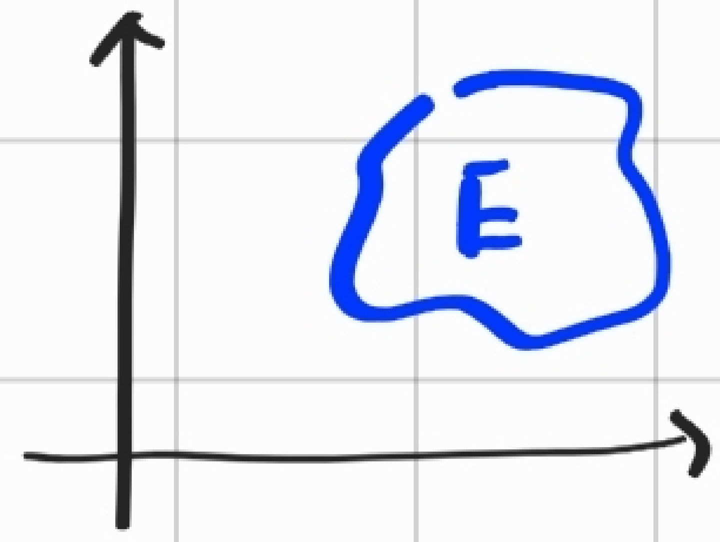
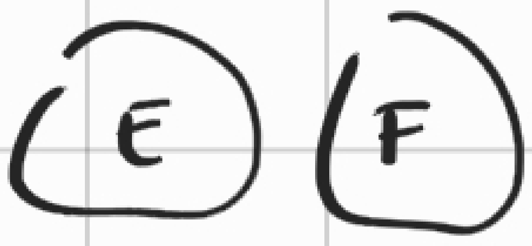
ANALISI MATEMATICA B
LEZIONE 56 - 22.2.2021

MISURA DI PEANO-JORDAN

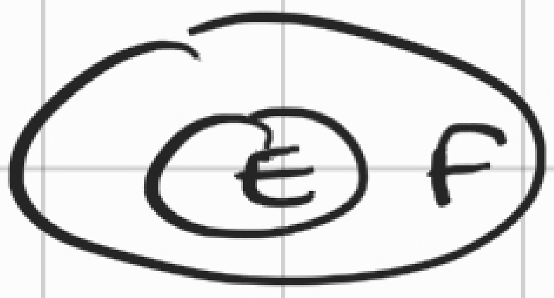


$E \subseteq \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

Vogliamo definire $m(E)$ l'area di E.

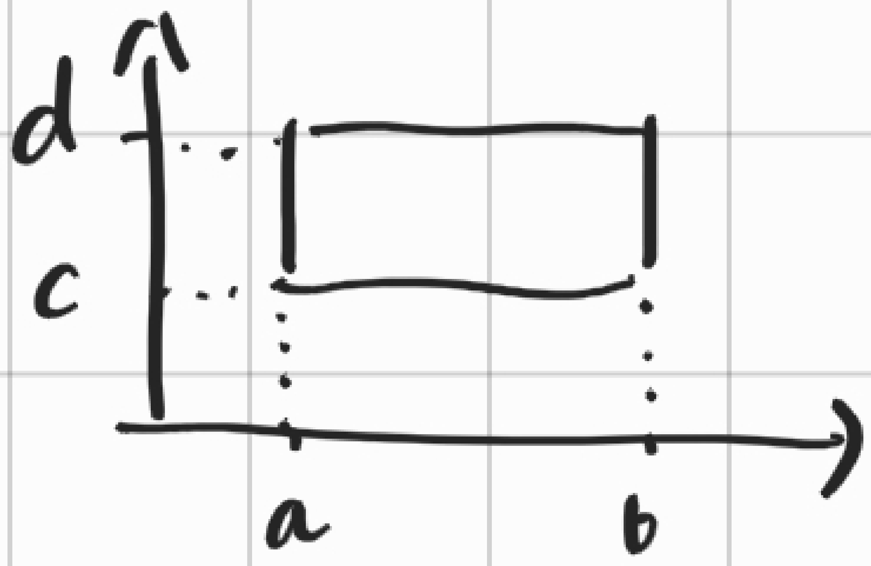


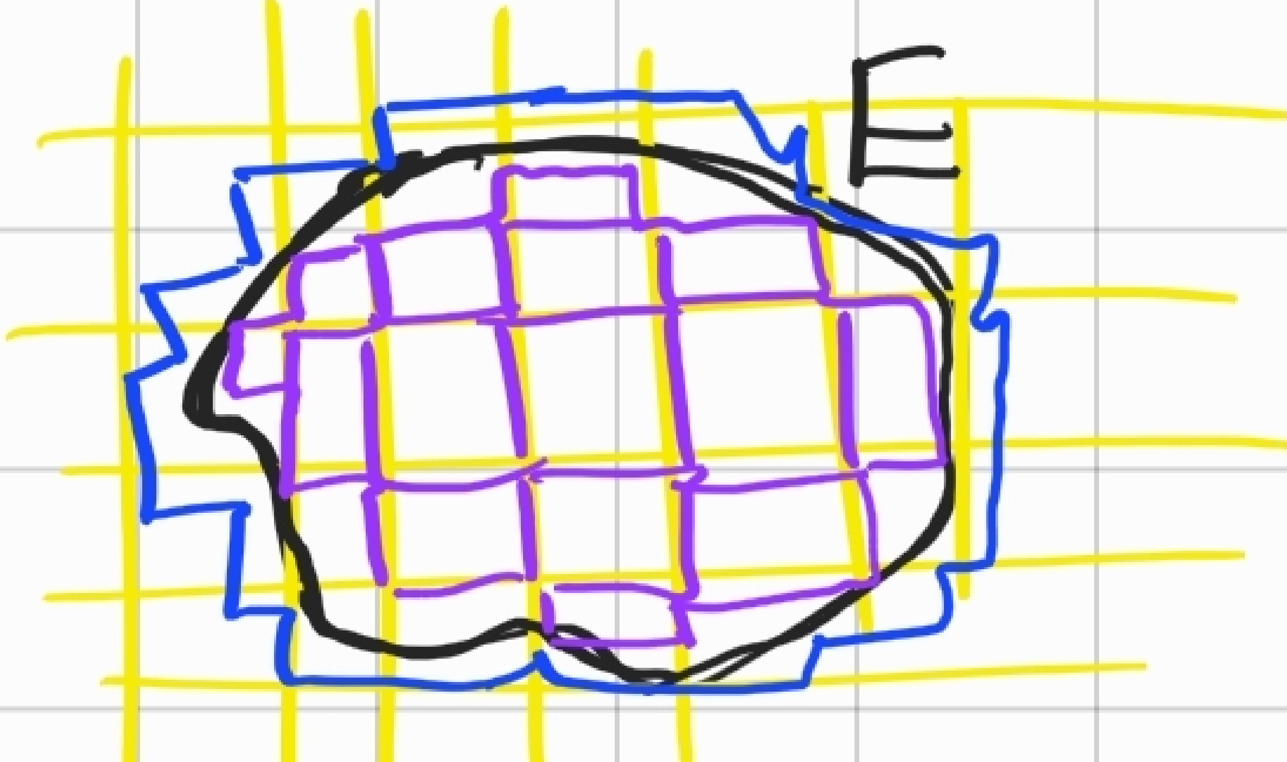
- 1. <u>additività</u> se $E \cap F = \emptyset \Rightarrow m(E \cup F) = m(E) + m(F)$
- 2. <u>monotonia</u> se $E \subseteq F \Rightarrow m(E) \leq m(F)$

