ANALISI MATEMATICA B
LEZIONE 34 - 11.12.2020

$$\sqrt{n!} \gg 2^{\sqrt{n}}$$

$$\frac{2^{\sqrt{n+1}}}{2^{\sqrt{n}}} \cdot \frac{\sqrt{\cancel{n!}}}{\sqrt{(n+1)\cancel{!}}} = \frac{2^{\sqrt{n+1}}}{\sqrt{n+1} \cdot 2^{\sqrt{n}}}$$

$$= \frac{2^{\sqrt{n+1}}}{\sqrt{2^{2\sqrt{n}} \cdot (n+1)}} = \frac{2^{\sqrt{n+1}}}{\sqrt{n}\, 2^{\sqrt{n}} \sqrt{1+\frac{1}{n}}}$$

$$\boxed{\sqrt{n} \cdot 2^{\sqrt{n}} \gg 2^{\sqrt{n+1}}}$$

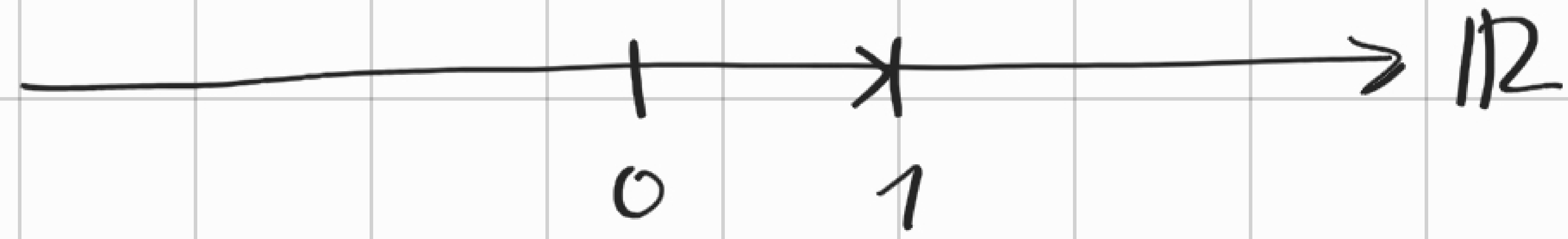
$$\frac{2^{\sqrt{n+1}}}{2^{\sqrt{n}}} = 2^{\sqrt{n+1}-\sqrt{n}}$$

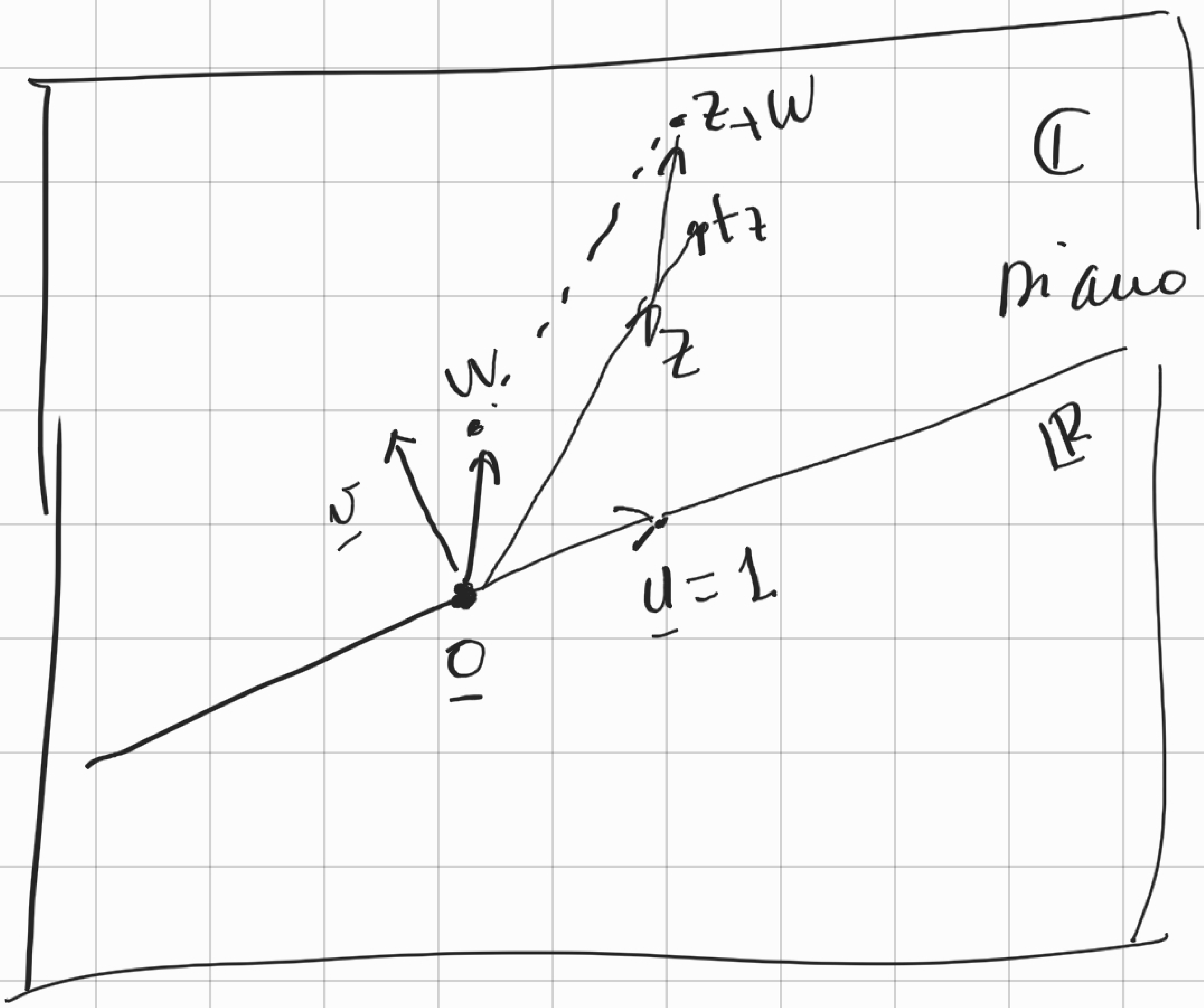
$$= 2^{\frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}} \longrightarrow 2^{\frac{1}{+\infty}} = 1$$

