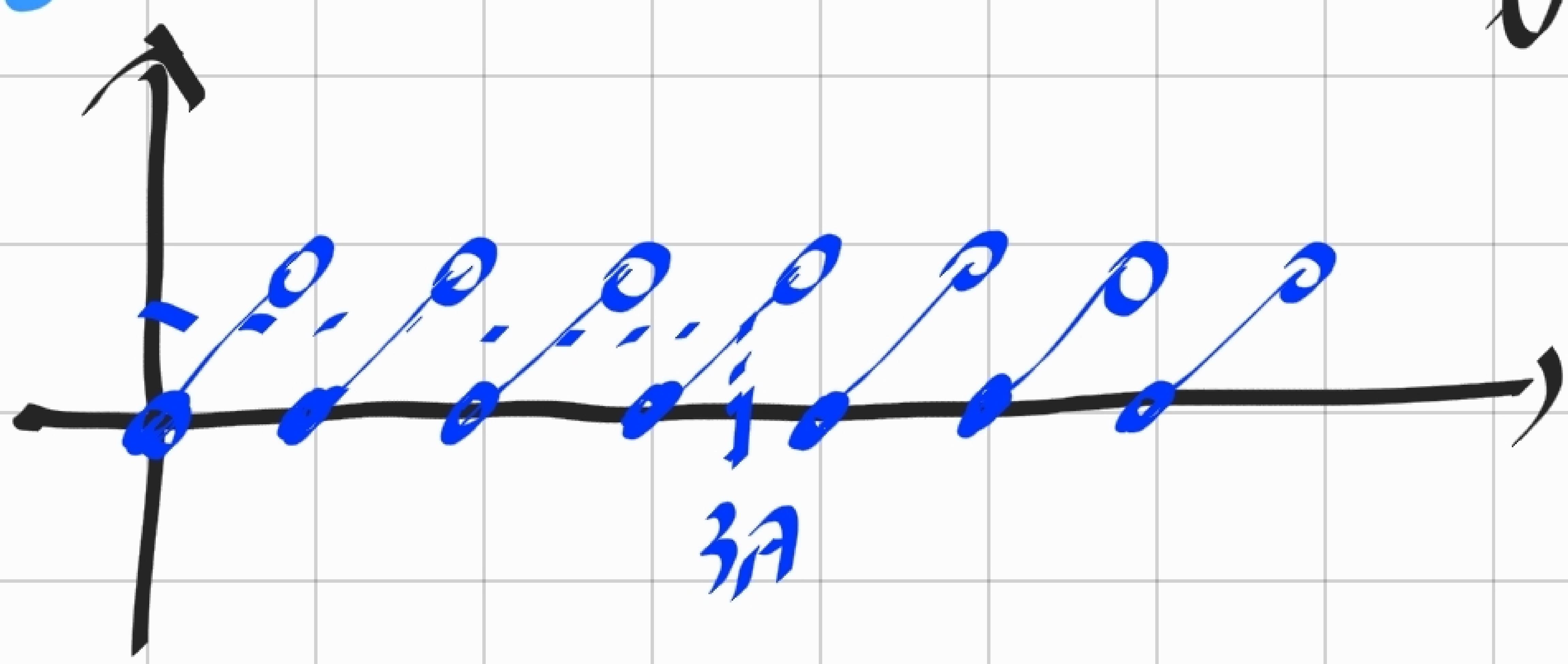
$$\Rightarrow f(x) \cdot g(x) \to 0$$

$\square$

Esempio $\displaystyle\lim_{x\to+\infty} \frac{x-\lfloor x\rfloor}{x} = 0$

$f(x)=\frac{1}{x}\to 0$ per $x\to+\infty$

$g(x) = x-\lfloor x\rfloor \leftarrow$ parte frazionaria



$$|g(x)|\le 1$$

$f(x)\to 0 \iff |f(x)|\to 0$

$\vdots \qquad\qquad \vdots$

$|f(x)-0|<\varepsilon \qquad ||f(x)|-0|<\varepsilon$

$|f(x)|<\varepsilon \qquad ||f(x)||<\varepsilon$

SUCCESSIONI

$\underline{a} = \mathbf{a} = \vec{a}$

$\underline{a} : \mathbb{N} \to \mathbb{R}$

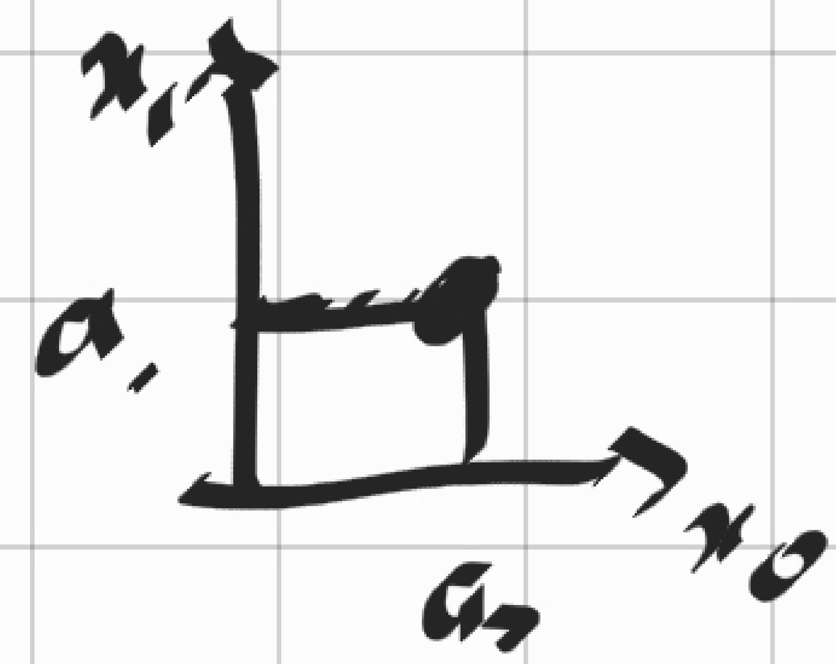
$f : A \to B$

$f \in B^A$

$\underline{a} \in \mathbb{R}^{\mathbb{N}}$

$$\mathbb{R}^2 \cong \mathbb{R} \times \mathbb{R} \ni (a_0, a_1)$$

$$\mathbb{R}^{\{0,1\}}$$



$$\underline{a} : \{0,1\} \to \mathbb{R}$$

$$0 \longmapsto a_0$$

$$1 \longmapsto a_1$$