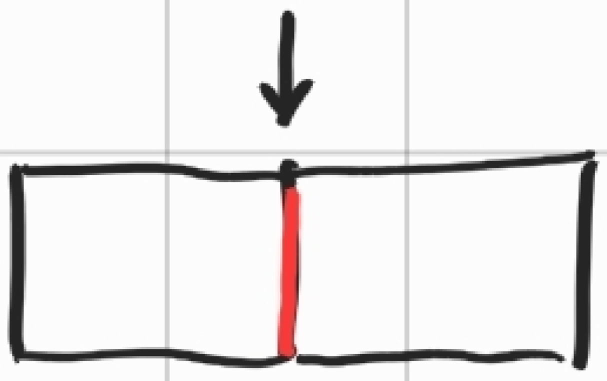


- 3. <u>normalizzazione</u> se $E = [a,b] \times [c,d]$

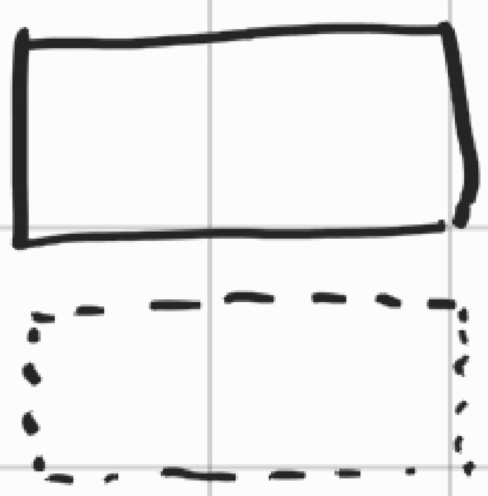
$$m(E) = (b-a) \cdot (d-c).$$







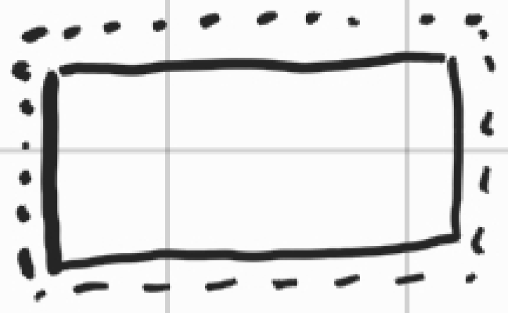
$$(a,b) \times (c,d) \subseteq E \subseteq [a,b] \times [c,d]$$



$$\Downarrow$$

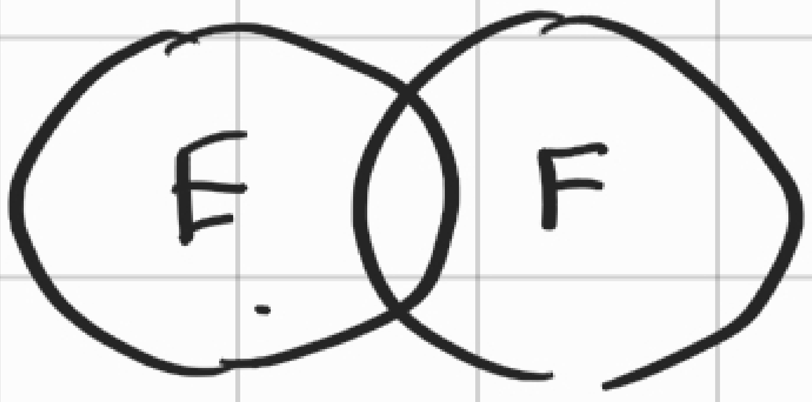
$$m(E) = (b-a) \cdot (d-c)$$

$$\underline{R = (a,b) \times (c,d) \supseteq [a+\varepsilon, b-\varepsilon] \times [c+\varepsilon, d-\varepsilon]}$$

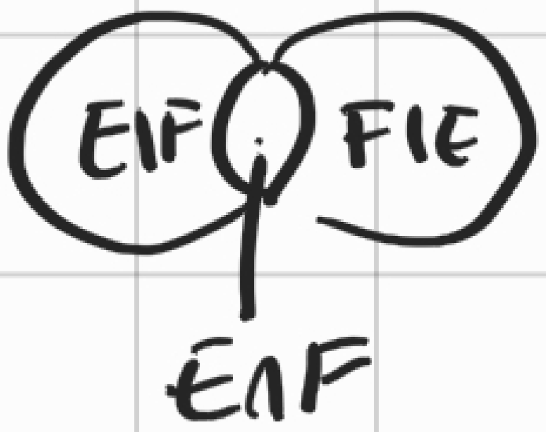


$m(R) \le (b-\varepsilon-a-\varepsilon)\cdot(d-\varepsilon-c-\varepsilon)$

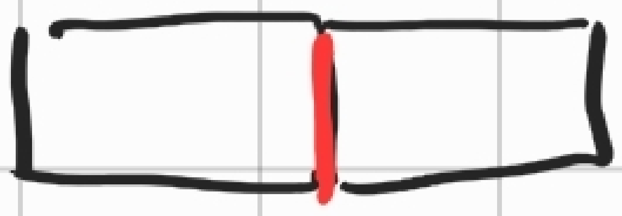
$= \underline{(b-a)(d-c)} + o(1)$ $\xrightarrow{\text{per } \varepsilon\to 0} 0$



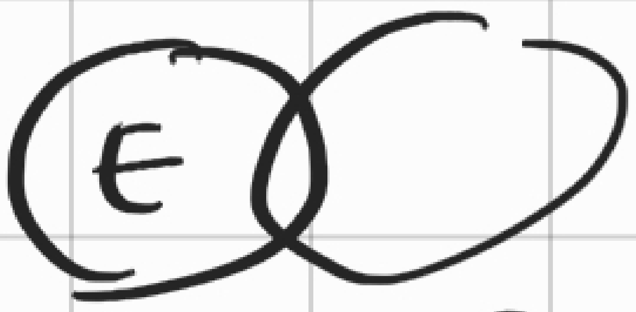
$m(E\cup F) = m((E\setminus F)\cup(E\cap F)\cup(F\setminus E))$