NUMERI COMPLESSI $\mathbb{C}$

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

$+, \cdot$ $-$ $:$ $\sqrt{\cdot}$ $z^2+1=0$ ha soluzione





piano di Gauss

$$\forall\ \underbrace{z = x\cdot\underline{u} + y\cdot\underline{v}}\ \longrightarrow\ (x,y)$$

$$\mathbb{C} \simeq \mathbb{R}\times\mathbb{R}$$

$\rightarrow w = a \cdot \underline{u} + b\underline{v}$

$$z + w = (x+a)\,\underline{u} + (y+b)\,\underline{v}$$

(la somma deve soddisfare le proprietà: commutativa, associativa.)

$t \in \mathbb{R}$ $\qquad z = x\underline{u} + y\underline{v}$

$$t \cdot z = (tx)\,\underline{u} + (ty)\,\underline{v}$$

Vogliamo definire $z \cdot w$

$$z \cdot w = (x\underline{u} + y\underline{v}) \cdot (a\underline{u} + b\underline{v})$$

$$= xa\,\underline{u}\cdot\underline{u} + yb\,\underline{v}\,\underline{v} + xb\,\underline{u}\,\underline{v} + ya\,\underline{v}\,\underline{u}$$

$$= (xa - yb)\,\underline{u} + (xb + ya)\,\underline{v}$$

Supponiamo che $\underline{u}$ sia elemento neutro della moltiplicazione.

$$\underline{u} \cdot \underline{u} = \underline{u} \qquad \underline{u} \cdot \underline{v} = \underline{v}$$

$$\underline{v}\,\underline{v} = ? \qquad \underline{v}^2 = ?$$

$$\underline{v}^2 = -\underline{u} \qquad z^2+1=0$$

$$\mathbb{C}, \underline{u}, \underline{v}, +, \cdot \quad \mathbb{C}$$

Identifichiamo $\quad x \cdot \underline{u} = x \cdot 1 = x$

$$\underline{u} = \underline{1} \in \mathbb{C} \qquad \mathbb{R}$$

$$1 \in \mathbb{R}$$

$$\mathbb{R} \subseteq \mathbb{C}. \qquad i = \underline{v} \qquad i^2 = -1.$$

$z = x + iy$ ← numero complesso.

reale   immaginario $x, y \in \mathbb{R}$

$w = a + ib$

$x = \mathrm{Re}\, z$ = parte reale di z

$y = \mathrm{Im}\, z$ = parte immaginaria

$\mathrm{Im}\, z \in \mathbb{R}$

$z + w = (x+a) + i(y+b)$

$z \cdot w = (x+iy)\cdot(a+ib)$

$= xa + iya + ixb + i^2 yb$

→ $i^2 = -1$

$= (xa - yb) + i(ya + xb)$

Si verifica che valgono le proprietà di quello:

- commutativa per somma e prodotto
- associativa per somme e prodotto.
- 0 elemento neutro per la somma. $= 0 + 0i$
- opposto per la somma $-z = -(x + iy) = (-x) + i(-y)$
- proprietà distributiva $(z + w) \cdot u = z \cdot u + w \cdot u$

$\mathbb{C}$ è un campo?

Dato $z \neq 0$ esiste $w \in \mathbb{C}$

t.c. $z \cdot w = 1$?

$z = x + iy$ $\qquad z \neq 0 \Leftrightarrow x^2 + y^2 > 0$

$$w = \frac{x - iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$z \cdot w = \frac{(x+iy)(x-iy)}{x^2+y^2} = \frac{x^2 - i^2y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2}{x^2+y^2} = 1.$$

