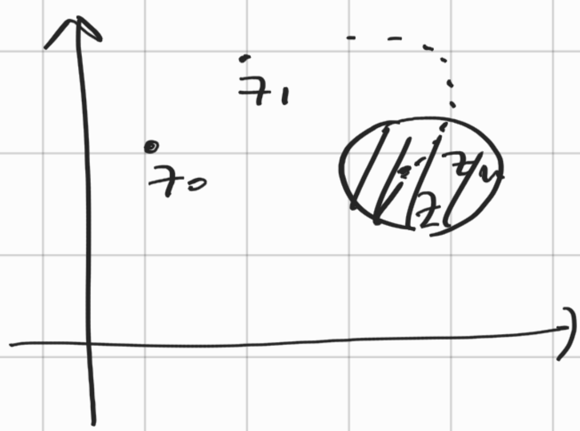
ANALISI MATEMATICA B
LEZIONE 37 — 8.1.2021

$$\exp(z) \underset{\text{def}}{=} e^z = \sum_{k=0}^{+\infty} \frac{z^k}{k!}, \quad z \in \mathbb{C}, \qquad e^x \underset{\text{teorema}}{=} \sum_{k=0}^{+\infty} \frac{x^k}{k!}, \quad x \in \mathbb{R}$$

topologia (limiti e continuità) su $\mathbb{C}$

Se $z_n \in \mathbb{C}$ $\quad z_n \to z$

(*) $\forall \varepsilon > 0 \ \exists \alpha \in \mathbb{N} : \forall n > \alpha : \quad |z_n - z| < \varepsilon$



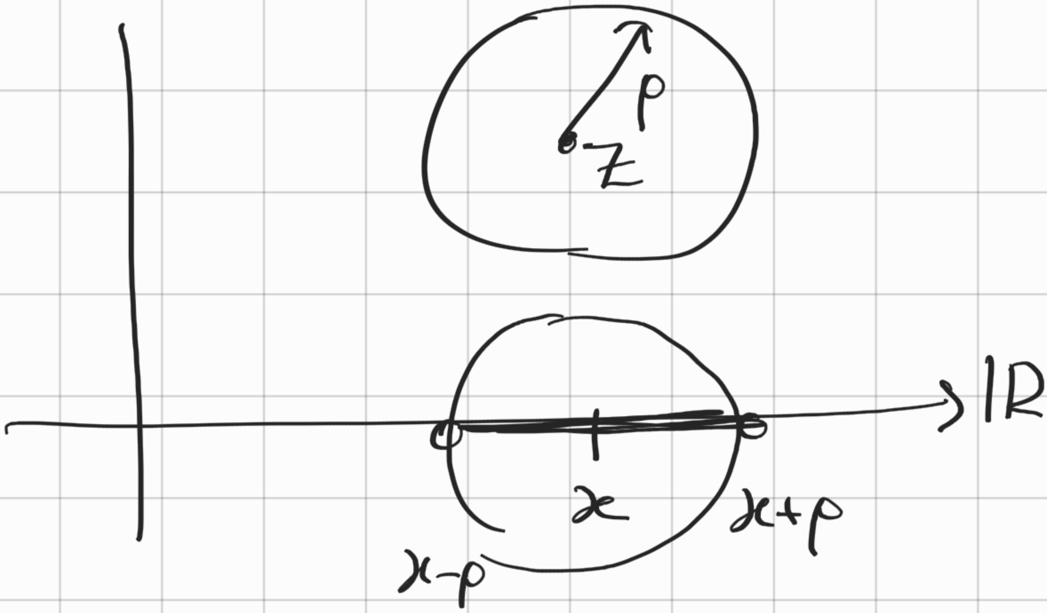
equivalentemente: $z_n \to z$ se $|z_n - z| \to 0$

Dal punto di vista topologico:

$$\forall U \in \mathcal{B}_z \ \exists V \in \mathcal{B}_\infty : \ n \in V \Rightarrow z_n \in U$$

$$\mathcal{B}_z = \{ B_\rho(z) : \rho > 0 \}$$

$$B_\rho(z) = \{ w \in \mathbb{C} : |w - z| < \rho \}$$



in alternativa:

