ANALISI MATEMATICA B
LEZIONE 49 – 5.2.2021

Teorema (formula di Taylor con resto di Peano)

$f : I \to \mathbb{R}$ $\quad x_0 \in I$, $f \in C^{n-1}(I)$, $\exists f^{(n)}(x_0)$, $I \neq \{x_0\}$

P polinomio di Taylor per f di ordine n centrato in x_0.

$$\lim_{x \to x_0} \frac{f(x) - P(x)}{(x - x_0)^n} = 0 \qquad \left[|f(x) - P(x)| << |x - x_0|^n \right]$$

dim $(x > x_0)$

$$\frac{f(x) - P(x)}{(x-x_0)^n} = \boxed{\frac{(f(x) - P(x)) - \overbrace{(f(x_0) - P(x_0))}^{=0}}{(x-x_0)^n - \underbrace{(x_0 - x_0)^n}_{=0}}}$$

f derivabile in (x_0, x) continua in $[x_0, x]$

$$\overset{\text{Cauchy}}{=} \boxed{\frac{f'(x_1) - P'(x_1)}{n(x_1 - x_0)^{n-1}}} = \frac{(f'(x_1) - P'(x_1)) - \overbrace{(f'(x_0) - P'(x_0))}^{=0}}{n(x_1 - x_0)^{n-1} - \underbrace{n(x_0 - x_0)^{n-1}}_{=0}}$$

$\exists x_1 \in (x_0, x)$

$$\overset{\text{Cauchy}}{=} \frac{f''(x_2) - P''(x_2)}{n(n-1)(x_2 - x_0)^{n-2}} = \dots$$

$\exists x_2 \in (x_0, x_1)$

$N(x) = f(x) - P(x)$

$D(x) = (x - x_0)^n$

$$\boxed{\frac{N(x) - N(x_0)}{D(x) - D(x_0)}} \overset{\text{Cauchy}}{\underset{\exists x_1 \in (x_0, x)}{=}} \boxed{\frac{N'(x_1)}{D'(x_1)}}$$

$x_0 < x_n < x_{n-1} < \dots < x_2 < x_1 \le x$

$$\dots = \frac{f^{(n-1)}(x_{n-1}) - P^{(n-1)}(x_{n-1})}{\underbrace{n(n-1)\dots 2}_{n-1 \text{ fattori}}\,(x_{n-1}-x_0)^1} = (*)$$

I° METODO Applico ancora Cauchy:

$$(*) = \frac{f^{(n)}(x_n) - P^{(n)}(x_n)}{n!} \longrightarrow \frac{f^{(n)}(x_0) - P^{(n)}(x_0)}{n!} = 0$$

se $x \to x_0^+$, $x_n \in (x_0, x)$, $x_n \to x_0$

Se $f^{(n)}$ è continua in x_0

II° METODO

$P^{(n-1)}(x) =$ polinomio di Taylor di ordine 1 per $f^{(n-1)}$

$= f^{(n-1)}(x_0) + f^{(n)}(x_0)(x - x_0)$

$$Q_n(x) = \sum_{k=0}^{n} \frac{g^{(k)}(x_0)}{k!}(x-x_0)^k$$

Taylor di ordine n per g in x_0

$$Q_1(x) = g(x_0) + g'(x_0)(x - x_0)$$

$$(*) = \frac{f^{(n-1)}(x_{n-1}) - \left[f^{(n-1)}(x_0) + f^{(n)}(x_0)(x_{n-1} - x_0)\right]}{n!\,(x_{n-1} - x_0)}$$

$$= \frac{1}{n!}\left[\frac{f^{(n-1)}(x_{n-1}) - f^{(n-1)}(x_0)}{x_{n-1} - x_0} - f^{(n)}(x_0)\right]$$

per $x \to x_0$ anche $x_{n-1} \to x_0$

Se $f^{(n-1)}$ è derivabile in x_0

$$\longrightarrow \frac{1}{n!}\left[f^{(n)}(x_0) - f^{(n)}(x_0)\right] = 0 \quad \square$$

Osservazione per $n = 1$, $P(x) = f(x_0) + f'(x_0)(x - x_0)$

$$\frac{f(x) - [f(x_0) + f'(x_0)(x - x_0)]}{x - x_0}$$

$$= \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \longrightarrow 0$$

è equivalente $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

Esempi

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$\lim_{x\to 0} \frac{e^x + \cos x - \sin x - 2}{x \cdot \ln(1+x^2)}$$

funzione — polinomio di Taylor di ordine 3 centrato in 0

$f(x)$ $\quad f(0) + f'(0)\cdot x + \frac{f''(0)}{2}\cdot x^2 + \frac{f'''(0)}{6} x^3$

e^x $\quad 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

$\cos x$ $\quad 1 + \cancel{0\cdot x} - \frac{x^2}{2} + \cancel{0\cdot\frac{x^3}{6}}$