$w = z^{-1} = \frac{1}{z}$

Esercizio $i \neq 0$ chi è $\frac{1}{i} = ?$

$i^2 = -1 \qquad i \cdot i = -1$

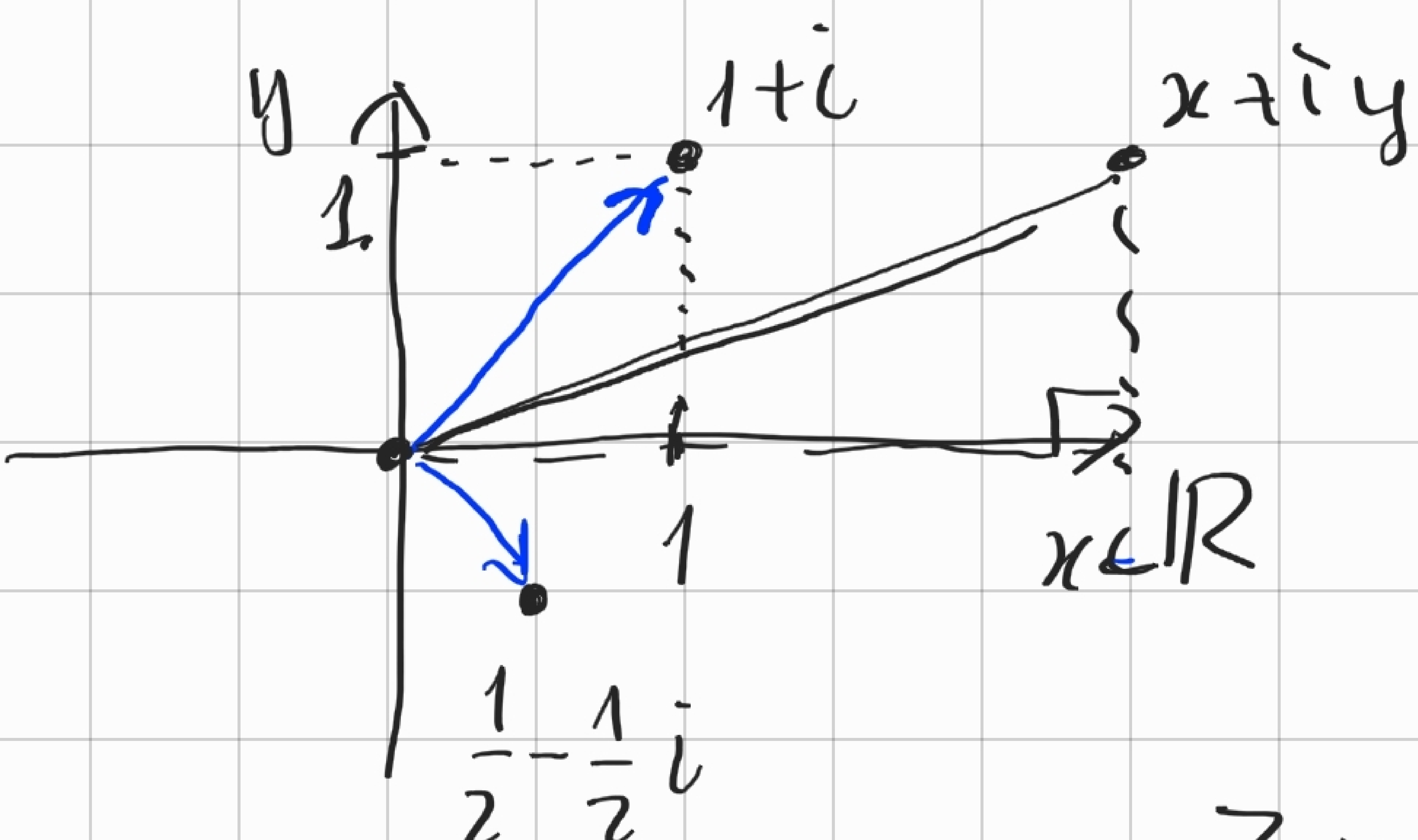
$$i \cdot (-i) = 1$$

$$\boxed{-i = \frac{1}{i}}$$

Esercizio

$$\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)}$$

$$= \frac{1-i}{1-i^2} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2} i$$



$$z = x + iy$$

Modulo $\quad |z| = \sqrt{x^2 + y^2}$

Se $z \in \mathbb{R}$ $\quad z = x + 0 \cdot i \; (y=0)$

$$z = x \qquad |z| = \sqrt{x^2 + 0^2} = \sqrt{x^2}$$

$$= \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x < 0 \end{cases}$$

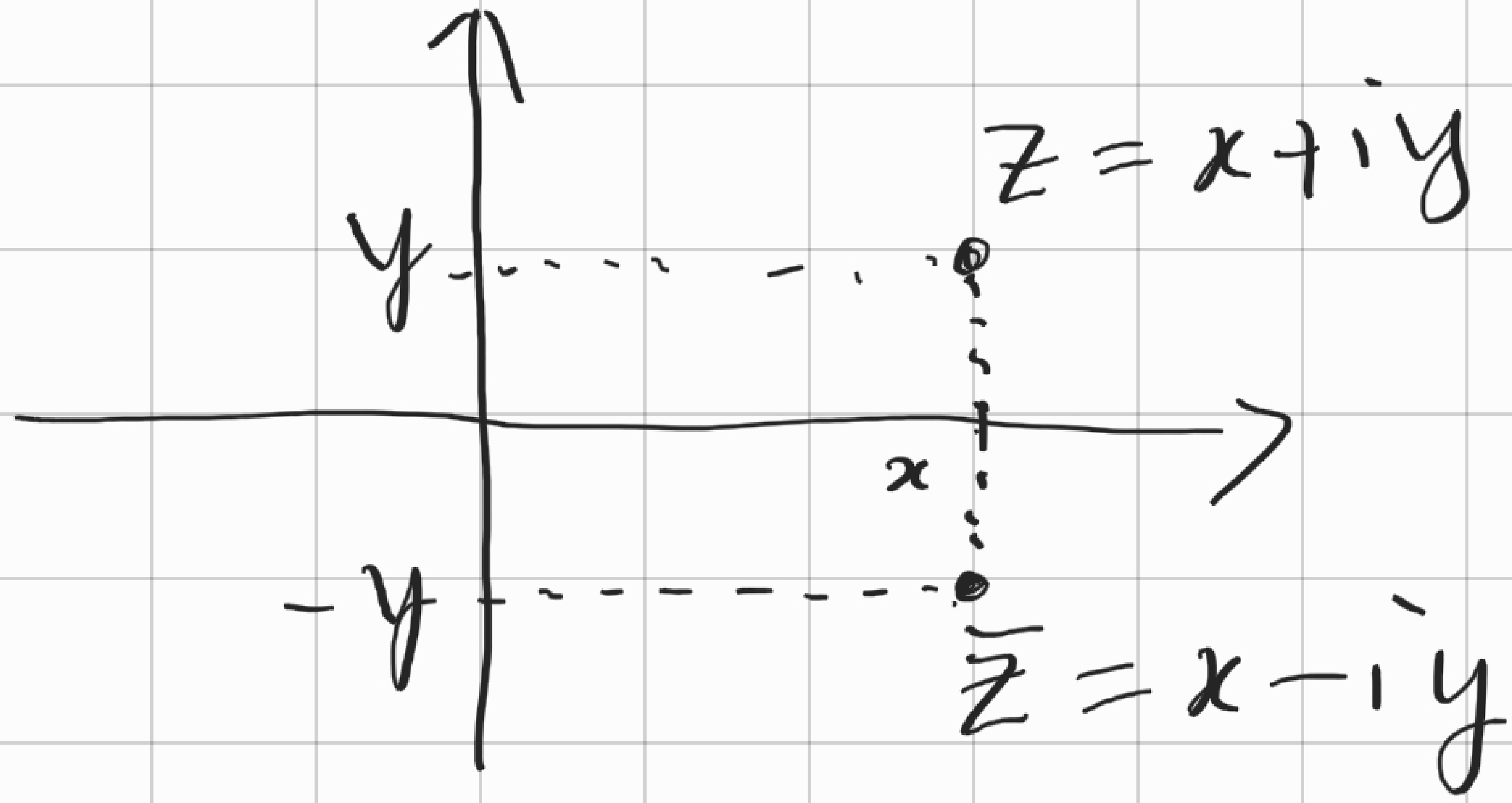
Il modulo di un numero reale non è altro che il valore assoluto.