$z_n = x_n + i y_n$ $\qquad$ $x_n = \mathrm{Re}\, z_n \in \mathbb{R}$

$\qquad\qquad\qquad\qquad$ $y_n = \mathrm{Im}\, z_n \in \mathbb{R}$

$$z_n \to z = x + iy \iff x_n \to x \text{ e } y_n \to y$$

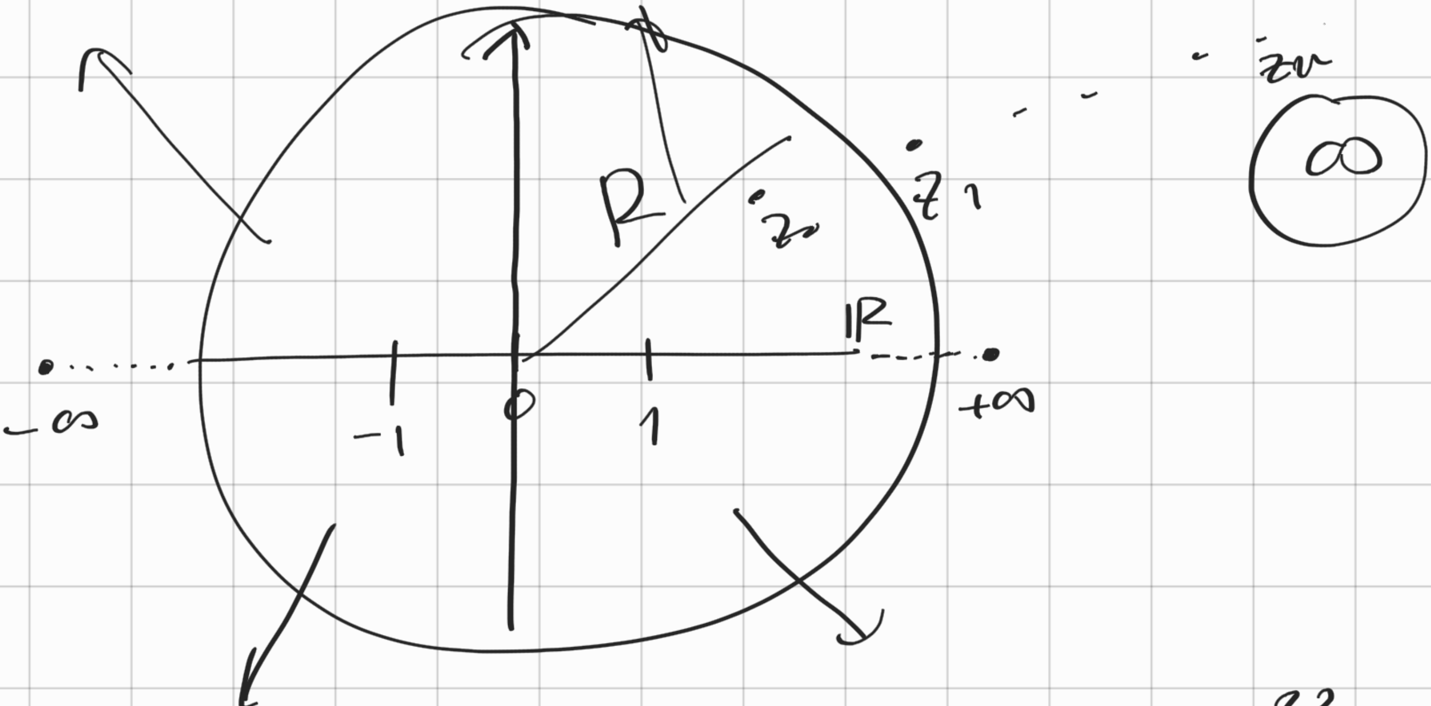
$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow$$