$$\underline{a} = \{0 \mapsto a_0,\ 1 \mapsto a_1\}$$

$$= (a_0, a_1).$$

$\underline{a} \in \mathbb{R}^n$ $\quad \underline{a} = (\underbrace{a_0, a_1, \dots, a_{n-1}}_{n \text{ componenti}})$

$\underline{a} \in \mathbb{R}^{\mathbb{N}}$ $\qquad$ $\underline{a} : \mathbb{N} \to \mathbb{R}$

$$\begin{aligned} 0 &\mapsto a_0 \\ 1 &\mapsto a_1 \\ &\vdots \\ n &\mapsto a_n \\ &\vdots \end{aligned}$$

$$\underline{a} = \underbrace{(a_0, a_1, a_2, \ldots\ldots, a_n, \ldots\ldots)}_{\# \mathbb{N}}$$

__Es__ $\qquad \underline{a}(n) = n^2$

$$\begin{aligned} \underline{a}(0) &= 0 \qquad & a_0 &= 0 \\ \underline{a}(1) &= 1 & a_1 &= 1 \\ \underline{a}(2) &= 4 & a_2 &= 4 \\ \underline{a}(3) &= 9 & &\vdots \\ &\vdots \end{aligned}$$

$\underline{a} = (0, 1, 4, 9, 16, \ldots, k^2, \ldots)$

<u>Esempio</u> a_n = n-esimo numero primo.

