ANALISI MATEMATICA B
LEZIONE 58 26.2.2021

Integrale di Riemann

Proprietà dell'integrale

MONOTONIA se $f \leq g$ $\int_a^b f \leq \int_a^b g$

ES se $f \geq 0$ $\int_a^b f \geq 0$

LINEARITA' se f, g sono limitate R-integrabili su $[a,b]$, $\lambda, \mu \in \mathbb{R}$ allora $\lambda f + \mu g$ è R-integrabile su $[a,b]$ e vale:

$$\int_a^b \lambda f + \mu g = \lambda \int_a^b f + \mu \int_a^b g$$

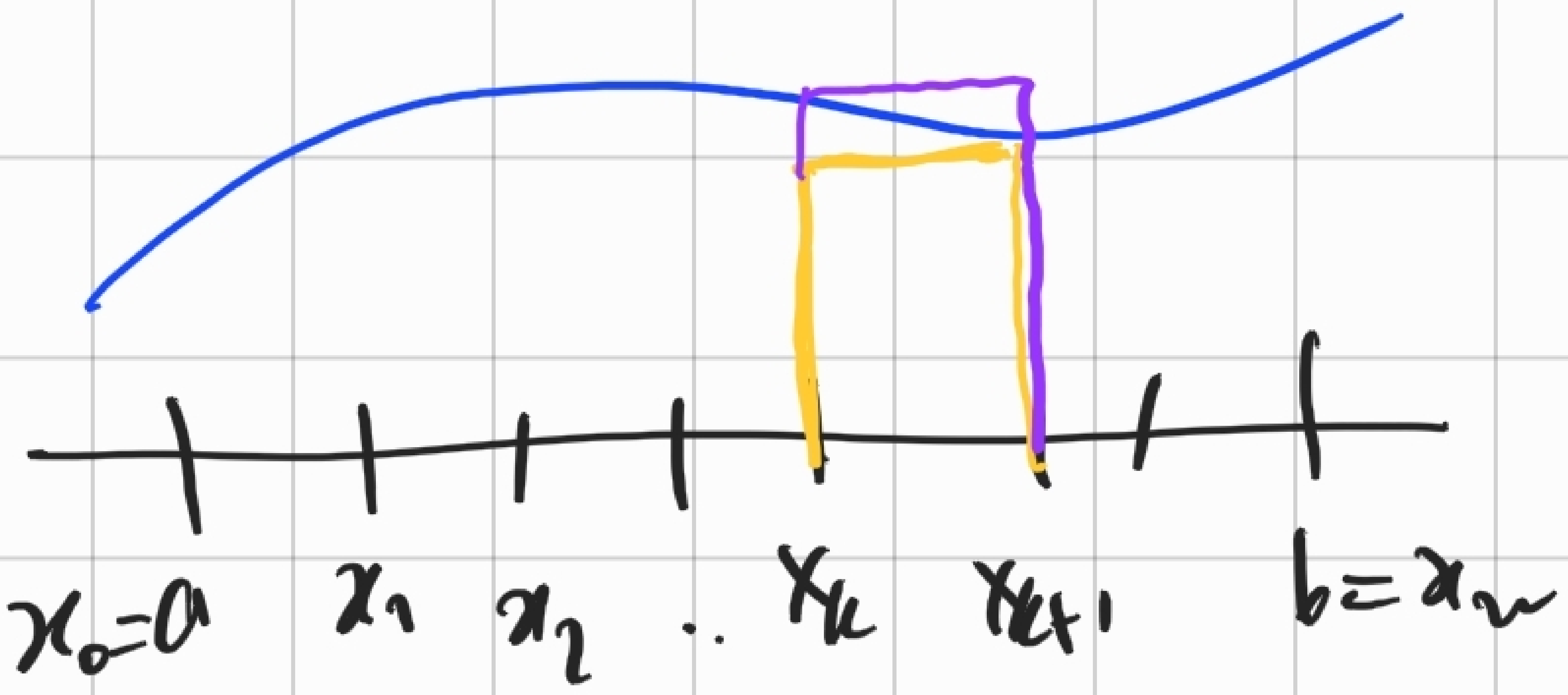
dim ① $\lambda = -1$ $\mu = 0$

$$\int_a^b (-f) = -\int_a^b f$$

P suddivisione di $[a,b]$

$P = \{x_0, x_1, \dots, x_n\}$ $\quad n = \# P \in \mathbb{N}$

$a = x_0 < x_1 < \dots < x_n = b$



$I = [x_k, x_{k+1}]$

$$\left.\begin{aligned} \sup_I (-f) &= -\inf_I f \\ \inf_I (-f) &= -\sup_I f \end{aligned}\right]$$

$$S^*(-f,P) = \sum_{k=0}^{n-1} (x_{k+1} - x_k) \sup_{[x_k, x_{k+1}]} (-f)$$

$$= -\sum_{k=0}^{n-1} (x_{k+1} - x_k) \inf f$$

$$= - S_*(f,P)$$

$$S_*(-f,P) = \dots = -S^*(f,P)$$

f R-integrabile $\Leftrightarrow \forall \varepsilon > 0 \ \exists P$

$$S^*(f,P) - S_*(f,P) < \varepsilon$$

$$\Updownarrow$$

$$-S_*(-f,P) + S^*(-f,P) < \varepsilon$$

$$\Updownarrow$$

$-f$ R-intégrable. $\iff$ $S^*(-f,P) - S_*(-f,P) < \varepsilon$

$$\int_a^b f = I^*(f) \le S^*(f,P)$$

$$S_*(f,P) \le I_*(f) = \int_a^b f$$

$$S^*(-f,P) \geq -I_*(f) \qquad S_*(-f,P) \le -I^*(f)$$

$$\left.\begin{array}{l} S_*(-f,P) \le -I^*(f) = -\int_a^b f \\ S^*(-f,P) \geq -I_*(f) = -\int_a^b f \end{array}\right]$$