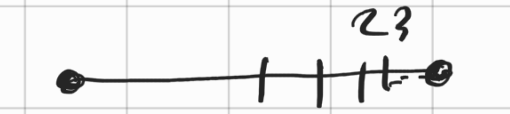
$$|z_n - z| \to 0 \qquad \underbrace{|x_n - x| \to 0} \qquad \underbrace{|y_n - y| \to 0}$$

$$\| $$

$$\sqrt{|x_n - x|^2 + |y_n - y|^2} \qquad \Leftarrow \quad \Rightarrow$$

$$\geq \max \{ |x_n - x|, |y_n - y| \}$$



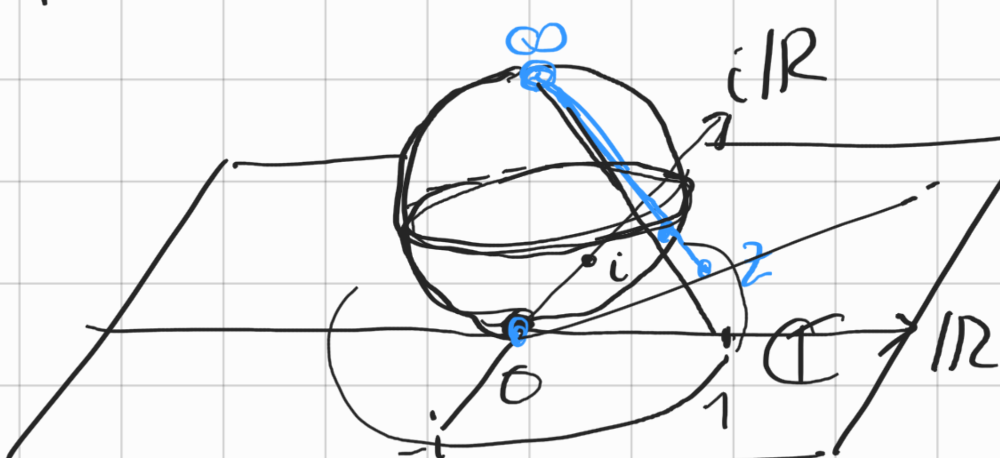


$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

$z_n \to \infty$ se $|z_n| \to +\infty$

$$\mathcal{B}_\infty = \left\{ \overline{\mathbb{C}} \setminus B_R(0) : R > 0 \right\}$$

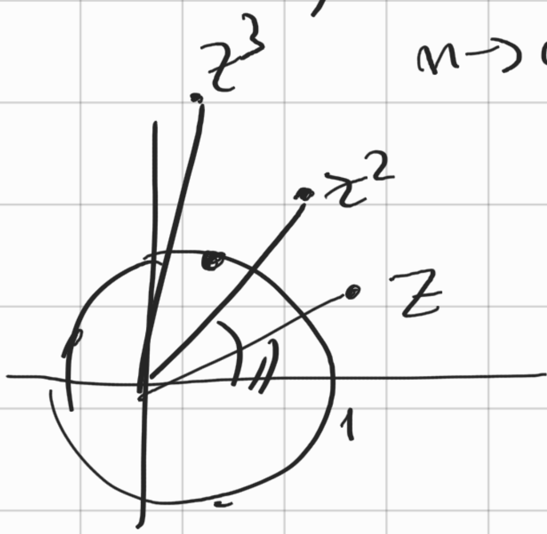
Sfera di Riemann:



proiezione stereografica

$\frac{1}{z}$ $\qquad \frac{1}{0} = \infty \qquad \frac{1}{\infty} = 0$

ES $\quad z \in \mathbb{C}, \quad \lim_{n\to\infty} z^n = \begin{cases} \infty & \text{se } |z|>1 \\ 1 & \text{se } z=1 \\ 0 & \text{se } |z|<1 \\ \nexists & \text{se } |z|=1,\ z\neq 1 \end{cases}$



$|z^n| = |z|^n$

$\frac{1}{z} \to \infty$ se $z \to 0$

ma $\lim \frac{1}{x} = \cancel{\exists}$ se $x \to 0$

$\pm\infty$

Continuità

$$f : A \subseteq \mathbb{C} \to \mathbb{C}$$

$z_0 \in A \qquad f$ è continua in z_0:

$$\forall \varepsilon > 0 : \exists \delta > 0 : |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