Coniugio

Se $z = x + iy$ definiamo

$$z^* = \overline{z} = x - iy$$

(si chiama z "coniugato")



<u>Osservazione</u>: $z = x + iy$

$$z \cdot \bar{z} = (x+iy)\cdot(x-iy)$$

$$= x^2 - i^2 y^2 = x^2 + y^2 = |z|^2$$

$$|z| = \sqrt{z \cdot \bar{z}}$$

$$z \in \mathbb{R} \iff \bar{z} = z$$

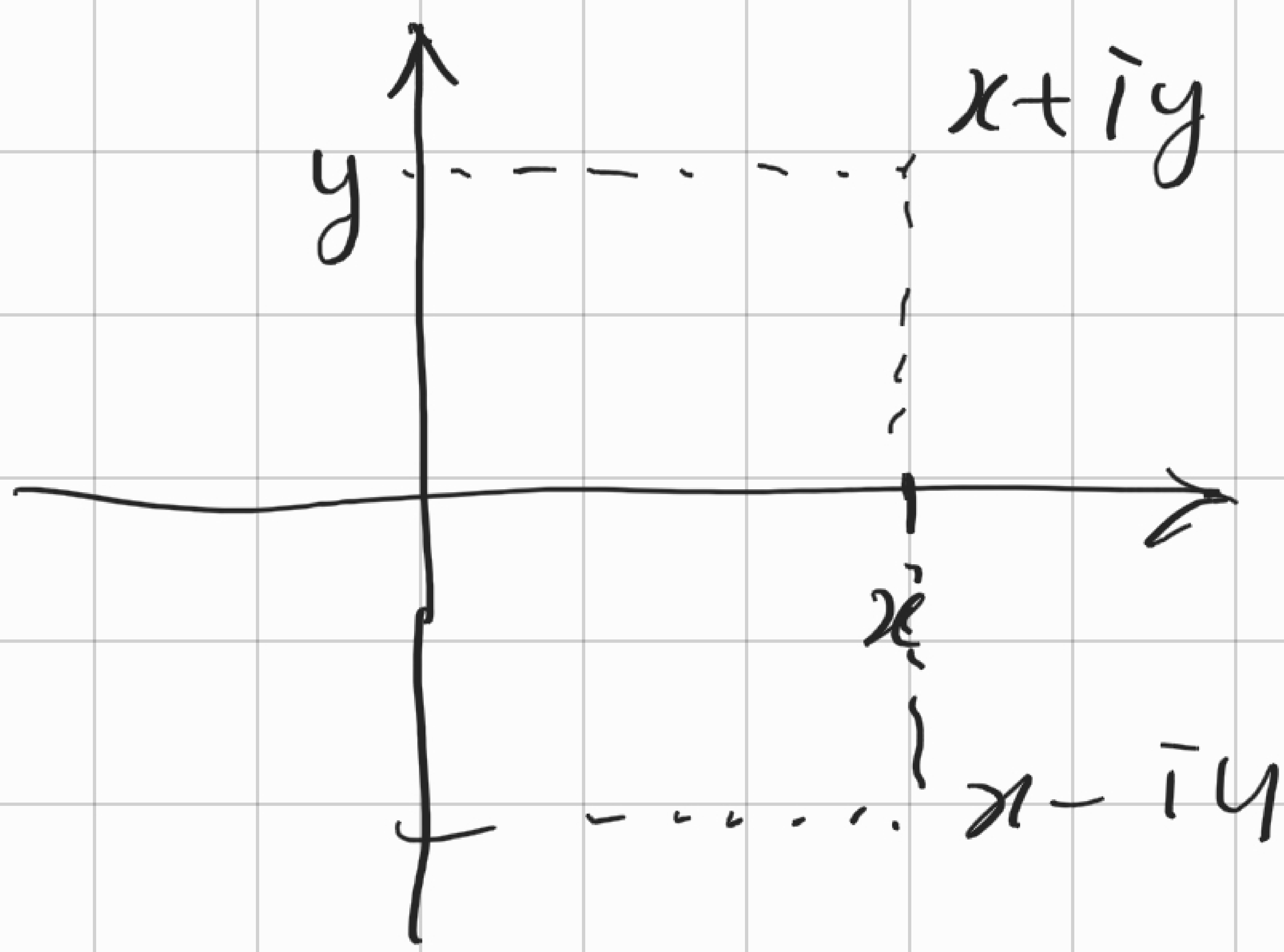
<u>Oss</u>

$$\frac{1}{z} = \frac{\bar{z}}{z \cdot \bar{z}} = \frac{\bar{z}}{|z|^2}$$

$$z = x + iy \quad \begin{cases} y = \mathrm{Im}\, z \\ x = \mathrm{Re}\, z \end{cases}$$