$\sin x$ $\quad \cancel{0} + x + \cancel{0\cdot\frac{x^2}{2}} - \frac{x^3}{6}$

$\ln(1+x)$ $\quad \boxed{0 + x - \frac{x^2}{2} + \frac{x^3}{3}}$

$$e^x = \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{6}} + a(x)$$

Taylor: $\frac{a(x)}{x^3} \to 0$ per $x \to 0$

$|a(x)| << x^3.$

$\cos x = 1 - \frac{x^2}{2} + b(x)$, $\quad \frac{b(x)}{x^3} \to 0$

$\sin x = x - \frac{x^3}{6} + c(x)$, $\quad \frac{c(x)}{x^3} \to 0$

$$\ln(1+x) = x + d(x), \quad \boxed{\frac{d(x)}{x} \to 0}$$

$$\ln(1+x^2) = x^2 + d(x^2)$$

$$\frac{e^x + \cos x - \sin x - 2}{x \cdot \ln(1+x^2)} = \frac{\cancel{1}\cancel{+x}\cancel{+\frac{x^2}{2}}+\frac{x^3}{6}+a(x)+\cancel{1}\cancel{-\frac{x^2}{2}}+b(x)\cancel{-x}+\frac{x^3}{6}-c(x)\cancel{-2}}{x \cdot (x^2+d(x^2))}$$

$$\frac{e(x)}{x^3} = \frac{a(x)}{x^3} + \frac{b(x)}{x^3} - \frac{c(x)}{x^3} \to 0$$

$$= \frac{\frac{x^3}{3} + e(x)}{x^3 + f(x)}$$

$$f(x) = x \cdot d(x^2)$$

$$\frac{f(x)}{x^3} = \frac{d(x^2)}{x^2} \to 0$$

$$= \frac{\frac{1}{3} + \frac{e(x)}{x^3}}{1 + \frac{f(x)}{x^3}} \to \frac{\frac{1}{3}+0}{1+0} = \frac{1}{3} \qquad \square$$

SE SVILUPPO TROPPO

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + g(x) \qquad \frac{g(x)}{x^3} \to 0$$

$$x \cdot \ln(1+x^2) = x^3 - \frac{x^5}{2} + \frac{x^7}{3} + \underbrace{x \cdot g(x^2)}_{h(x)}$$

$$\frac{h(x)}{x^7} = \frac{x\, g(x^2)}{x \cdot x^6} = \frac{g(x^2)}{x^6} = \frac{g(t)}{t^3} \to 0 \qquad t = x^2$$

$$\frac{e^x + \cos x - \sin x - 2}{x \cdot \ln(1+x^2)} = \frac{\frac{x^3}{3} + e(x)}{x^3 - \frac{x^5}{2} + \frac{x^7}{3} + h(x)}$$

$$= \frac{\frac{1}{3} + \cancel{\frac{e(x)}{x^3}}}{1 - \cancel{\frac{x^2}{2}} + \cancel{\frac{x^4}{3}} + \cancel{\frac{h(x)}{x^3}}\,\frac{x^4}{x^4}}$$

$$|h(x)| << x^7 << x^3$$

$$\longrightarrow \frac{\frac{1}{3}+0}{1+0} = \frac{1}{3}$$

FATICA INUTILE

SE SVILUPPO TROPPO POCO

$$\frac{e^x+\cos x-\sin x-2}{x\cdot \ln(1+x^2)} = \frac{\cancel{1}+\cancel{x}+l(x)+\cancel{1}+m(x)-\cancel{x}-n(x)\cancel{-2}}{x^3+f(x)}$$

$$= \frac{l(x)+m(x)-n(x)}{x^3+f(x)} = \frac{p(x)}{x^3+f(x)}$$

$$\frac{l(x)}{x}\to 0, \quad \frac{n(x)}{x}\to 0$$

$$\frac{m(x)}{x}\to 0$$

$$= \frac{\frac{p(x)}{x^3}}{1+\cancel{\frac{f(x)}{x^3}}} \;??$$

0

$$\frac{p(x)}{x} = \frac{l(x)}{x}+\frac{m(x)}{x}-\frac{n(x)}{x}\longrightarrow 0$$

$$\frac{p(x)}{x^3} = \frac{\frac{p(x)}{x}}{x^2} = \frac{0}{0} = ???$$

OTTENGO UNA FORMA INDETERMINATA

NOTAZIONE o-piccolo

Scriveremo $f(x)=o(x^n)$, per $x\to 0$

se $\frac{f(x)}{x^n}\to 0$ $\quad f(x)<<x^n$ per $x\to 0$

Più in parole: $\boxed{f(x) = o(g(x))}$ per $x \to x_0$

se $\dfrac{f(x)}{g(x)} \to 0$ per $x \to x_0$

$f(x) << g(x)$ per $x \to x_0$

Taylor: $\boxed{f(x) = P(x) + o((x-x_0)^n)}$

$f(x) - P(x) = o((x-x_0)^n)$

$\dfrac{f(x) - P(x)}{(x-x_0)^n} \to 0$

$\ln(1+x) = x + o(x)$

$\ln(1+x^2) = x^2 + o(x^2)$ $\quad x \to 0$

$$\frac{e^x + \cos x - \sin x - 2}{x \cdot \ln(1+x^2)} = \frac{\cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) + \cancel{1} - \frac{x^2}{2} + o(x^3) - \cancel{x} + \frac{x^3}{6} - o(x^3) - \cancel{2}}{x \cdot (x^2 + o(x^2))}$$

$x \cdot o(x^2) = o(x^3)$

$$= \frac{\frac{x^3}{3} + o(x^3)}{x^3 + o(x^3)}$$

$$= \frac{\frac{1}{3} + \frac{o(x^3)}{x^3}}{1 + \frac{o(x^3)}{x^3}} \to \frac{1}{3} \qquad \square$$

$$\frac{o(x^3)}{x^2} = \frac{o(x^3)}{x^3} \cdot x$$

$$o(x^3) - o(x^3) = o(x^3)$$

$$7 \cdot o(x^3) = o(x^3)$$

$$-1 \cdot o(x^3) = o(x^3)$$

$$o(x^3) + o(x^4) = o(x^3)$$

$$o(x^4) = o(x^3)$$

$$\longrightarrow$$

$$o(x^3) \neq o(x^4)$$

$$\longrightarrow$$