<u>Esempio</u> $a_n = n!$

termini

$\underline{a} = (1, 1, 2, 6, 24, 120, \ldots)$

<u>Esempio</u>

$$\begin{cases} a_0 = \alpha \\ a_{n+1} = f(a_n) \end{cases}$$

$$\begin{cases} a_0 = 1 \\ a_{n+1} = \dfrac{1}{1+a_n} \end{cases}$$

$a_0 = 1$

$a_1 = \dfrac{1}{1+1} = \dfrac{1}{2}$

$a_2 = \dfrac{1}{1+\frac{1}{2}} = \dfrac{2}{3}$

$a_3 = \dfrac{1}{1+\frac{2}{3}} = \dfrac{3}{5}$

Esempio $a_n = \sum_{k=1}^{n} b_k$

$$a_n = \sum_{k=1}^{n} \frac{1}{k!}$$

$a: \mathbb{N} \to \mathbb{R}$ $A = \mathbb{N}$

l'unico punto di accumulazione per A è $+\infty$

$n \in \mathbb{N}$

Possiamo fare: $\lim_{n \to +\infty} a_n$

solo

Scriveremo a volte:

$\lim a_n$ $\lim_{n} a_n$

$a_n \to l$

Carattere di una successione

$\lim a_n$ — esiste
— non esiste

a_n — regolare se $\exists \lim a_n$
— indeterminata se $\nexists \lim$

carattere

a_n regolare — convergente se $l \in \mathbb{R}$
— divergente se l infinito.

$a_n \to l$

indeterminata — limitata $\sup |a_n| < +\infty$
— illimitata $\sup |a_n| = +\infty$

a termini positivi se $a_n > 0$ $\forall n$.

a segni alterni

$$\begin{cases} a_n > 0 & \text{se } n \text{ pari} \\ a_n < 0 & \text{se } n \text{ dispari} \end{cases}$$

<u>Esempio</u>

$$a_n = (-1)^n = \begin{cases} 1 & \text{se } n \text{ pari} \\ -1 & \text{se } n \text{ dispari} \end{cases}$$

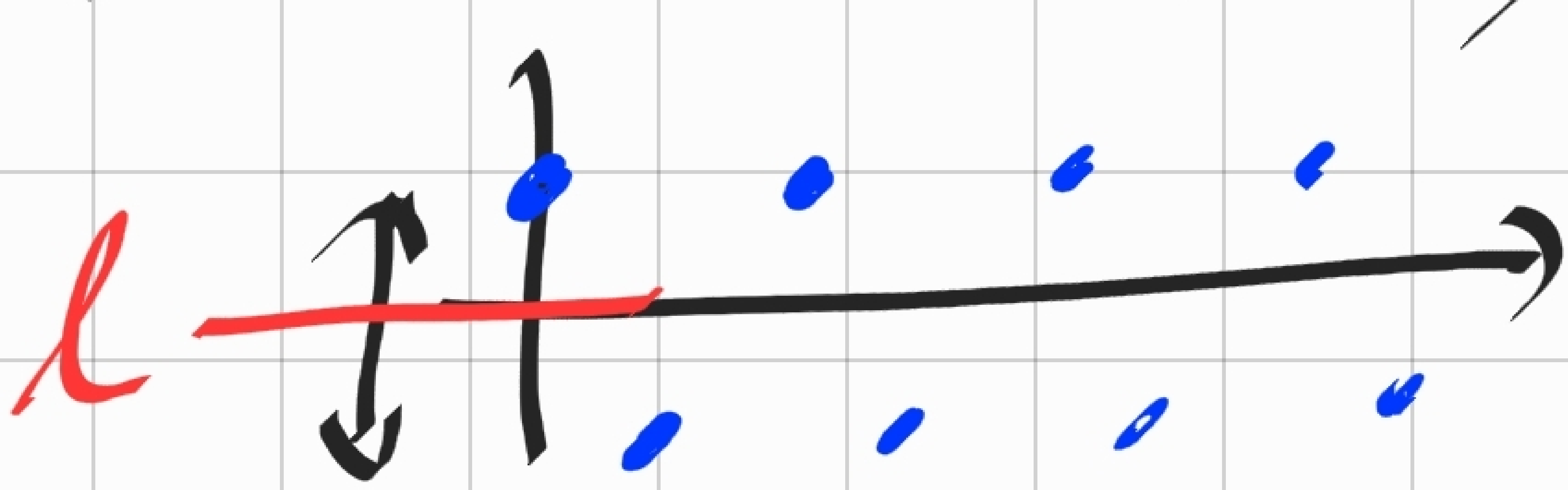
è a segni alterni

è indeterminata.