$$\begin{cases} z+\bar{z} = (x+iy)+(x-iy) = 2x \\ z-\bar{z} = (x+iy)-(x-iy) = 2iy \end{cases}$$

$$\begin{cases} \operatorname{Re} z = \dfrac{z+\bar{z}}{2} \\ \operatorname{Im} z = \dfrac{z-\bar{z}}{2i} \end{cases}$$



$z \longmapsto \bar{z}$ è un isomorfismo di $\mathbb{C}$.

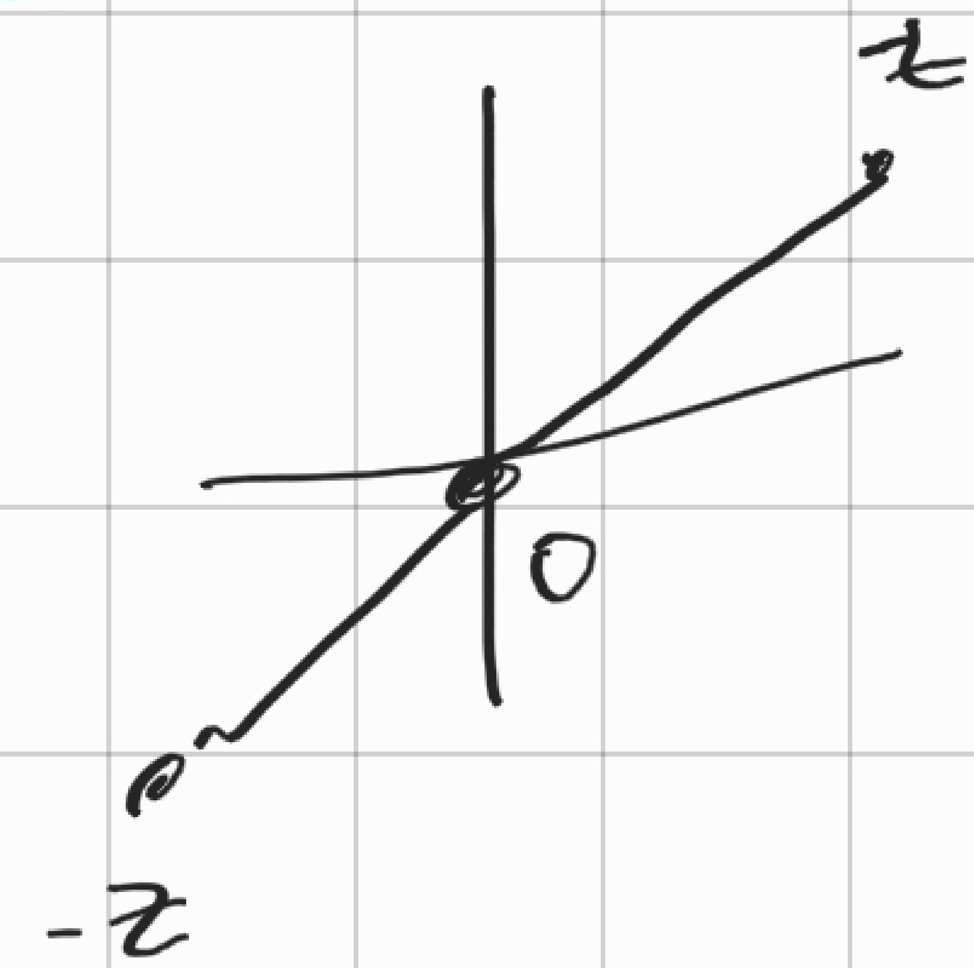
$i = \sqrt{-1}$ $\qquad$ $i^2 = -1$

$-i = \sqrt{-1}$ $\qquad$ $(-i)^2 = \left((-1)\cdot i\right)^2 = (-1)^2 \cdot i^2$

$= 1 \cdot i^2 = i^2 = -1$

$j = -i$ $\qquad$ $-z = (-1)\cdot z$ $\qquad$ $(-z)^2 = z^2$

Proprietà del modulo



- $\left| |z| \right| = |z|$
- $|-z| = |z|$
- $|z \cdot w| = |z| \cdot |w|$
- $|z+w| \leq |z| + |w|$ conseguenti
- $|z-w| \leq |z-v| + |v-w|$