$= m(E\setminus F)+m(E\cap F)+m(F\setminus E)$



$= m(E)+m(F)-m(E\cap F)$



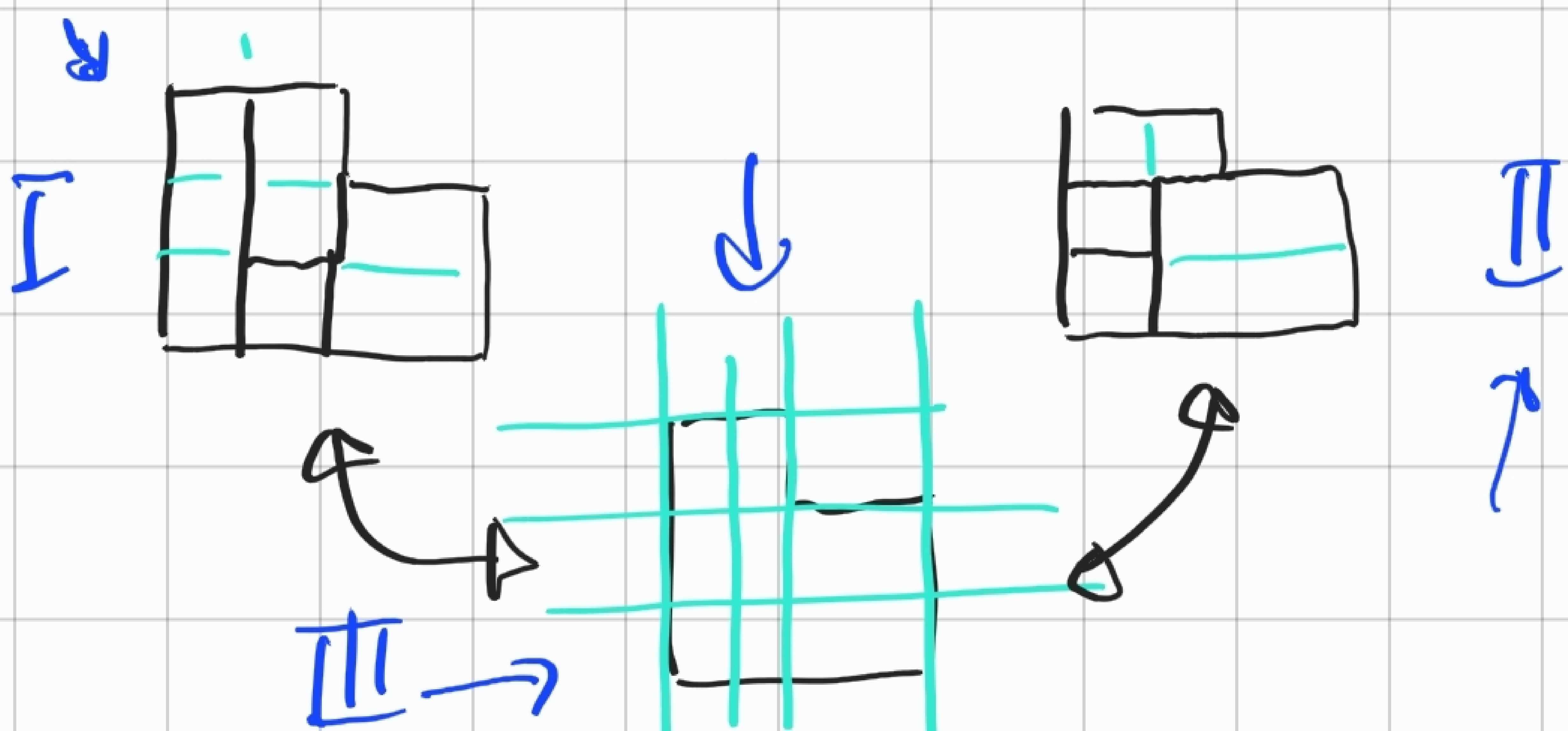
$$\begin{cases} E=(E\setminus F)\cup(E\cap F)\\ F=(F\setminus E)\cup(E\cap F)\end{cases}$$

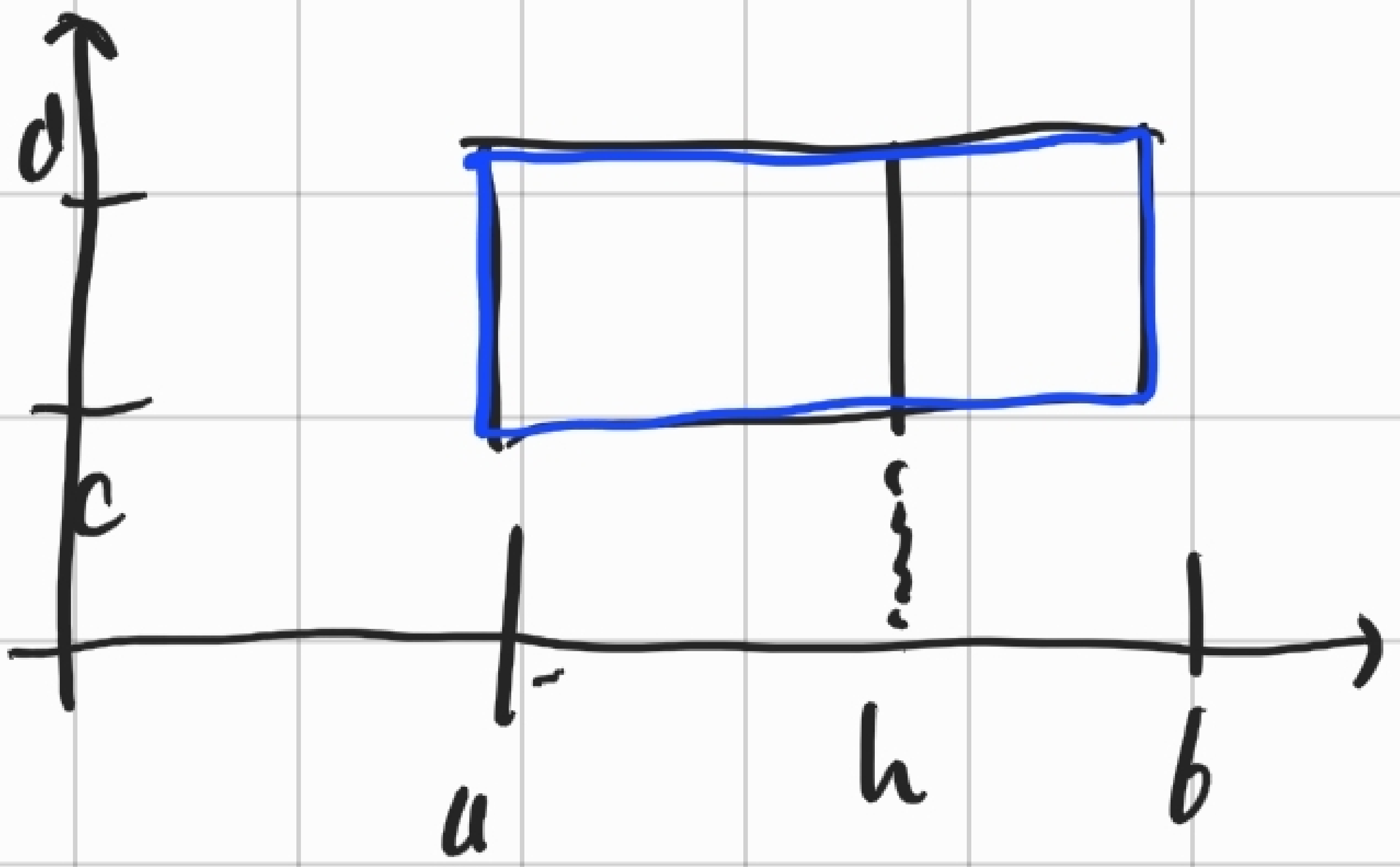
Se E è un plurirettangolo $E=\bigcup_{k=1}^{N} R_k$

$m(R_k\cap R_j)=0$

allora $m(E) = \sum_{k=1}^{N} m(R_k)$

<u>Questione</u>: è una buona definizione?