<u>dim</u> Se per assurdo $(-1)^n \to l$

$$-1 \le a_n \le 1 \Rightarrow -1 \le l \le 1$$

$$\forall \varepsilon > 0 \ \exists \alpha : n > \alpha \Rightarrow |(-1)^n - l| < \varepsilon$$

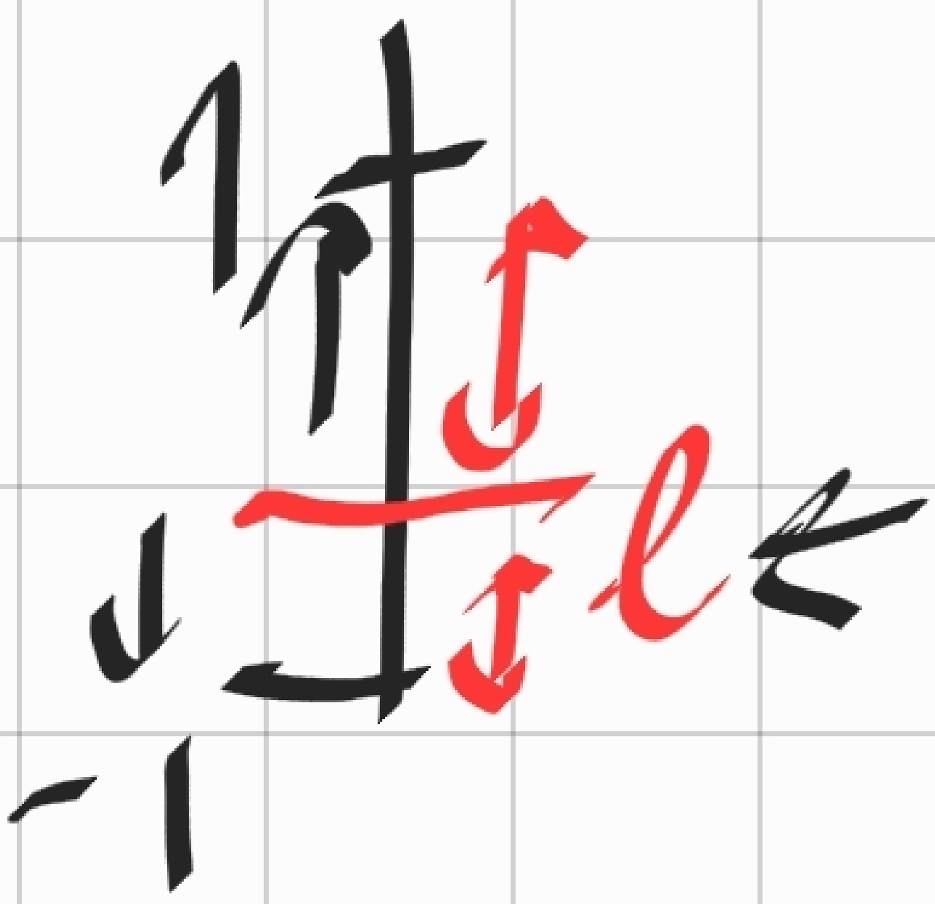


Scelgo $\varepsilon = 1$: def: $|a_n - \ell| < 1$

freq: $a_n = 1$

freq: $a_n = -1$

$$2 = |1-(-1)| \le \underbrace{|1-\ell|}_{\text{def}} + |\ell-(-1)| < 2\varepsilon < 2$$