triangolare

Verifichiamo $|z \cdot w|$

$$|z \cdot w| = |(x+iy) \cdot (a+ib)|$$

$$= |(ax-by) + i(ya+xb)|$$

$$= \sqrt{(ax-by)^2 + (ay+bx)^2}$$

$$= \sqrt{a^2x^2 + b^2y^2 - \cancel{2axby} + a^2y^2 + b^2x^2 + \cancel{2aybx}}$$

$$= \sqrt{a^2(x^2+y^2) + b^2(x^2+y^2)}$$

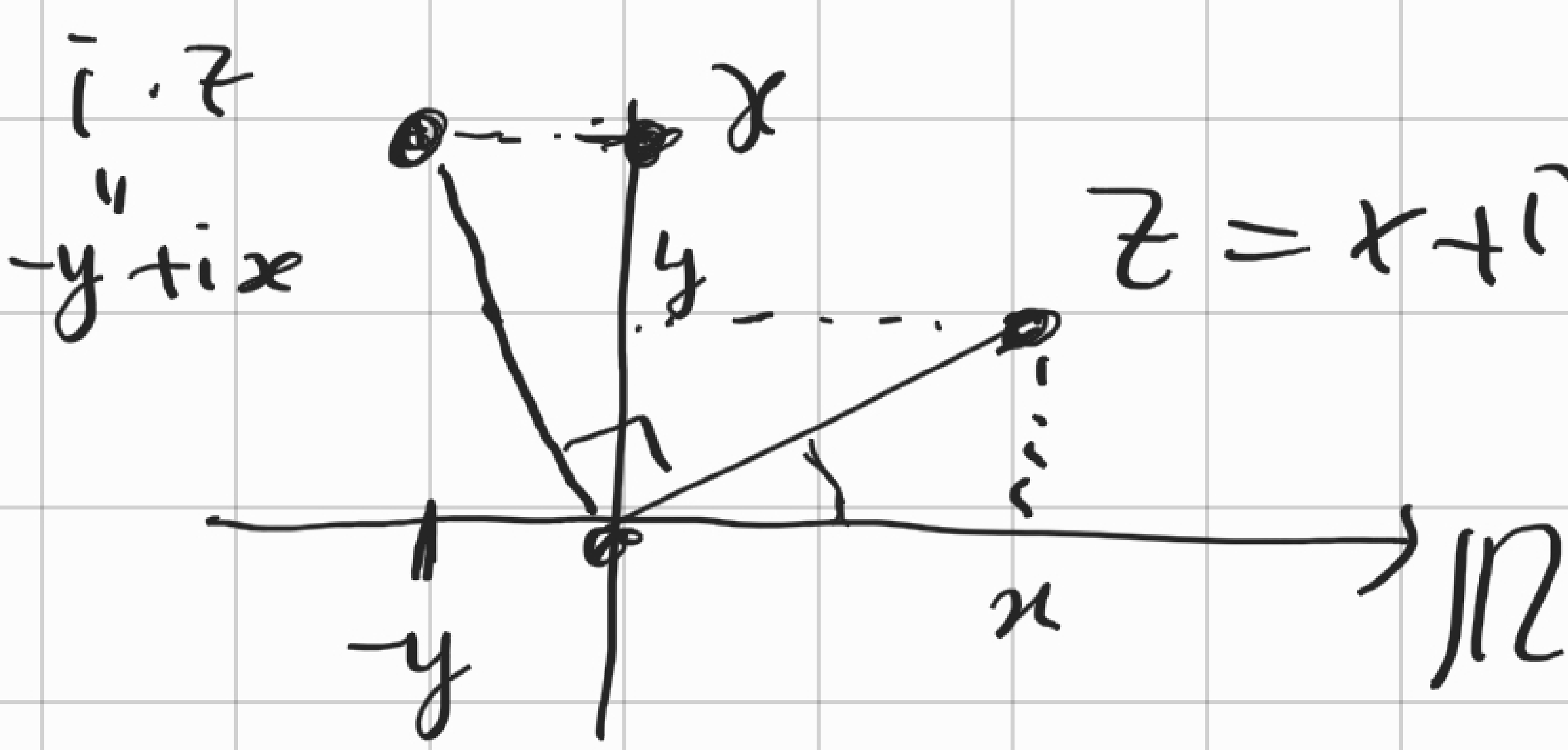
$$= \sqrt{(x^2+y^2) \cdot (a^2+b^2)}$$

$= \sqrt{x^2+y^2} \cdot \sqrt{a^2+b^2}$

$= |z| \cdot |w|$ 17

$i^2 = -1 \qquad i^3 = i^2 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = (-1)^2 = 1$.