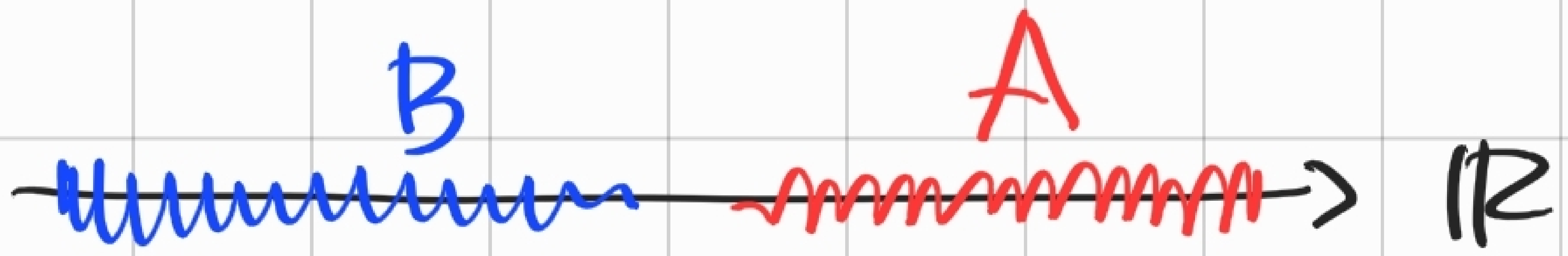
$$(b-a)\cdot(d-c)$$
$$=[(h-a)+(b-h)]\cdot(d-c)$$
$$=(h-a)\cdot(d-c)+(b-h)\cdot(d-c)$$

Def se $E \subseteq \mathbb{R}^2$, E limitato

$$m^*(E) = \inf \left\{ m(P) : \begin{array}{l} P \text{ plurirettangolo} \\ P \supseteq E \end{array} \right\} \quad A$$

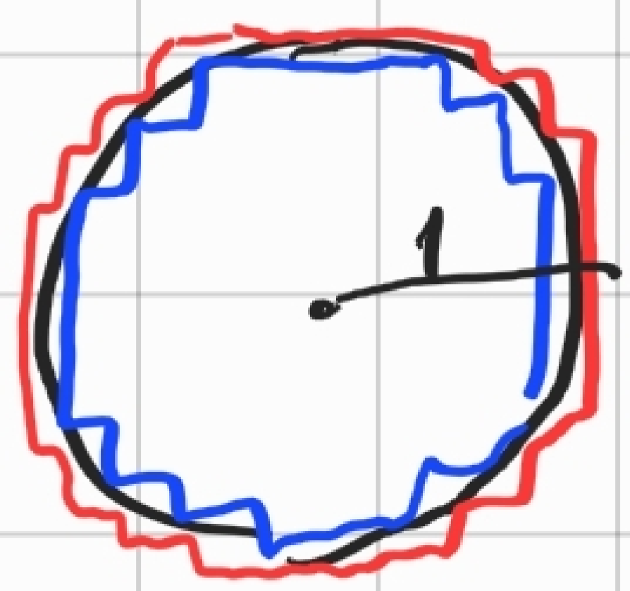
$$m_*(E) = \sup \left\{ m(P) : \begin{array}{l} P \text{ plurirettangolo} \\ P \subseteq E \end{array} \right\} \quad B$$

