IL TEOREMA FONDAMENTALE DELL'ALGEBRA

30. 1. 2021

<u>Teorema</u> Se $P(z)$ è un polinomio a coefficienti complessi, $\deg P > 0$, allora esiste $z_0 \in \mathbb{C}$: $P(z_0) = 0$

($\mathbb{C}$ è algebricamente chiuso).

<u>Esempio</u> $P(x) = x^2 + 1 \geq 1$ se $x \in \mathbb{R}$

ma $P(i) = 0$.

<u>Corollario</u> $\deg P = n$ esistono $z_1, z_2, \ldots, z_n \in \mathbb{C}$, $a \in \mathbb{C}$

$$P(z) = a \cdot (z - z_1) \cdot (z - z_2) \cdots (z - z_n).$$

<u>Teor</u> (B-W) Se $z_n \in \mathbb{C}$ esiste z_{n_k} sottosucc. regolare $z_{n_k} \to z \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

<u>dim</u>

$$z_n = x_n + i y_n \qquad x_n, y_n \in \mathbb{R}$$

$\exists\ x_{n_k} \to x \in [-\infty, +\infty]$

da y_{n_k} $\exists\ y_{n_{k_j}} \to y \in [-\infty, +\infty]$

ma anche $x_{n_{k_j}} \to x$.

Dunque se $x \in \mathbb{R}$ e $y \in \mathbb{R}$

allora $z_{n_{k_j}} \to z = x + iy$.

altrimenti:

$$|z_{n_{k_j}}|^2 = |x_{n_{k_j}}|^2 + |y_{n_{k_j}}|^2 \longrightarrow +\infty$$

$$z_{n_{k_j}} \longrightarrow \infty \in \overline{\mathbb{C}} \qquad \square$$

<u>Proposizione</u> P polinomio, $\deg P > 0$

$$\lim_{z \to \infty} P(z) = \infty$$

ovvero $\lim_{z \to \infty} |P(z)| = +\infty$

<u>dim</u>

$$P(z) = \sum_{k=0}^{n} a_k z^k \qquad \deg P = n$$

$z \neq 0$

$$= \underbrace{z^n}_{\infty} \Big[a_n + \underbrace{\sum_{k=0}^{n-1} \frac{a_k}{z^{k-n}}}_{0} \Big]$$

$$\left(|z^n| = |z|^n \to +\infty \right) \qquad \to \infty . \qquad \square$$

Proposizione $P(z)$ è un polinomio $\deg P > 0$ allora

$|P(z)|$ ha minimo su $\mathbb{C}$.

$$|P| : \mathbb{C} \to \mathbb{R}$$
$$z \mapsto |P(z)|$$

dim Sia z_n successione minimizzante per $|P(z)|$. Cioè

$$|P(z_n)| \to \inf_{z \in \mathbb{C}} |P(z)| = s \in \overline{\mathbb{R}}$$

Infatti se $t > s$ esiste z : $|P(z)| < t$

se $t_n > s$ $t_n \to s$

$$s \leq |P(z_n)| < t_n$$
$$\downarrow \qquad \downarrow$$
$$s \qquad s$$

✓

Per B-W $\exists\ z_{n_k} \to z \in \overline{\mathbb{C}}$

se fosse $z_{n_k} \to \infty$ $\quad P(z_{n_k}) \to \infty$

$$|P(z_{n_k})| \to +\infty \neq s$$

$$z_{n_k} \to z_0 \in \mathbb{C}$$

$|P(z)|$ è una funzione continua.