$$\Rightarrow \int_a^b -f = -\int_a^b f.$$

② $\lambda \geq 0$. $\mu = 0$ $\qquad \int_a^b \lambda f = \lambda \int_a^b f.$

$\boxed{\lambda = 0 \text{ évident}}$

$$\sup_A \lambda f = \lambda \sup_A f \qquad \lambda \geq 0$$

$$\inf_A \lambda f = \lambda \inf_A f$$

$$S^*(\lambda f, P) = \sum (x_{k+1} - x_k) \sup_{[x_k, x_{k+1}]} \lambda f$$

$$= \lambda S^*(f,P)$$

$$S_*(\lambda f, P) = \lambda S_*(f,P)$$

$$I^*(\lambda f) = \lambda I^*(f) = \lambda \int_a^b f$$

$$I_*(\lambda f) = \lambda I_*(f) = \lambda \int_a^b f$$

(3) Se $\lambda \in \mathbb{R}$ $\mu = 0$ OK.

(4) $$\int_a^b (f+g) = \int_a^b f + \int_a^b g . \qquad \lambda = 1 \quad \mu = 1$$

f, g R-integrabili.

$$\sup_A (f+g) \le \sup_A f + \sup_A g$$

$$\inf_A (f+g) \ge \inf_A f + \inf_A g$$

$$S_*(f,P) + S_*(g,P) \le S_*(f+g,P) \qquad S^*(f+g,P) \le S^*(f,P) + S^*(g,P)$$

$\forall \varepsilon > 0$ $\exists P$ $\quad S^*(f,P) - S_*(f,P) < \frac{\varepsilon}{2}$

$\exists Q$ $\quad S^*(g,Q) - S_*(g,Q) < \frac{\varepsilon}{2}$

$P \cup Q$

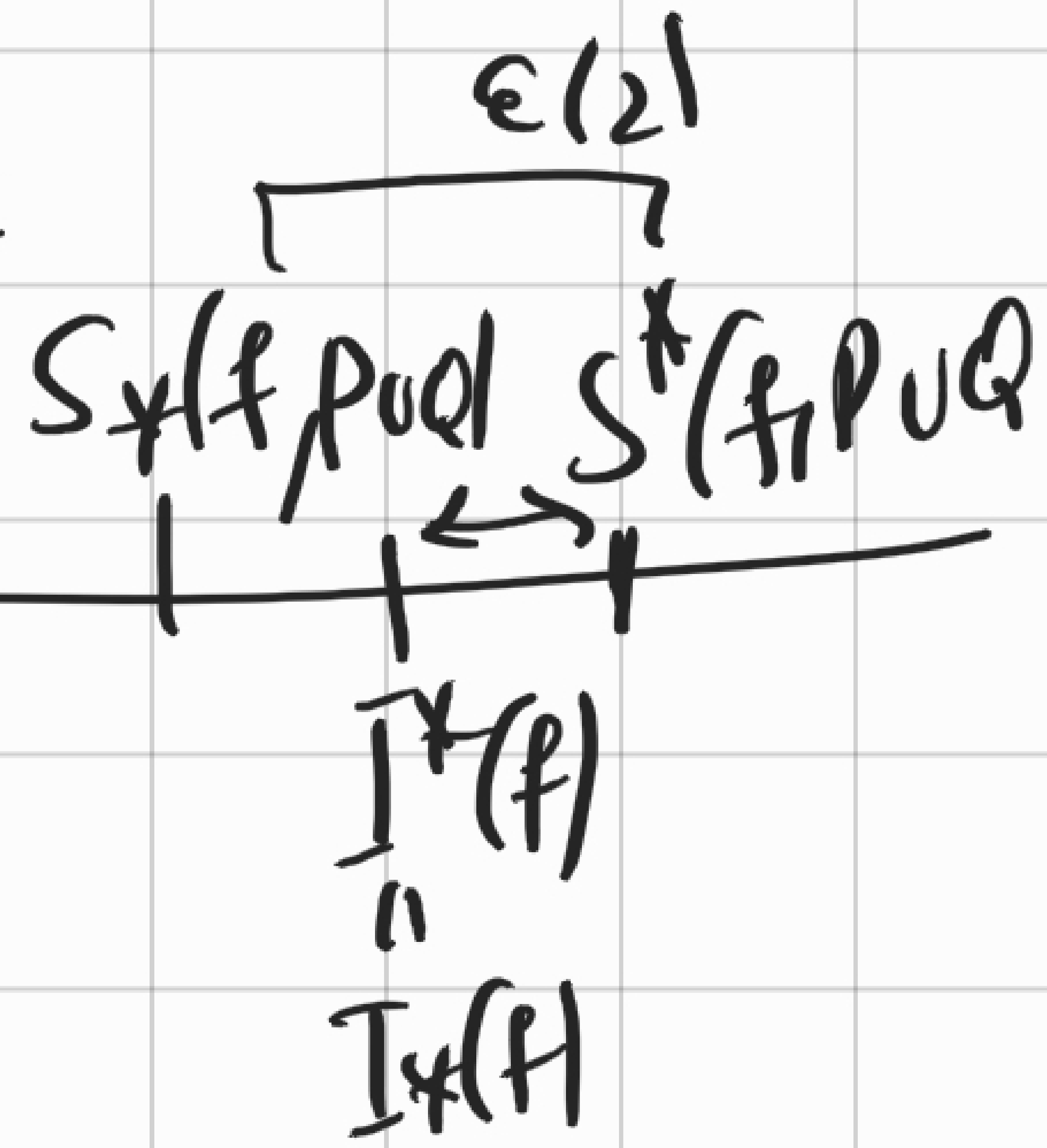
$$S^*(f+g, P\cup Q) - S_*(f+g, P\cup Q) \le \left[\begin{array}{l} S^*(f, P\cup Q) + S^*(g, P\cup Q) \\ - S_*(f, P\cup Q) - S_*(g, P\cup Q) \end{array} \right] < \varepsilon$$