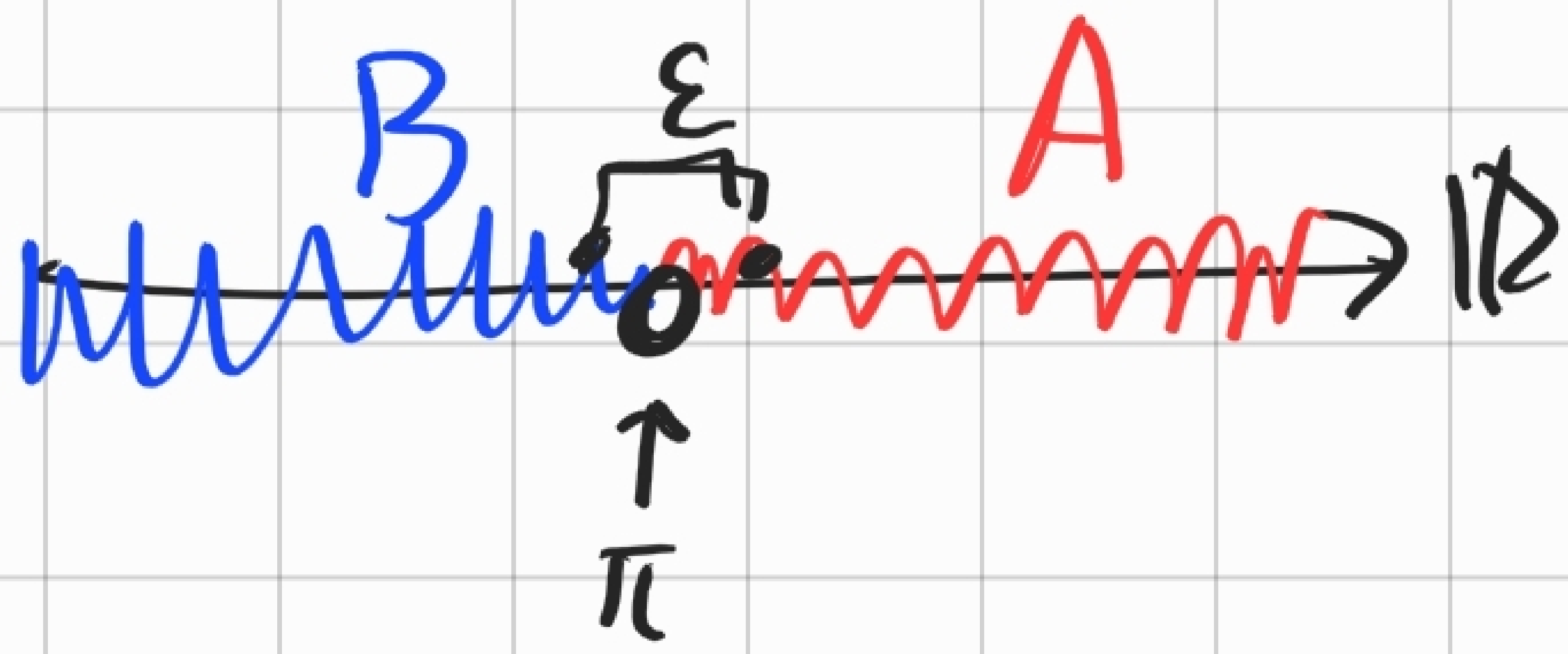
$$m^*(E) \geq m_*(E)$$

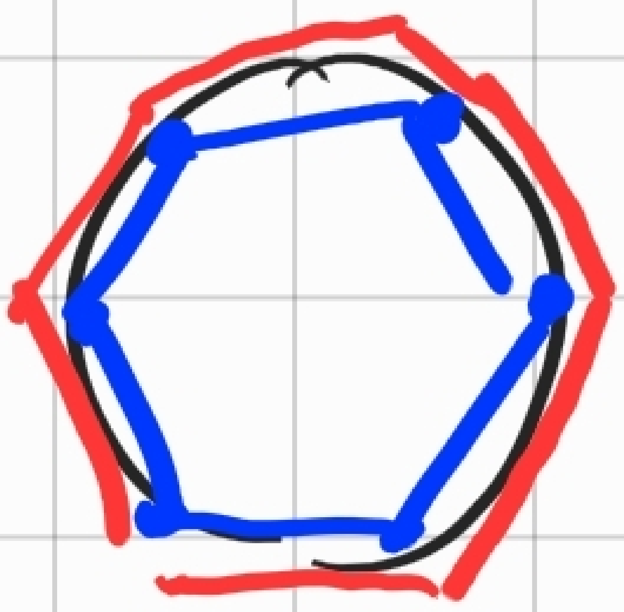


Se $m^*(E) = m_*(E)$ diremo che E è misurabile secondo Peano-Jordan e poniamo:

$$m(E) = m^*(E) = m_*(E).$$







<u>Teorema</u> (formula dell'area)

Se $L : \mathbb{R}^2 \to \mathbb{R}^2$ è lineare, se

$E \subseteq \mathbb{R}^2$ è P.J.-misurabile allora

$L(E)$ è P.J-misurabile e

$$m(L(E)) = |\det L| \cdot m(E)$$

Possibile def. di $\det L$:

(1) det è multilineare

sulle colonne di L

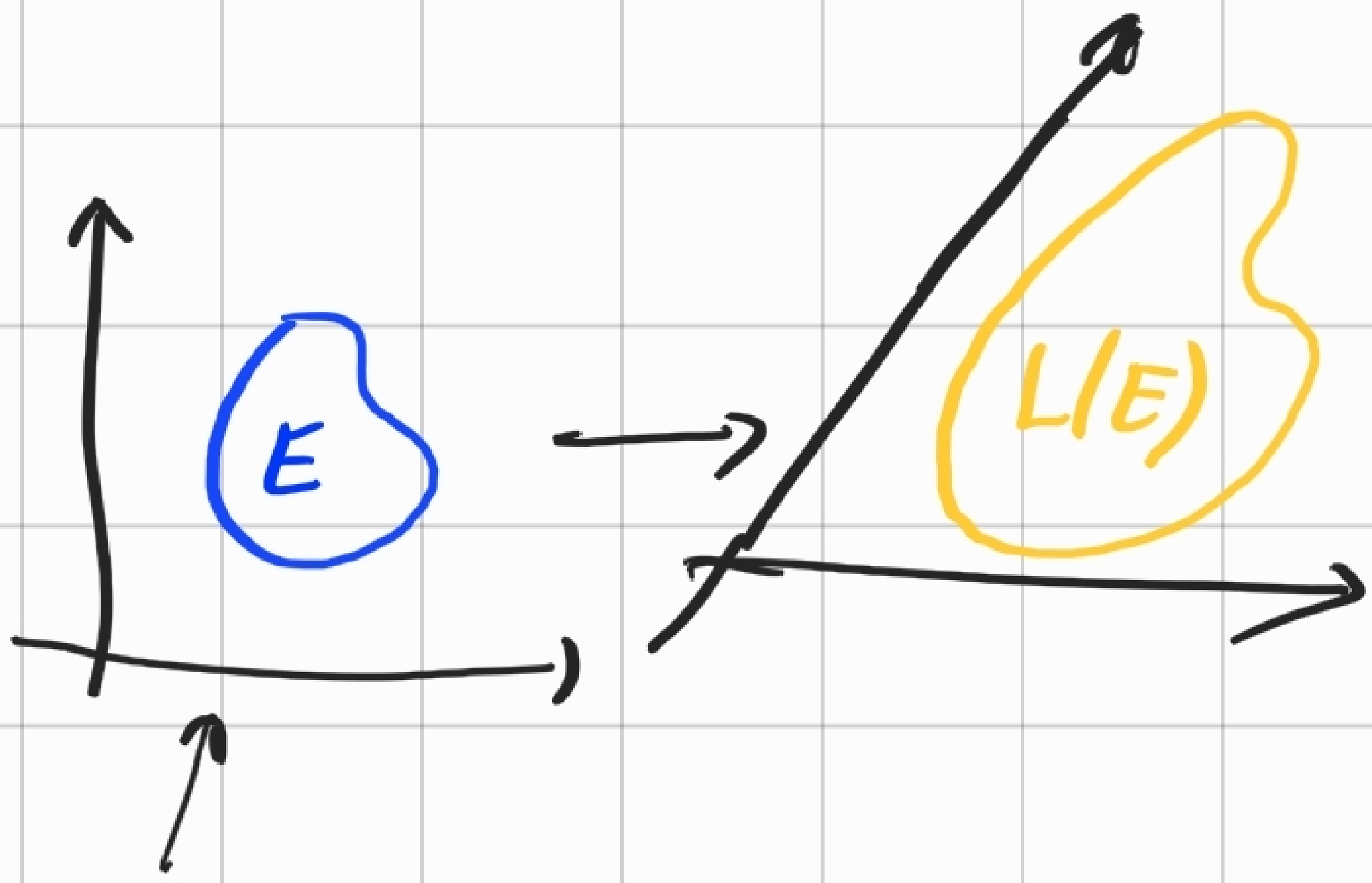
(2) cambio segno se scambio due colonne

(3) vale 1 sull'identità.

dim Per gradi su L.