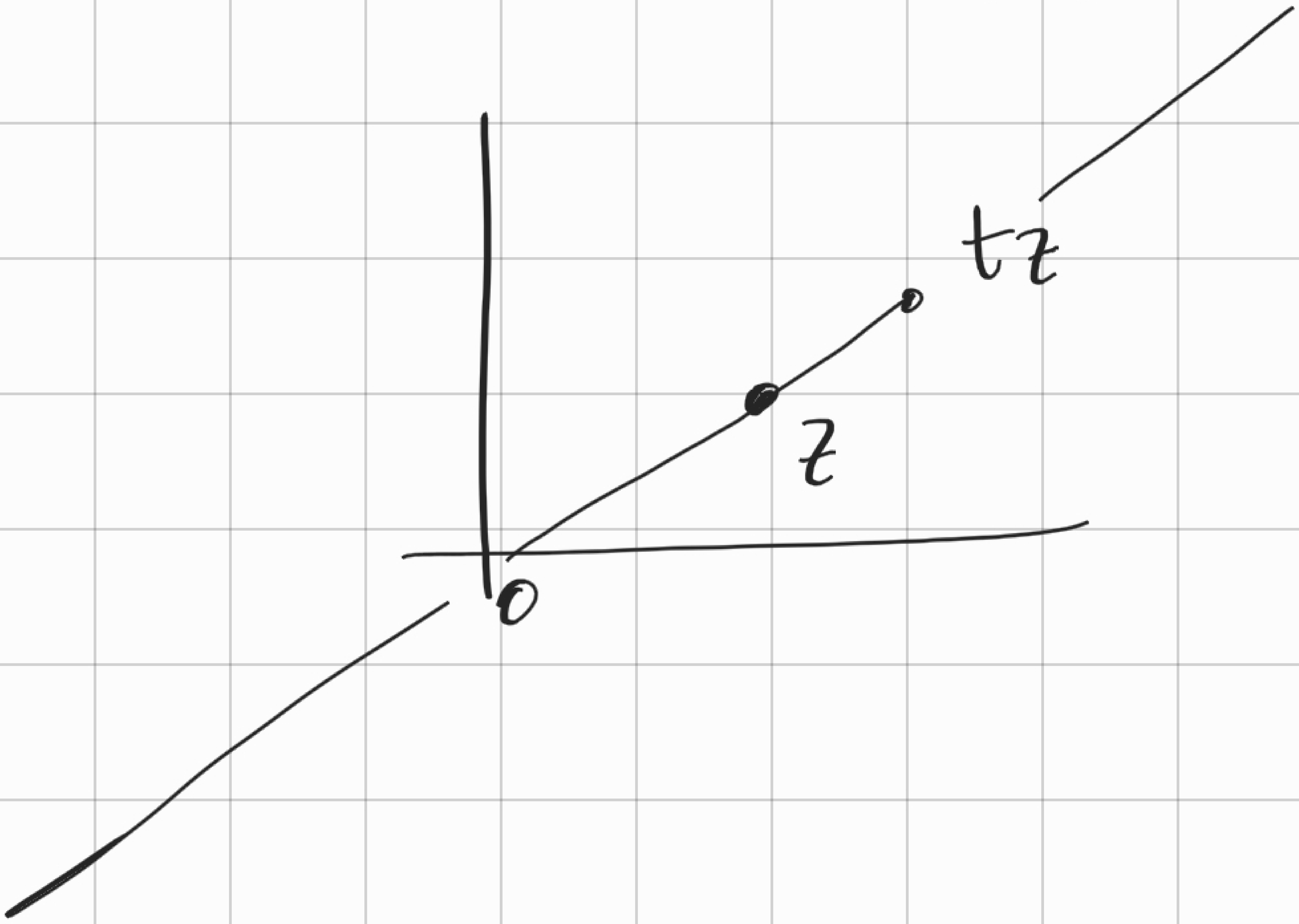
$z = x + iy$

$i \cdot z = ix + i^2 y = -y + ix$

$i \cdot z =$ "z ruotato di 90° a sinistra". $|i| = 1$

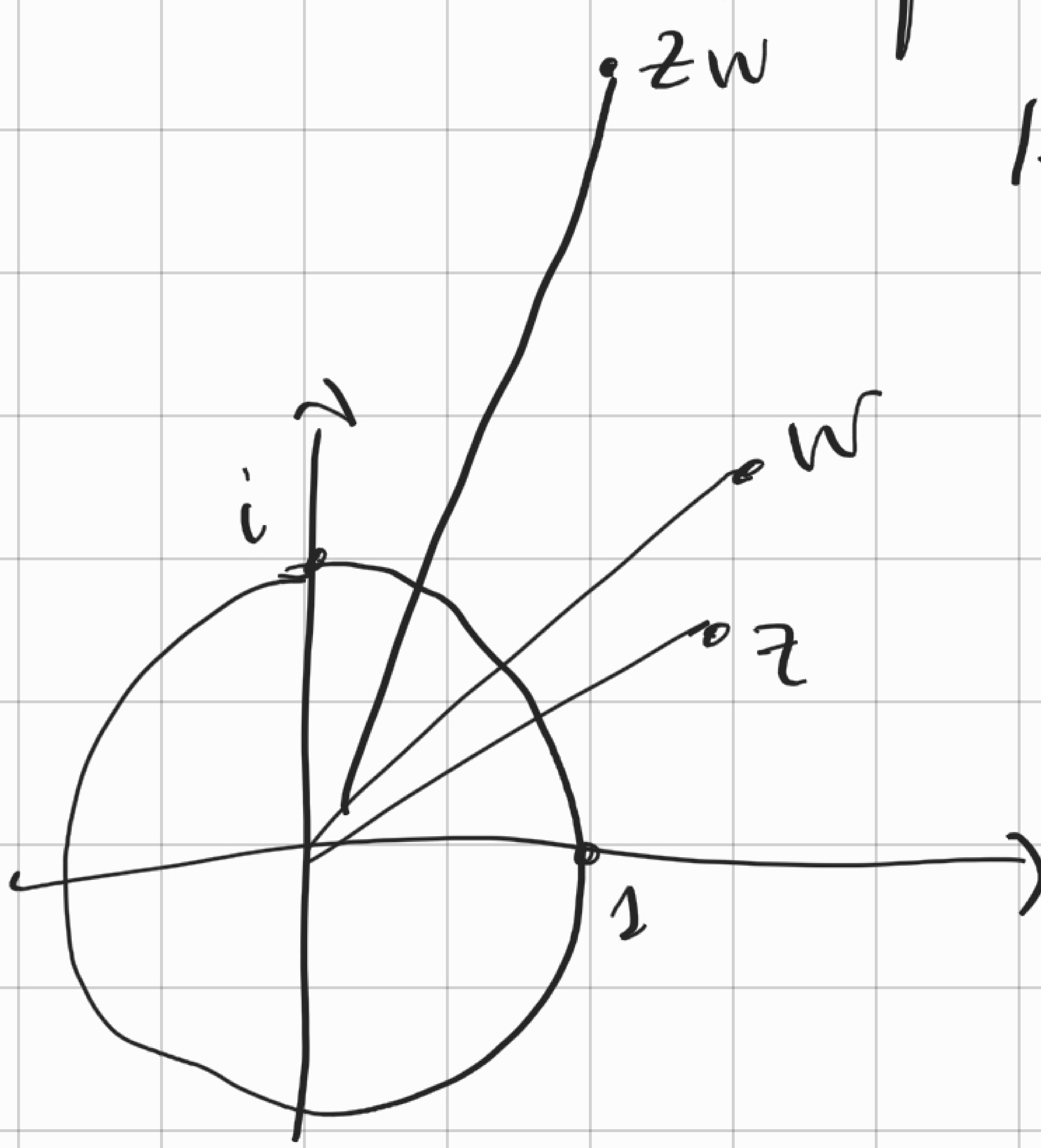
$|z \cdot w| = |z| \cdot |w|$

$t \in \mathbb{R}$ $tz =$ "riscalamento di z"