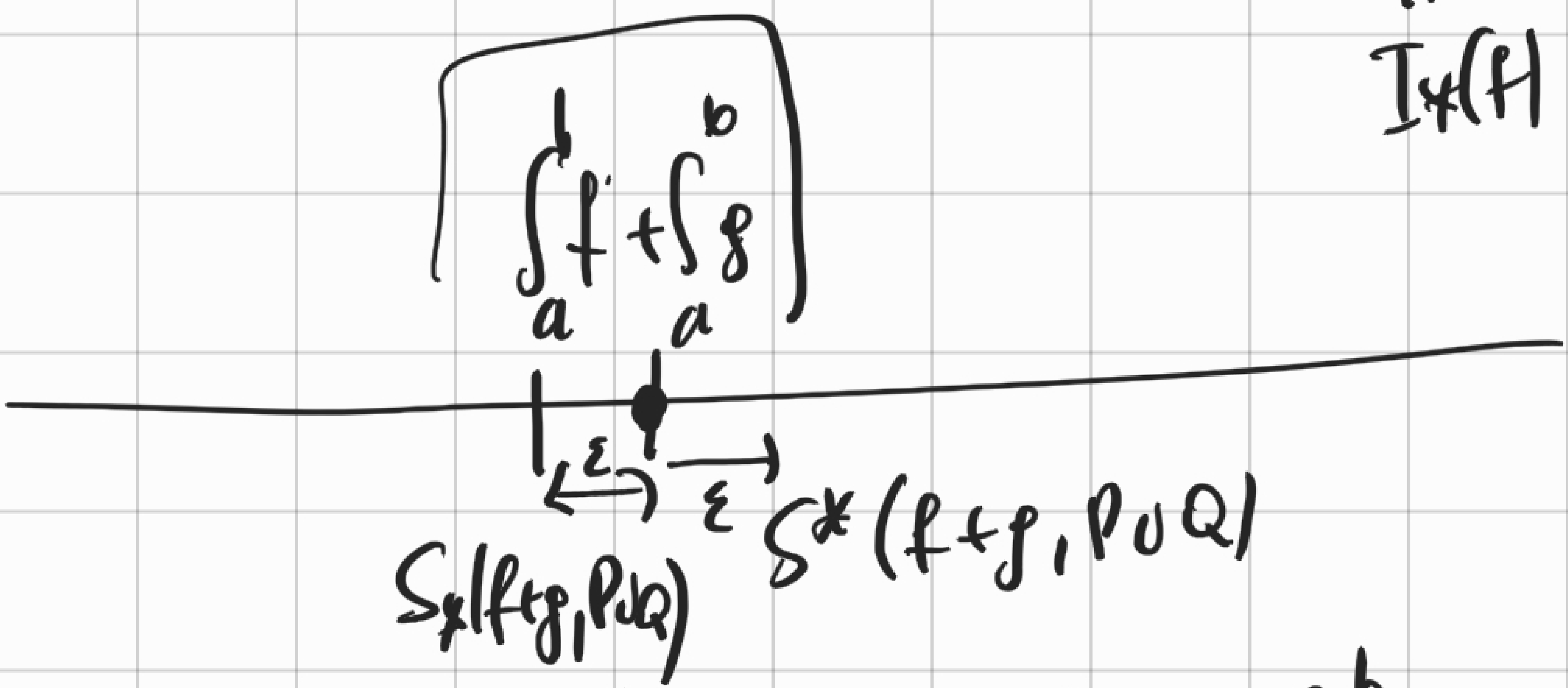
$f+g$ è integrabile

$$S^*(f+g,P\cup Q) \le \underbrace{S^*(f,P\cup Q)} + S^*(g,P\cup Q)$$

$$\le \underbrace{I^*(f)+\frac{\varepsilon}{2}} + I^*(g)+\frac{\varepsilon}{2}$$

$$S_*(f+g,P\cup Q) \ge \underset{I^*(f)}{\underset{\shortparallel}{I_*(f)}} + \underset{I^*(g)}{\underset{\shortparallel}{I_*(g)}} - \varepsilon$$