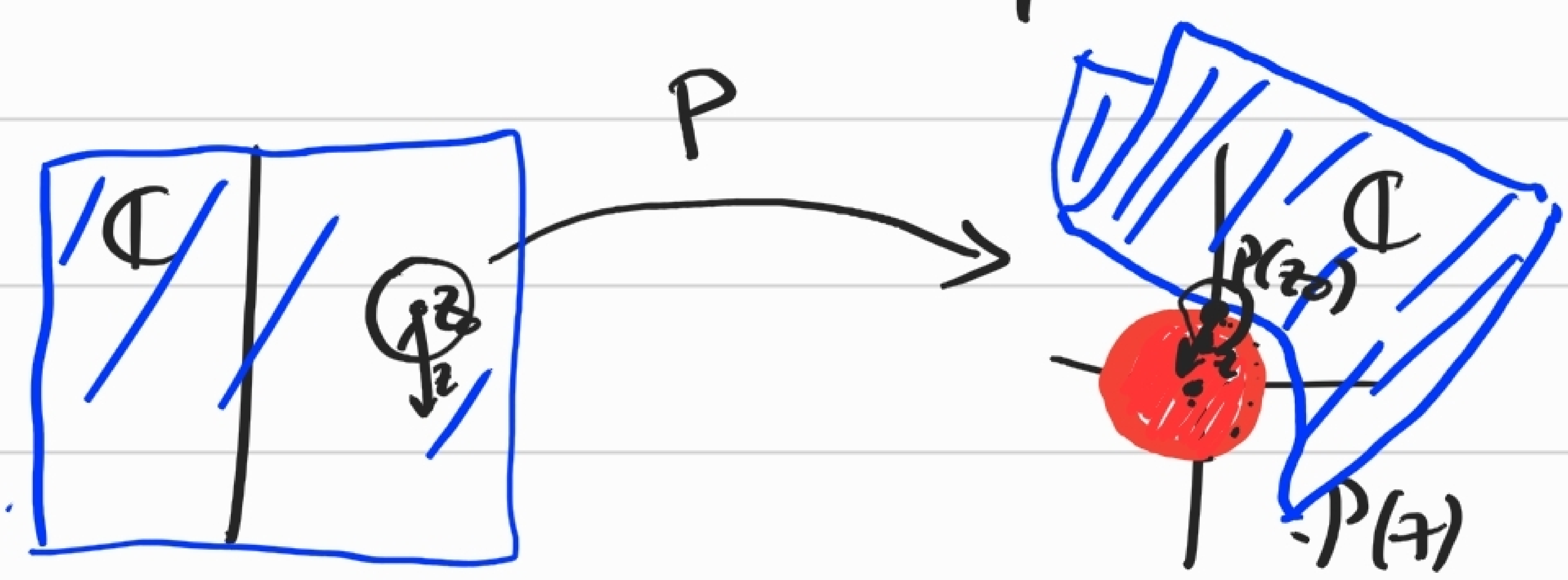
$$|P(z_{n_k})| \to |P(z_0)| = \min_{z \in \mathbb{C}} |P(z)|. \quad \square$$

$$\downarrow \qquad \qquad \|$$

$$s = \inf_{z \in \mathbb{C}} |P(z)|$$

dim (teo fondamentale dell'algebra)