(1) $L = id$. ovvio

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



(2) $L = \lambda \cdot id$ ovvio $\quad L = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \det L = \lambda^2$

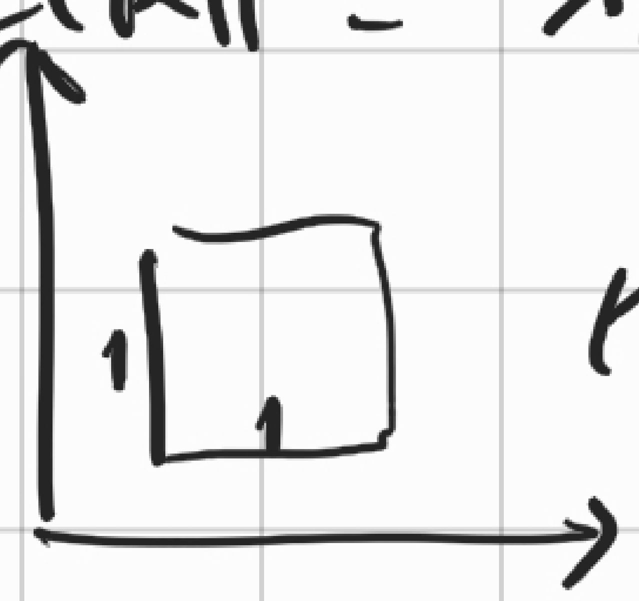
(3) $L = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \begin{array}{l} x \mapsto \lambda x \\ y \mapsto \mu y \end{array}$

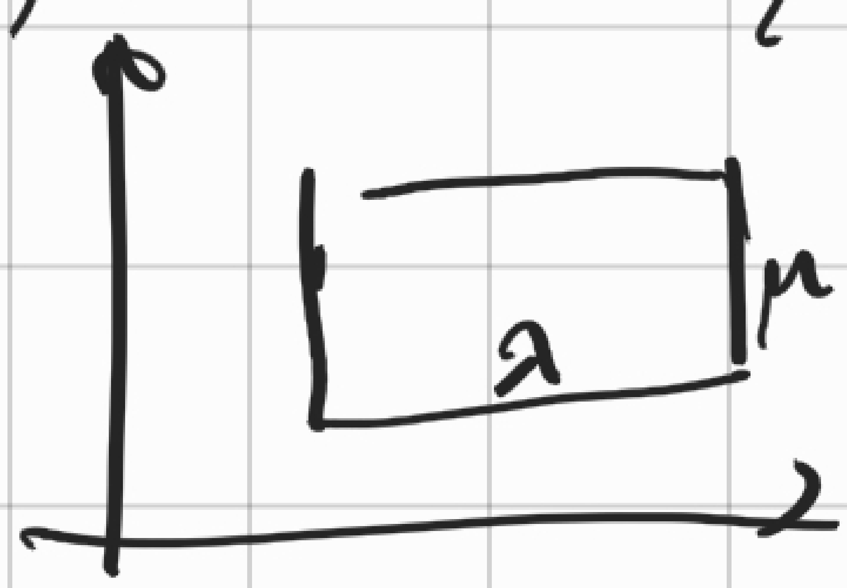
$\lambda, \mu \geq 0$

$R = [a,b] \times [c,d]$

$L(R) = [\lambda a, \lambda b] \times [\mu c, \mu d]$

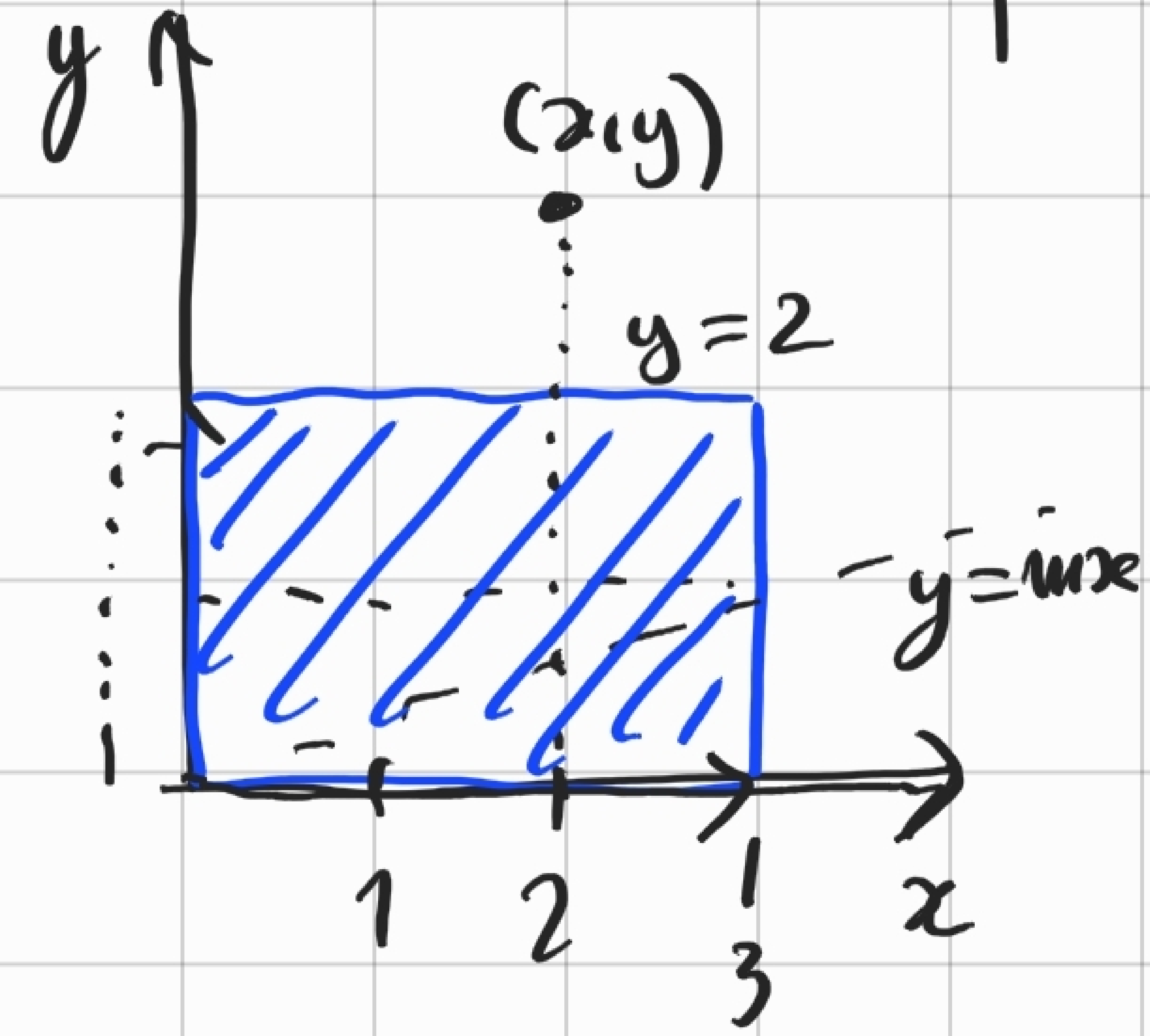
$m(L(R)) = \lambda |b-a| \cdot \mu |c-d| = \lambda \mu \; m(R) = \det L \cdot m(R)$