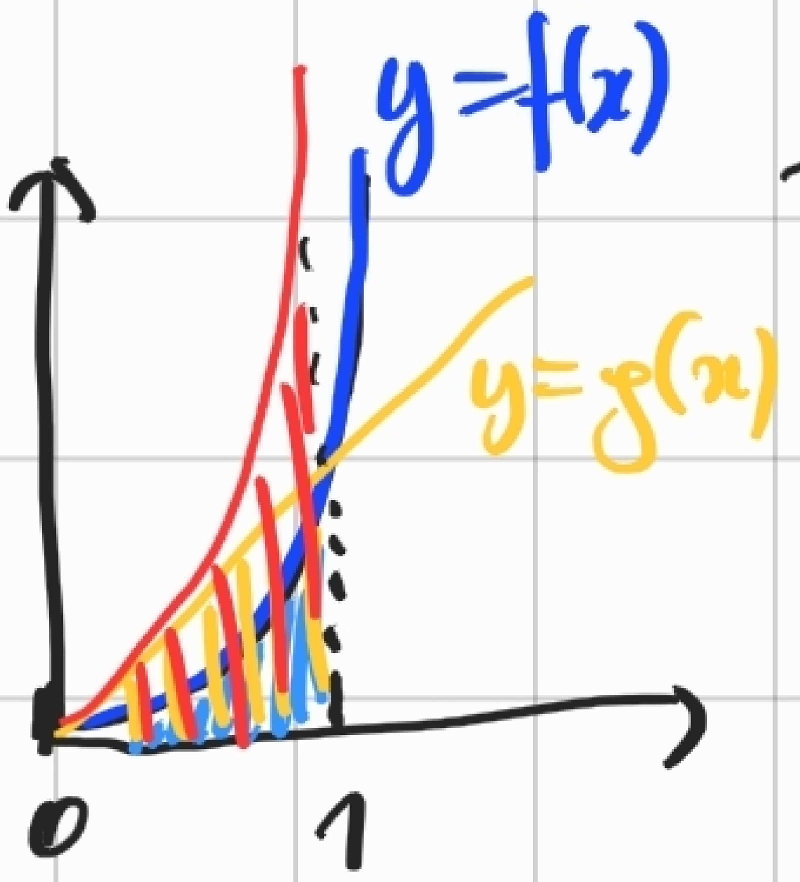


$$\Rightarrow \quad \int_a^b (f+g) = \int_a^b f + \int_a^b g .$$

Conclusión:
$$\int_a^b \lambda f = \lambda \int_a^b f$$
$$\int_a^b \mu g = \mu \int_a^b g$$

$$\int_a^b (\lambda f+\mu g) = \int_a^b \lambda f + \int_a^b \mu g = \lambda \int_a^b f + \mu \int_a^b g . \quad \square$$

ES



$f(x)=x^2$ $\qquad$ $g(x)=x$

Teorema $\quad$ Se $f:[a,b]\to\mathbb{R}$ limitata, R-integrabile $\Rightarrow |f|$ è R-integrabile.

[e vale: $\int_a^b |f| \geq \left| \int_a^b f \right|$]

dim

$$(*) = S^*(|f|,P) - S_*(|f|,P) = \sum_{k=0}^{n-1} (x_{k+1}-x_k)\left[\sup_{[x_k,x_{k+1}]} |f| - \inf_{[x_k,x_{k+1}]} |f|\right]$$

$$\left[\begin{array}{l} \sup_A |f| - \inf_A |f| \leq \underbrace{\sup_A f - \inf_A f}_{\substack{f(x)\,-\,f(y) \\ \geq \\ |f(x)| - |f(y)|}} \\ \\ |a-b| \geq \big||a|-|b|\big| \end{array}\right]$$