$$z \in B_\delta(z_0) \Rightarrow f(z) \in B_\varepsilon(f(z_0))$$

$$f(B_\delta(z_0)) \subseteq B_\varepsilon(f(z_0))$$

Se f è continua in z_0

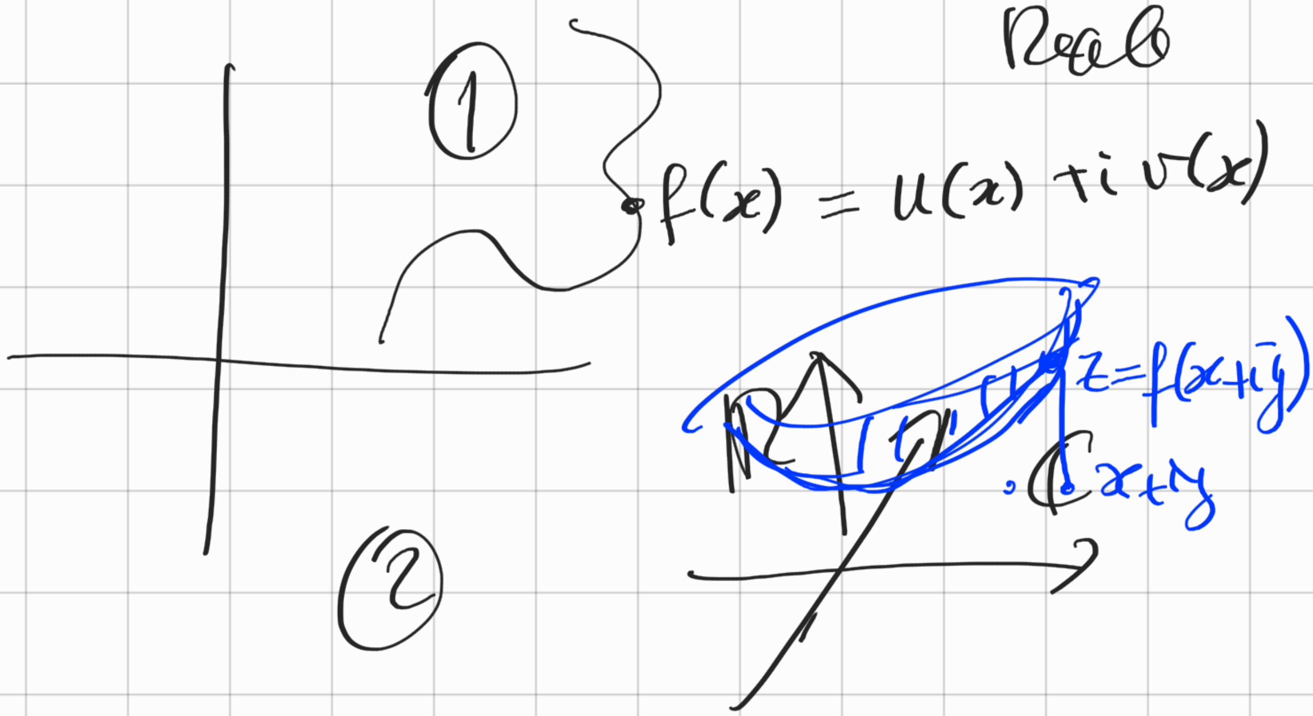
$$z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$$

Casi particolari di $f: \mathbb{C} \to \mathbb{C}$ — Analisi Complessa

(1) $f: \mathbb{R} \to \mathbb{C}$

(2) $f: \mathbb{C} \to \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$ — Analisi Reale



Serie a termini complessi

$z_n \in \mathbb{C}$ $\qquad \sum_{n=0}^{+\infty} z_n$ converge a S

$$S_n = \sum_{k=0}^{n} z_k \qquad S_n \to S.$$

<u>Teorema</u> (convergenza assoluta)

Se $\sum |z_n|$ converge

allora $\sum z_n$ converge.

<u>dim</u> $\sum z_n = \sum (x_n + i y_n)$

$\sum |z_n|$ converge $\left\{ \begin{array}{l} |x_n| \le |z_n| \\ |y_n| \le |z_n| \end{array} \right.$

$\Downarrow \qquad \Downarrow$

$\sum |x_n|$ converge $\qquad \sum |y_n|$ converge

$$\begin{cases} \sum x_n = x & \text{converge} \\ \sum y_n = y & \text{converge} \end{cases}$$

$$\sum z_n = \sum (x_n + i y_n)$$

$$= \sum x_n + i \sum y_n$$

$$= x + i y . \qquad \square$$

ES $\qquad \boxed{z \in \mathbb{C}}$

$$\sum_{k=0}^{+\infty} \frac{z^k}{k!} \quad \text{converge}$$

in quanto $\left| \frac{z^k}{k!} \right| = \frac{|z|^k}{k!}$

criterio del rapporto $(z \neq 0)$

$$\frac{|z|^{k+1}}{(k+1)!} \Big/ \frac{|z|^k}{k!} = \frac{|z|}{k+1} \longrightarrow 0$$

la serie $\sum \frac{|z|^k}{k!}$ converge

quindi $\sum \frac{z^k}{k!}$ converge.