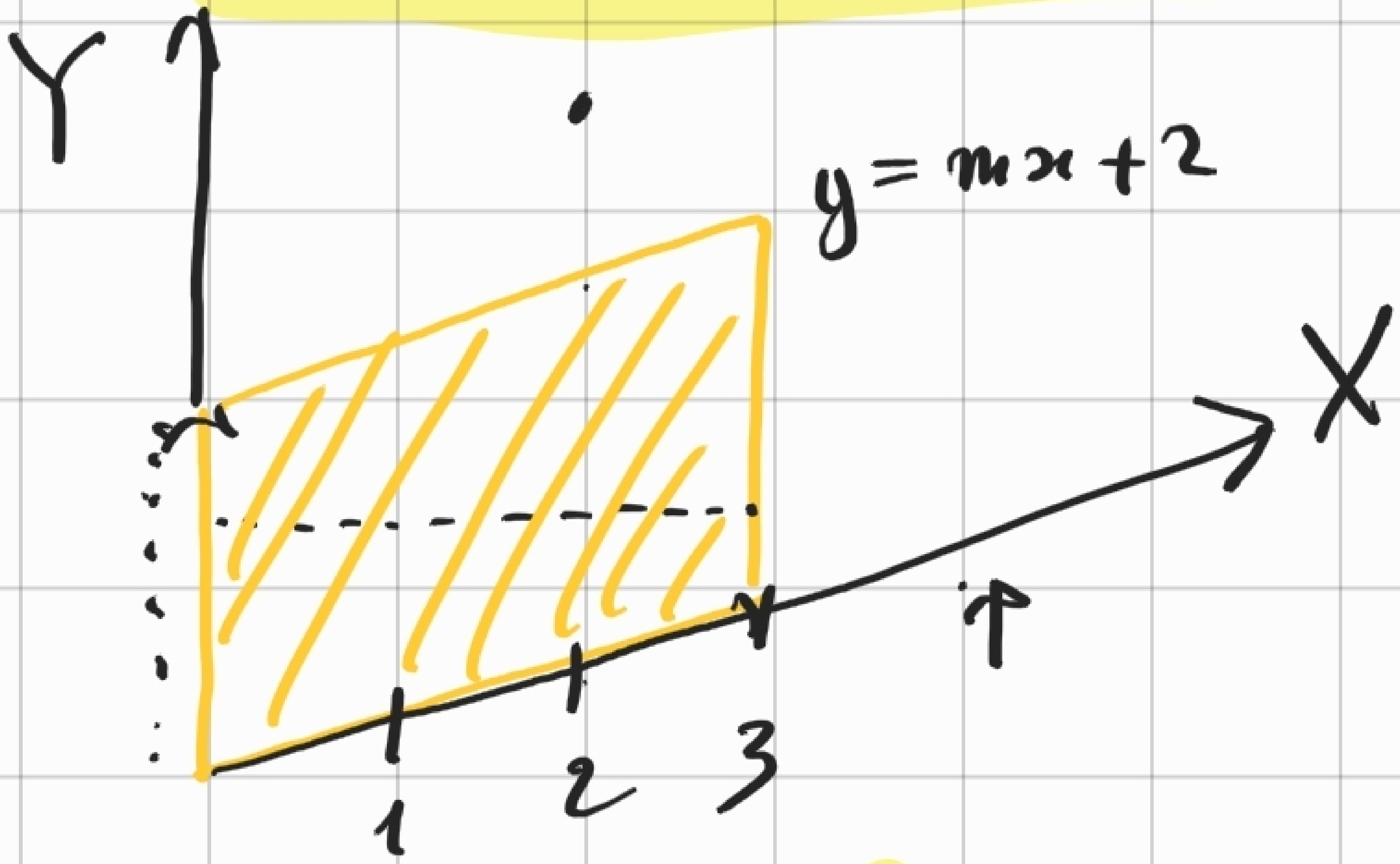




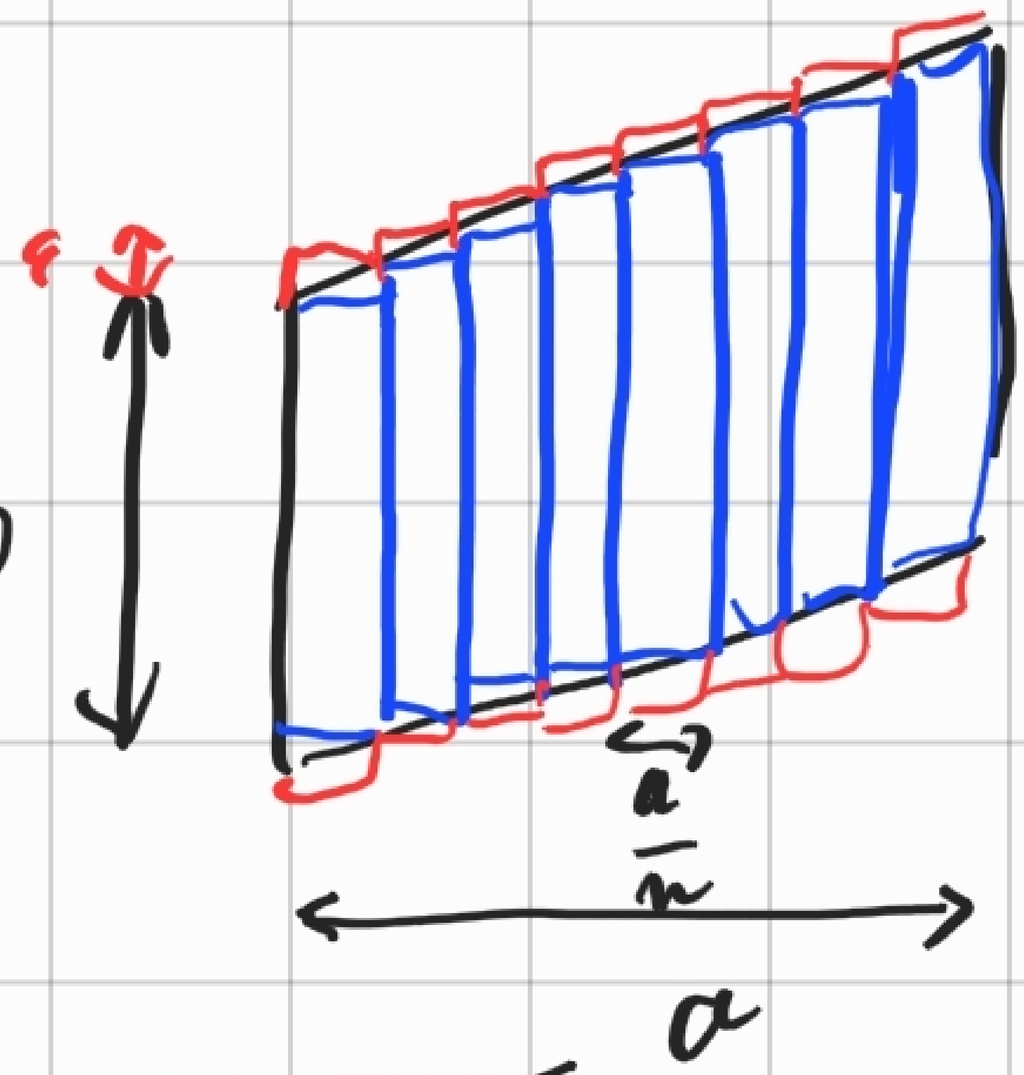
in parole $\qquad \lambda, \mu \in \mathbb{R} \qquad |\det L|$.