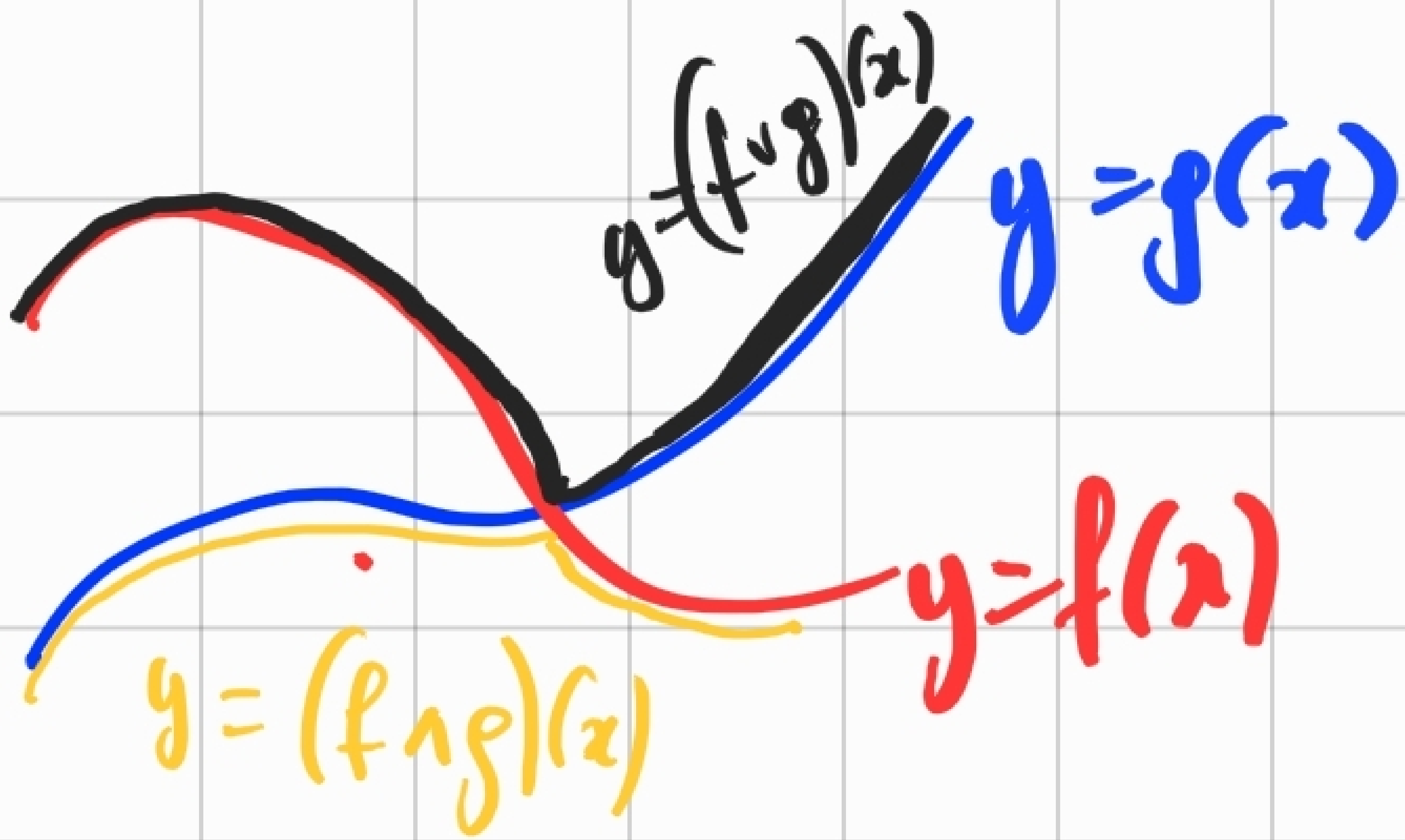
$$(*) \leq S^*(f,P) - S_*(f,P) < \varepsilon \qquad \square$$

Di conseguenza: se f, g integrabili anche:

$$\max\{f,g\} = f \vee g$$

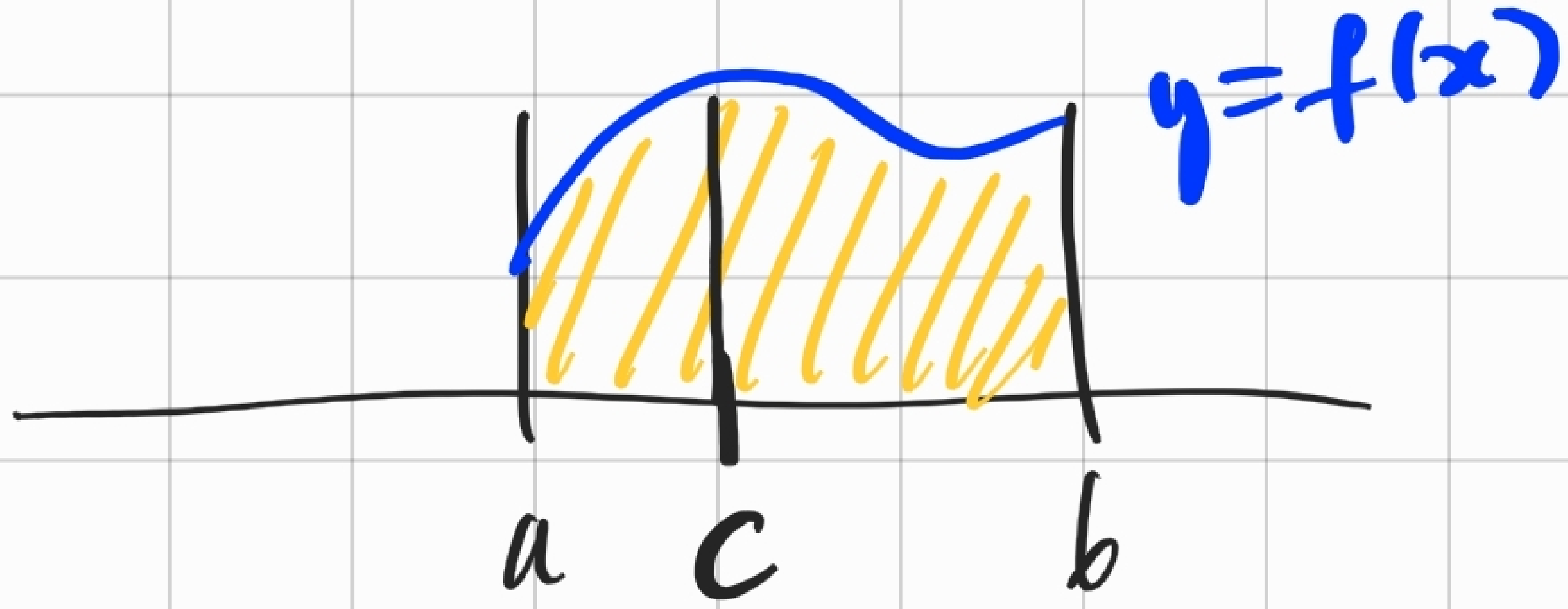
$$\min\{f,g\} = f \wedge g$$

sono integrabili



$$\begin{cases} f \wedge g = \dfrac{f+g-|f-g|}{2} \\ f \vee g = \dfrac{f+g+|f-g|}{2} \end{cases}$$