Esempio

$$z_n = (-2)^n$$

$$|z_n| = 2^n \to +\infty$$

$$z_n \to \infty$$

$z_n = \mathrm{Re}\, z_n = x_n$ ma $\lim x_n$ non esiste.

potremmo dire che $(-2)^n \to \pm\infty$

Serie di potenze

$$f(z) = \sum_{k=0}^{+\infty} a_k \cdot z^k \qquad a_k \in \mathbb{C}$$

$$S_n = \sum_{k=0}^{n} a_k \cdot z^k \qquad \infty$$

$S_n \in \mathbb{C}$ per $n \to +\infty$ $S_n \to S \in \overline{\mathbb{C}}$

ES $\quad a_k = \frac{1}{k!} \qquad \sum \frac{z^k}{k!}$

Denotiamo con

$$A = \{ z \in \mathbb{C} : \sum a_k z^k \text{ converge} \}$$

$$f : A \subseteq \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto \sum_{k=0}^{+\infty} a_k z^k.$$

ES $\quad a_k = \frac{1}{k}, \qquad \sum_{k=0}^{+\infty} \frac{z^k}{k}$

[se $z = x \in \mathbb{R}$ $\quad \sum \frac{x^k}{k}$ converge se $x \in [-1, 1)$]

convergenza assoluta:

$$\sum \frac{|z|^k}{k}$$

criterio della radice: $\sqrt[k]{\frac{|z|^k}{k}} = \frac{|z|}{\sqrt[k]{k}} \longrightarrow |z|$

se $|z| < 1$ la serie converge assolutamente

se $|z| > 1$ la serie non converge.

(Se $\sum a_n$ converge $a_n \in \mathbb{C}$ $\overset{\text{condizione necessaria}}{\Longrightarrow}$ $a_n \to 0 \Rightarrow |a_n| \to 0$)

Se $|z|=1$?

$$f(z)=\sum_{k=1}^{+\infty}\frac{z^k}{k}$$

serie armonica $z=1$

serie armonica a segni alterni

$A=\{z\in\mathbb{C} : \sum \frac{z^k}{k} \text{ converge}\}$

$$\boxed{\{z : |z|<1\} \subseteq A \subseteq \{z : |z|\le 1\}}$$

Se $z=1$ diverge $\quad 1\notin A$

Se $z=-1$ converge $\quad 1\in A$

per gli altri z con $|z|=1$?

(si applica il criterio Dirichlet)

Teorema Sia $a_n\in\mathbb{C}$. L'insieme A di convergenza della serie di potenze

$$\sum_{k=0}^{+\infty} a_k\cdot z^k$$

è della forma

$$\{z : |z|<R\}\subseteq A\subseteq\{z : |z|\le R\}$$

con $R \in [0, +\infty]$.

Inoltre se $|z| < R$ la serie converge assolut.

R si chiama raggio di convergenza.

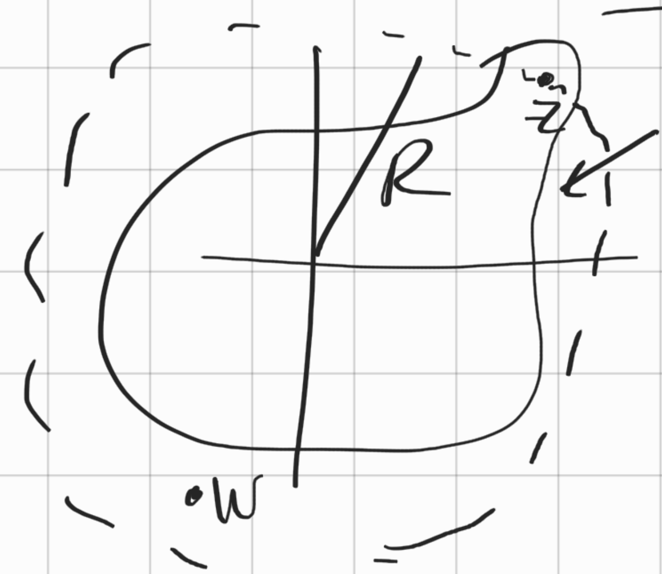
<u>Lemma</u> Se $\sum a_k z^k$ converge.