In generale $z \cdot w$ sul piano di Gauss

$|zw| = |z||w|$



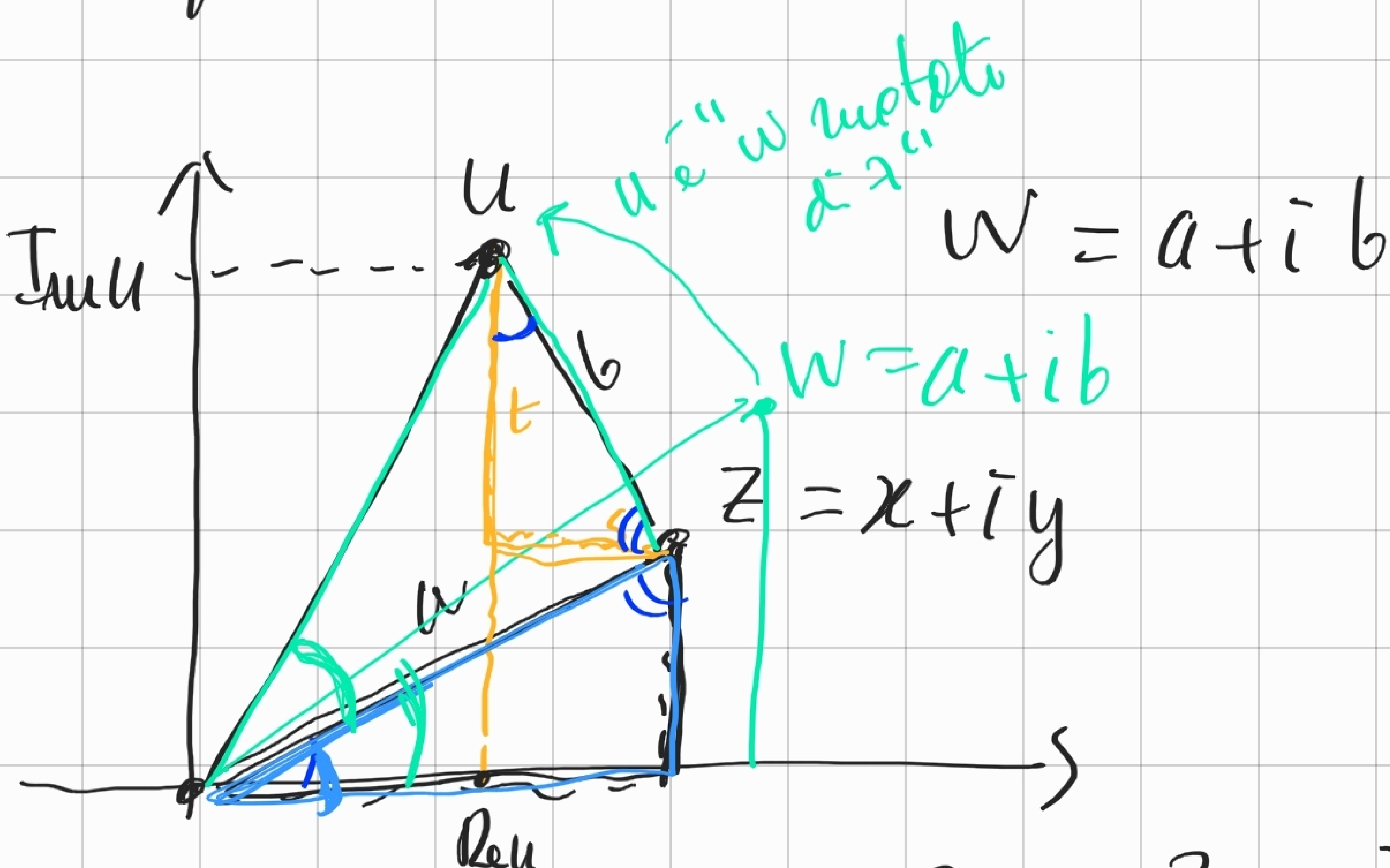
zw ha come distanza dall'origine il prodotto delle distanze

$|zw| = |z| \cdot |w|$

e ha come angolo: (argomento)

la somma degli angoli

(argomenti) di z e w.



supponiamo $a^2 = x^2 + y^2$

(riscalando opportunamente

w possiamo farlo)

$u = (x - s) + i(t + y)$

$$\frac{s}{y} = \frac{t}{x} = \frac{b}{a}$$

$$\begin{cases} s = \dfrac{by}{a} \\ t = \dfrac{bx}{a} \end{cases}$$

$$u = \left(x - \frac{by}{a}\right) + i\left(\frac{bx}{a} + y\right)$$

$$= \frac{ax - by}{a} + i\left(\frac{bx + ay}{a}\right)$$

$$= \frac{\operatorname{Re}(w \cdot z)}{a} + i\,\frac{\operatorname{Im}(w \cdot z)}{a}$$

$$= \frac{w \cdot z}{a}$$

$\square$