Additività rispetto al dominio



Se $f : [a,b] \to \mathbb{R}$ limitata, $c \in [a,b]$

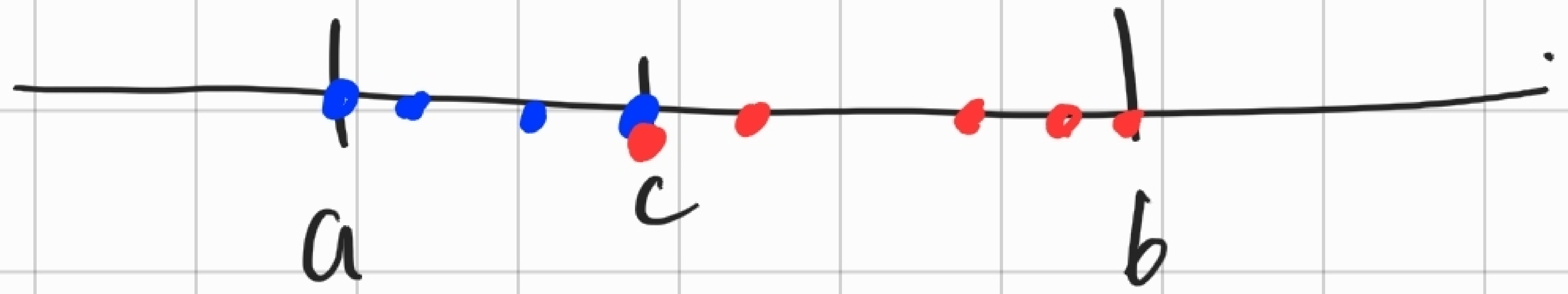
f è R-integrabile in $[a,b]$

$\Updownarrow$

f è R-integrabile in $[a,c]$ e in $[c,b]$

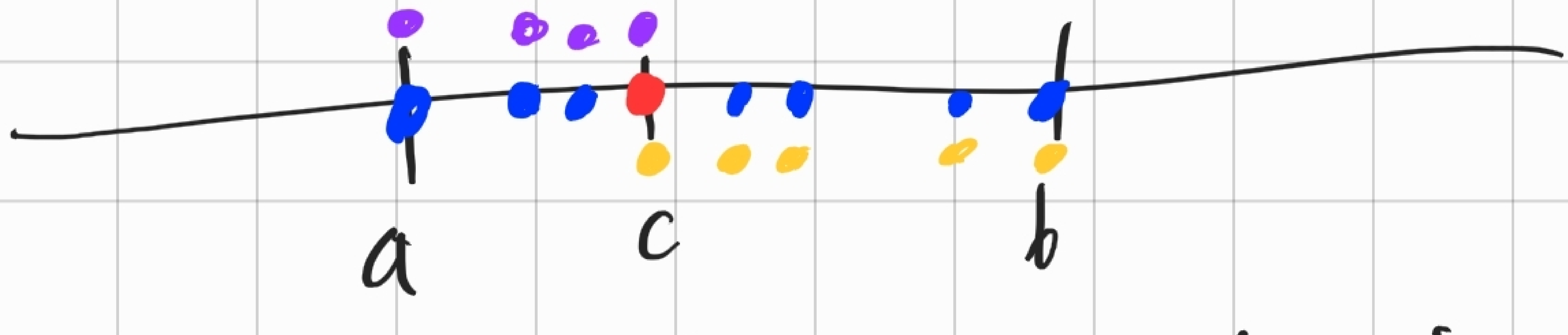
e vale $$\int_a^b f = \int_a^c f + \int_c^b f.$$

dim



P suddivisione di $[a,c]$ e Q suddivisione di $[c,b]$
allora $P \cup Q$ è una suddivisione di $[a,b]$