Anzi se $a_k z^k$ è limitata

(cioè $\exists L$ $|a_k z^k| \leq L$ $\forall k \in \mathbb{N}$)

Allora se $w \in \mathbb{C}$, $|w| < |z|$ allora

$$\sum a_k w^k \text{ converge } \underline{\text{ordinatamente}}.$$



<u>dim</u>

$$|a_k z^k| \leq L$$

$\shortparallel$

$$|a_k| \cdot |z|^k \leq L$$

Se $|w| < |z|$

$$|a_k w^k| = |a_k| \cdot |w|^k$$

$$= \underbrace{|a_k| \cdot |z|^k}_{} \cdot \left(\frac{|w|}{|z|}\right)^k$$

$$\leq L \cdot q^k \quad \text{con} \quad q = \frac{|w|}{|z|} < 1$$

$$\sum L \cdot q^k = L \cdot \sum q^k$$

converge se $q < 1$

$\sum |a_k w^k|$ converge per confronto

$\sum a_k w^k$ converge assolut. $\square$.

Dirichlet

$$\sum \frac{\cos(\alpha n)}{n} \qquad \underbrace{\alpha \neq 2k\pi}$$

$$\sup \mathbb{Z} = +\infty$$
$$\inf \mathbb{Z} = -\infty$$

se ponessi

$$\sup \mathbb{Z} = \infty$$
$$\inf \mathbb{Z} = \infty$$

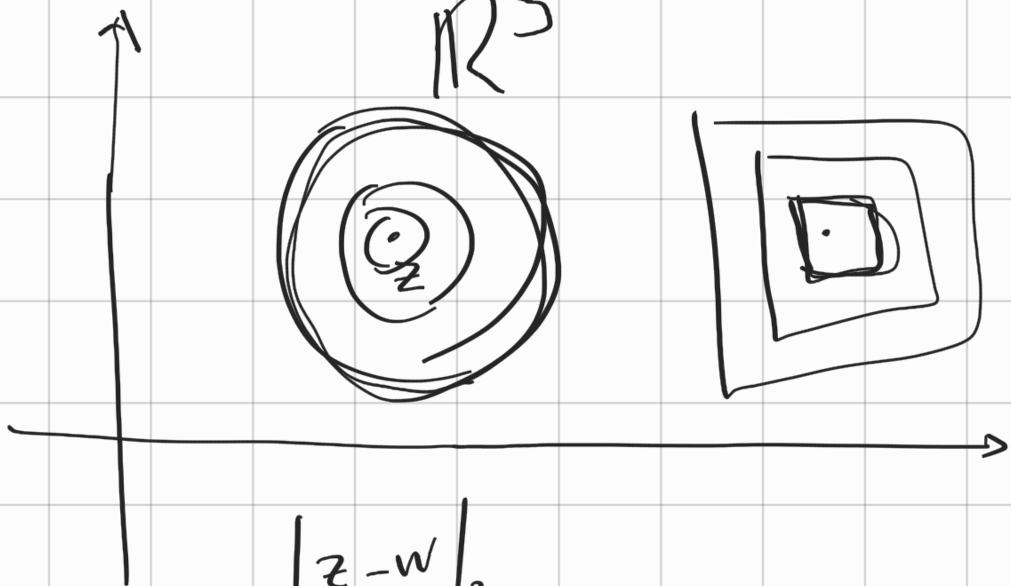
$a \in A \qquad a \leq \sup A$

$-\infty < 2 < +\infty$

$2 > \infty$?

$2 < \infty$?

$\mathbb{R}^3$



$|z-w|_2$

$|z-w|_1 = \max\{|\mathrm{Re}\, z - \mathrm{Re}\, w|, |\mathrm{Im}\, z - \mathrm{Im}\, w|\}$