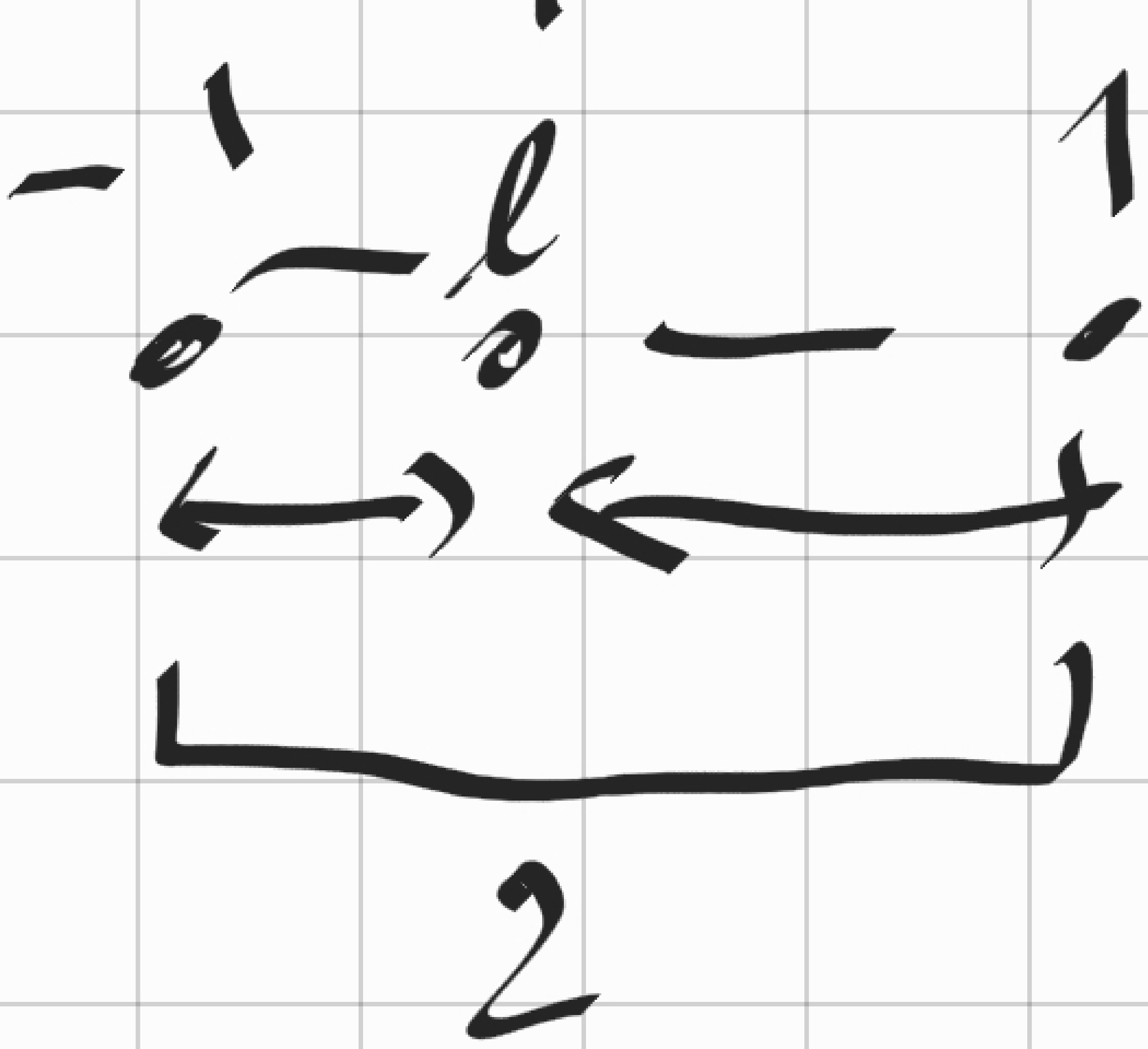


$2 < 2$ assurdo $\square$

$\exists \alpha : n > \alpha \quad |a_n - \ell| < \varepsilon$

esiste n_1 pari $> \alpha \quad |1-\ell| < \varepsilon$

esiste n_2 dispari $> \alpha \quad |-1-\ell| < \varepsilon$



frequentemente $a_n = 1$ $a_n = -1$

definitivamente $|a_n - \ell| < \varepsilon$

---

frequentemente $|1 - \ell| < \varepsilon$

$|-1-\ell| < \varepsilon$

Notazione invece di scrivere

$\underline{a}$ si scrive a_n

$\underline{a} = (a_0, a_1, a_2, \dots)$

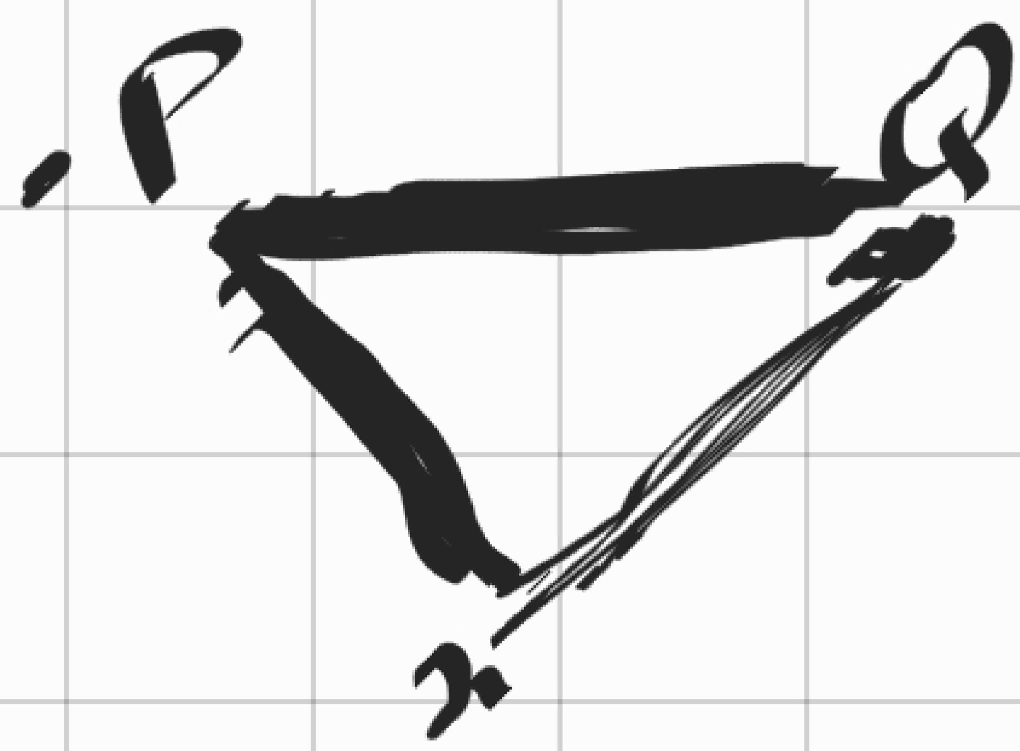
$\underline{a} = \underbrace{(a_n)_{n \in \mathbb{N}}}$

a_{10}

$\underline{a}$ è a_n

Come $f(x)$ invece di f

$x \mapsto f(x)$



$$\Rightarrow d(P,Q) \leq \underline{d(P,x)} + \underline{d(x,Q)}$$

$$|P-Q| \leq |P-x| + |x-Q|$$

$$|1-(-1)| \leq |1-\ell| + |\ell-(-1)|$$

$$|x+y| \leq |x| + |y|$$

$$-|x| - |y| \leq x+y \leq |x| + |y|$$

$$-|x| \leq x \leq |x|$$

$$-|x| \leq x \leq |x|$$

$$-|y| \leq y \leq |y|$$

$$-|x|-|y| \leq x+y \leq |x|+|y|$$

$$\Updownarrow$$

$$|x+y| \leq |x|+|y|$$

$$|x-y| = |(x-z)+(z-y)|$$

$$\leq |x-z| + |z-y|$$

$$-1 \leq l \leq 1$$

$$1-l + l-(-1) = 2$$

$$\shortparallel$$

$$|1-l| + |l-(-1)|$$