Se P è una suddivisione di $[a,b]$

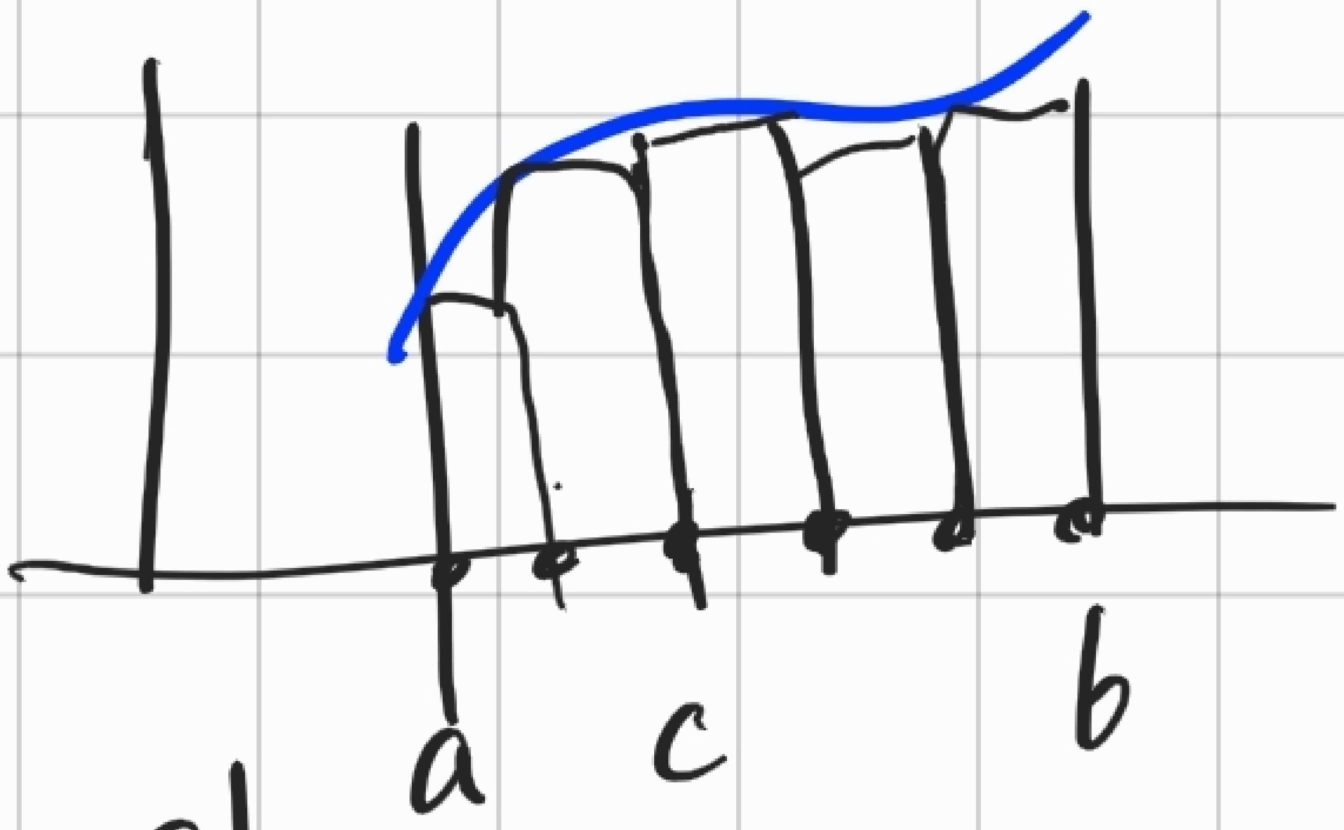


$P_1 =$ $P \cap [a,c] \cup \{c\}$ è una suddivisione di $[a,c]$
$P_2 =$ $P \cap [c,b] \cup \{c\}$ è una suddivisione di $[c,b]$

→ P suddivisione di $[a,c]$ | f int. su $[a,c]$ e su $[c,b]$

→ Q " " $[c,b]$

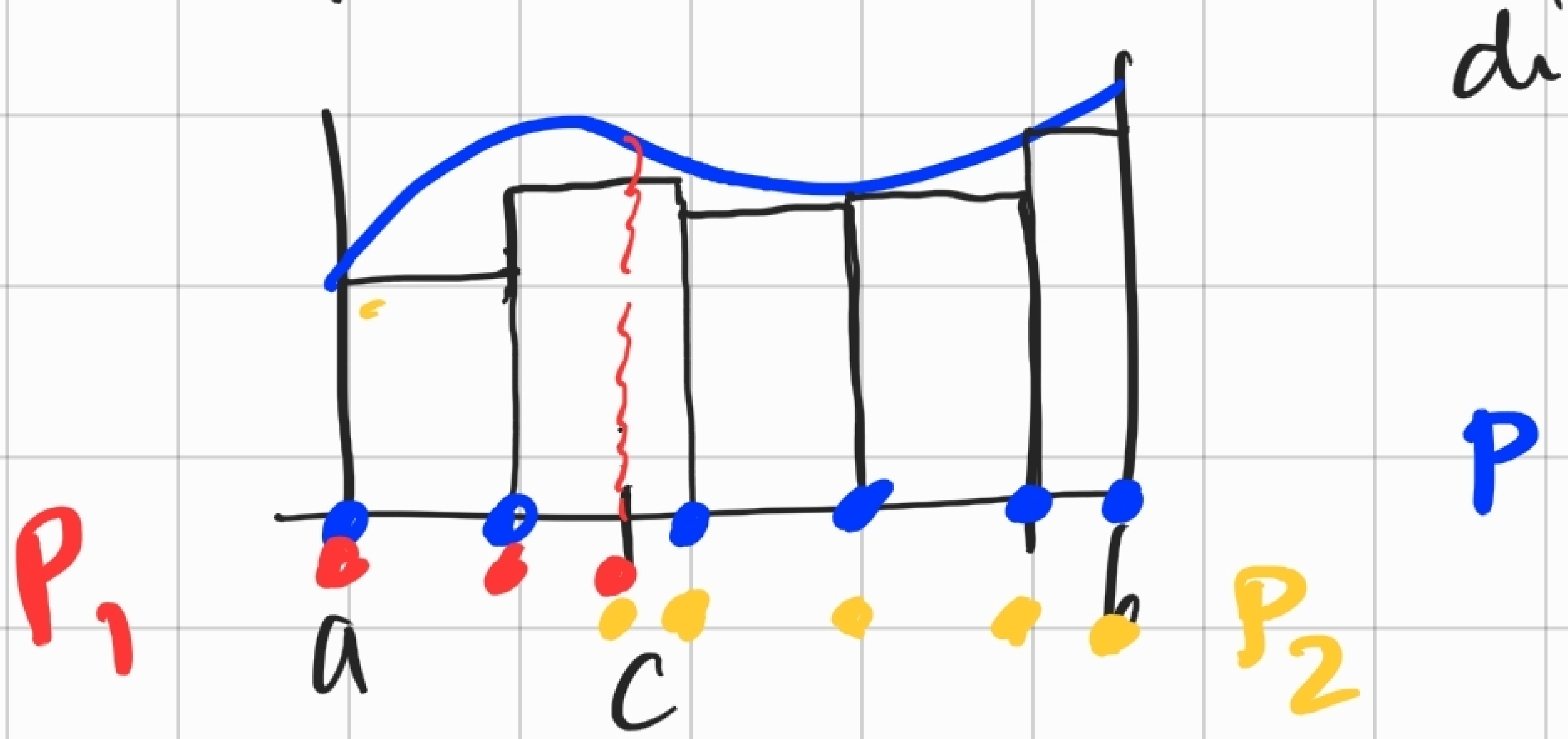
$$S^*(f, P\cup Q) = S^*(f,P) + S^*(f,Q)$$
$$S_*(f, P\cup Q) = S_*(f,P) + S_*(f,Q)$$