idea



z_0 punto di minimo per $|P(z)|$

$P(z_0)$ se $P(z_0) \neq 0$

idea: trovare z tale che $|P(z)| < |P(z_0)|$

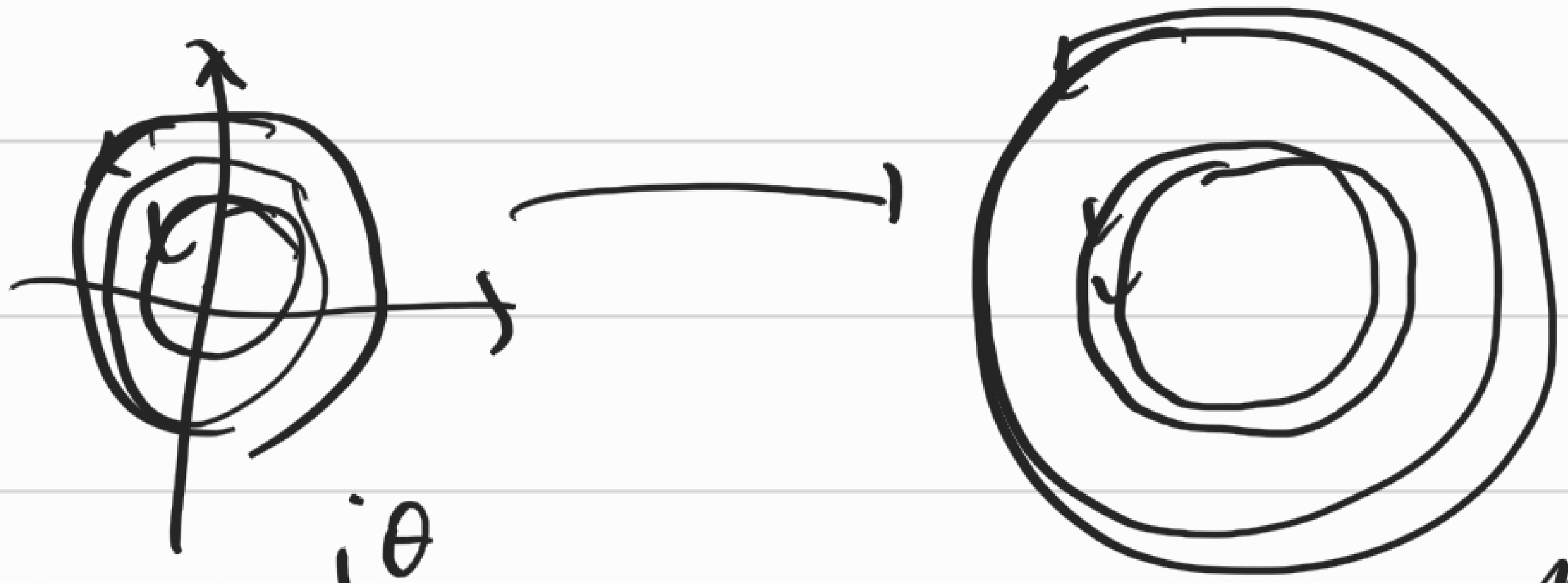
vicino a z_0

$$P(z) = P(z_0) + \underset{\neq 0}{c} \cdot (z - z_0)^m + \ldots\ldots$$

piccoli

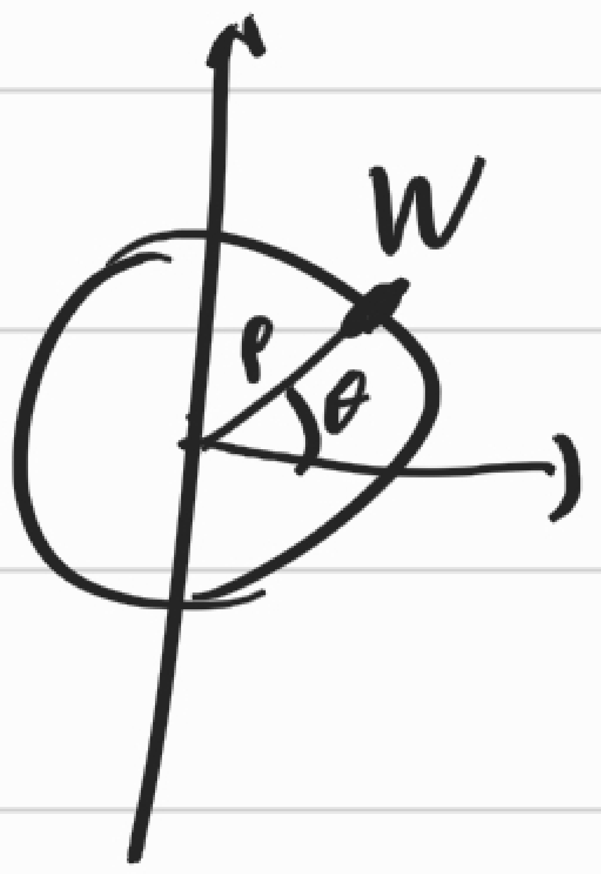
$w = z - z_0$

$w \longrightarrow w^n$



$w = \rho e^{i\theta}$

$\rho = |w|$



$w^m = (\rho e^{i\theta})^m = \rho^m e^{im\theta}$

<u>Fondamentale</u> z_0 t.c. $|P(z_0)|$ è minimo

Per assurdo suppongo $P(z_0) \neq 0$.