(4) $\boxed{L = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}}$ $\qquad \boxed{\begin{pmatrix} X \\ Y \end{pmatrix} = L \cdot \begin{pmatrix} x \\ y \end{pmatrix}}$





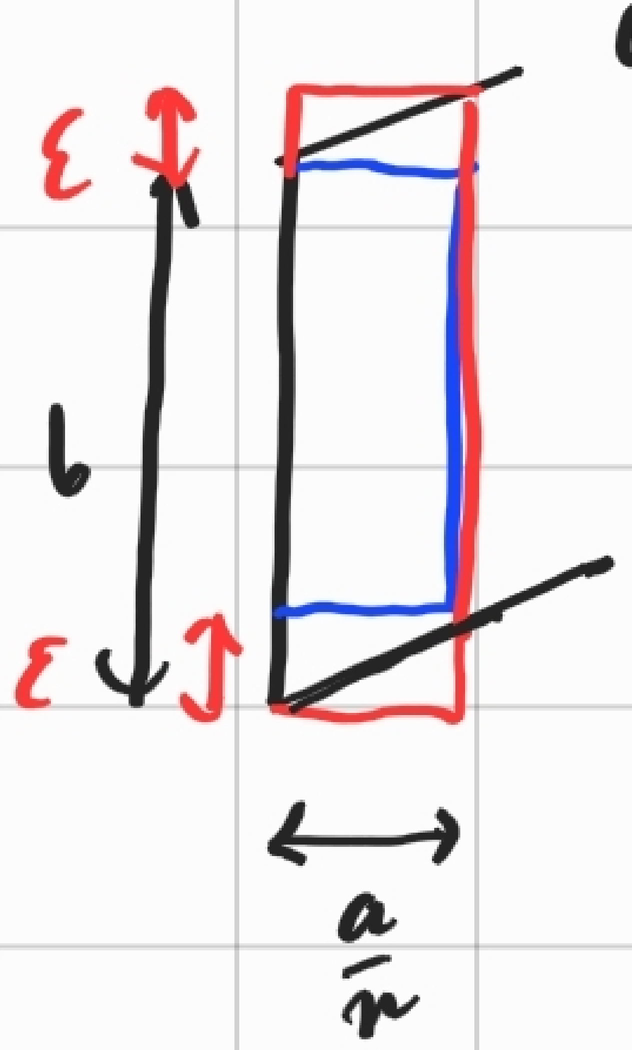
$R = [0,a] \times [0,b] \longrightarrow \begin{cases} X = x \\ Y = y + mx \end{cases}$



$n \cdot \frac{a}{n}(b-\varepsilon) \le m(L(R)) \le n \cdot \boxed{\frac{a}{n}(b+\varepsilon)}$

$\varepsilon = m\frac{a}{n}$

$n \to +\infty \qquad m(L(R)) = a \cdot b = m(R).$



(5) $L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad \begin{cases} Y = x \\ X = y \end{cases}$

ok

(6) Se L qualunque

$$L = \boxed{(\quad)(\quad)\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}\boxed{(\quad)(\quad)}$$

aggiungo ad una riga un multiplo di un'altra.

scambi di righe

simili di altro

ognuno un multiplo

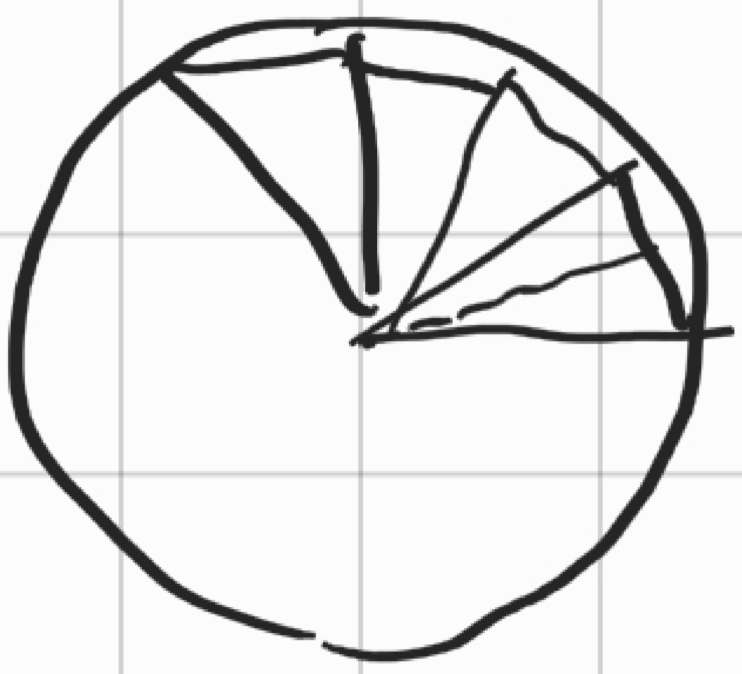
$$\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

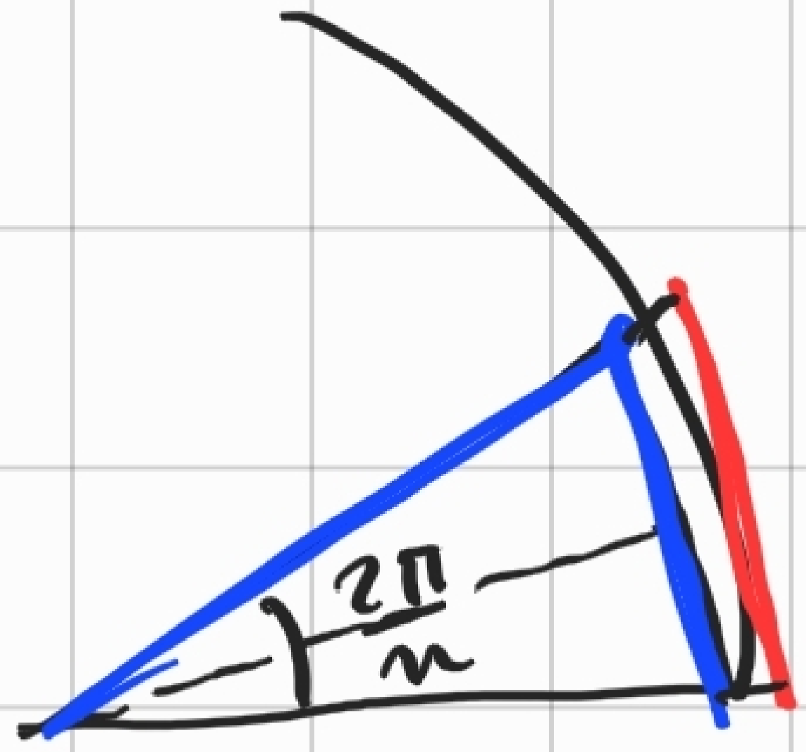
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + m\begin{pmatrix} 0 & 0 \\ \underline{a\ b} \end{pmatrix} = \begin{pmatrix} a & b \\ c+ma & d+mb \end{pmatrix}$$

aggiungo alla seconda riga un multiplo della prima.

$\frac{1}{2} a \cdot b$


h
h
a




2π/n