$$P(z) = \sum_{k=0}^{n} a_k z^k = \sum_{k=0}^{n} b_k (z - z_0)^k$$

$w = z - z_0$

$$= Q(z - z_0)$$

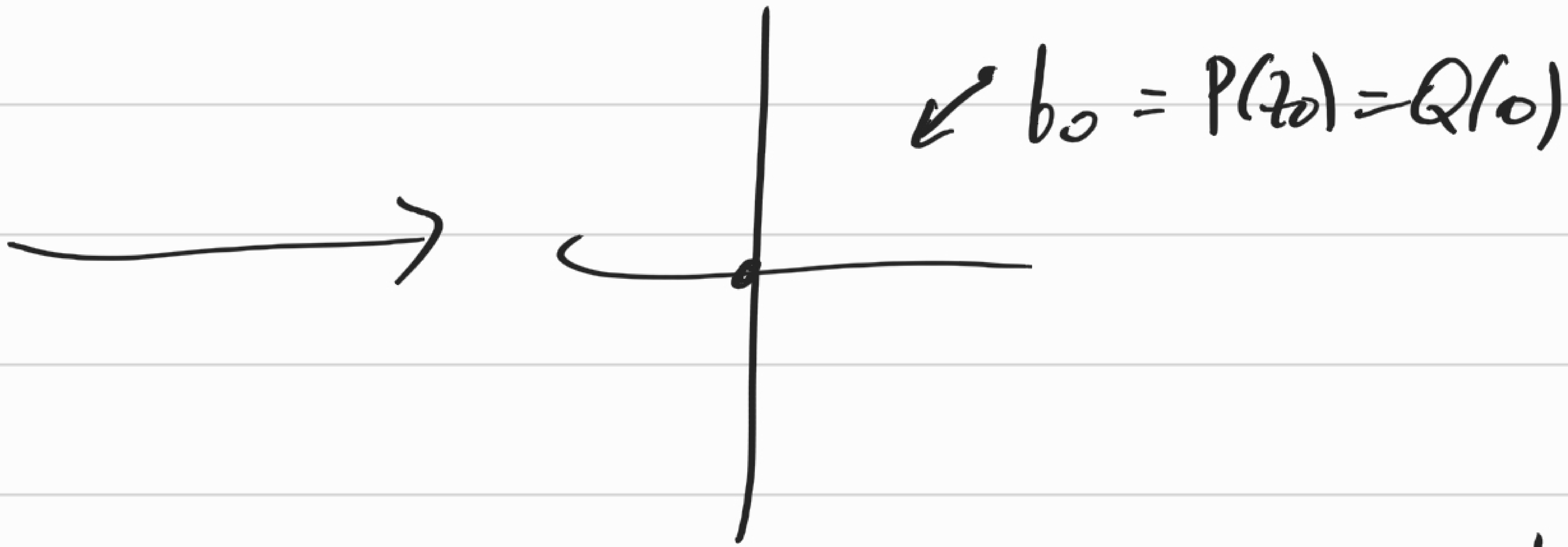
$$Q(w) = P(z_0 + w)$$

$$Q(w) = \sum_{k=0}^{n} b_k w^k = b_0 + b_m w^m + \sum_{k=m+1}^{n} b_k w^k.$$

$\underline{b_m \neq 0}$

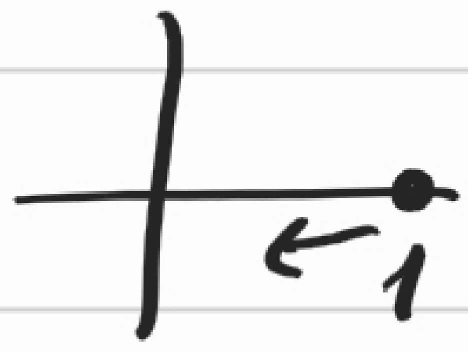
$b_0 = Q(0) = P(z_0) \neq 0$ (per ipotesi)

Voglio trovare w vicino 0 tale che $|Q(w)| < |b_0|$



$$|Q(w)| = |b_0| \cdot \left| 1 + \frac{b_m}{b_0} w^m + \sum_{k=m+1}^{n} \frac{b_k}{b_0} w^k \right|$$

$\frac{b_m}{b_0} = r \cdot e^{i\alpha}$ con $r > 0$



scelgo $w = \rho e^{i\theta}$ in modo che $\frac{b_m}{b_0} w^m$ reale negativo.

$$\| \quad r e^{i\alpha} \rho^m e^{im\theta} = r \rho^m \underbrace{e^{i(\alpha + m\theta)}}$$

scelgo θ in modo che $e^{i(\alpha + m\theta)} = -1$

$\alpha + m\theta = \pi \qquad \theta = \frac{\pi - \alpha}{m}$

$$|Q(\underbrace{\rho e^{i\theta}})| = |b_0| \cdot \left| 1 - r\rho^m + \sum_{k=m+1}^{n} \frac{b_k}{b_0} \rho^k e^{ik\theta} \right|$$

$$\leq |b_0| \cdot \left[|1 - r\rho^m| + \sum_{k=m+1}^{n} \left| \frac{b_k}{b_0} \right| \rho^k \right]$$

se ρ è piccolo $\boxed{1 - r\rho^m > 0}$

$$= |b_0| \cdot \left[1 - r\rho^m + \rho^{m+1} \cdot \underbrace{\sum_{k=m+1}^{n} \left| \frac{b_k}{b_0} \right| \cdot \rho^{k-m+1}}_{R(\rho)} \right]$$

se $\rho \to 0$ $R(\rho) \to R(0)$

$$= |b_0| \cdot \left| 1 - \rho^m \left(r - \underbrace{\rho\, R(\rho)}_{\substack{\downarrow \\ 0 \text{ per } \rho \to 0}} \right) \right| = (*)$$

scelgo ρ abbastanza piccolo in modo che $\boxed{|\rho R(\rho)| < \frac{r}{2}}$

$$r - \rho R(\rho) > r - \frac{r}{2} = \frac{r}{2}$$

$$|P(z_0 + \rho e^{i\theta})| = |Q(\rho e^{i\theta})| \leq |P(z_0)| \cdot \left(1 - \rho^m \frac{r}{2}\right)$$

$$< |P(z_0)| \quad \text{assurdo}$$

perché $|P(z_0)|$ era minimo. $\square$