$$\Downarrow$$

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Se f integrabile su $[a,b]$, P suddivisione di $[a,b]$



$$S^*(f, P\cup\{c\}) \leq S^*(f,P)$$
$$S_*(f, P\cup\{c\}) \geq S_*(f,P)$$
$$\|$$
$$S^*_*(f,P_1) + S^*_*(f,P_2)$$

$\square$

<u>Convenzione</u> Se $f:[a,b]\to\mathbb{R}$ limitata R-integrabile denoteremo con:

$$\int_b^a f = -\int_a^b f$$

(analoga a $\int_a^b (-f) = -\int_a^b f$)

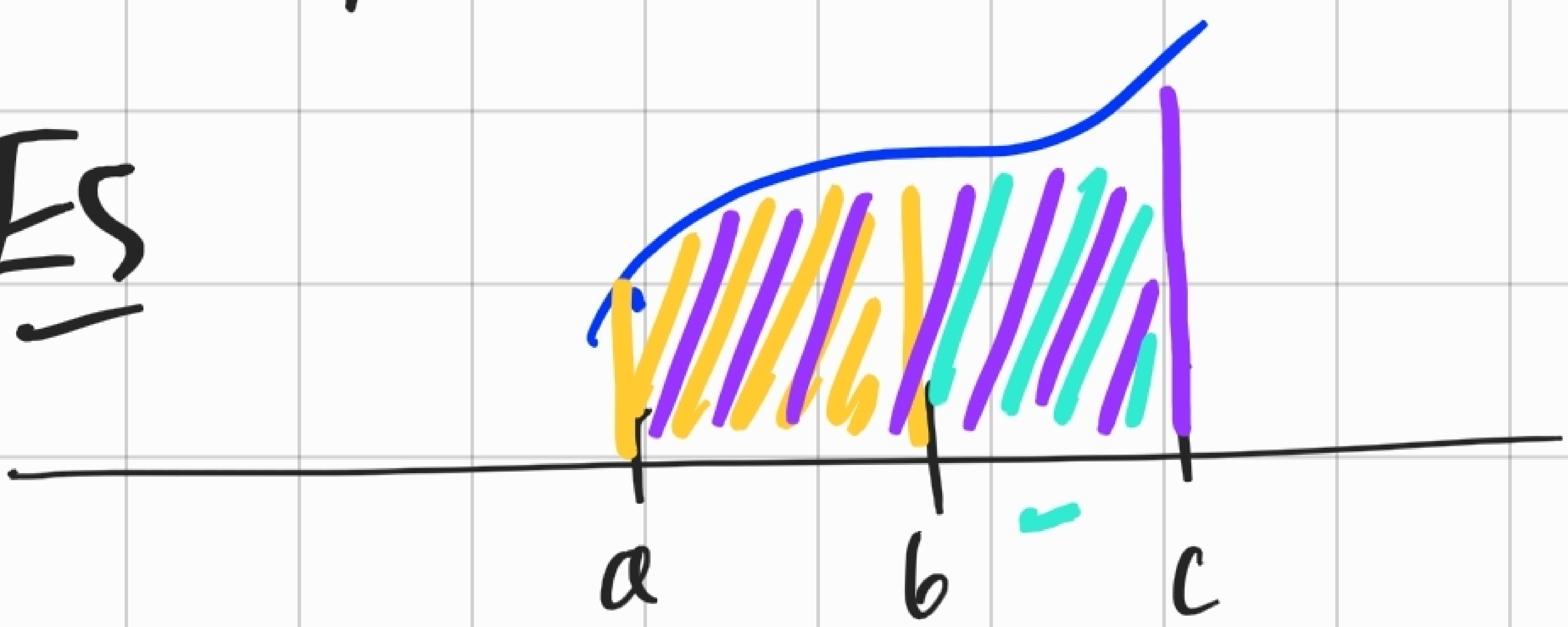
Con questa convenzione:

$$\boxed{\int_a^b f} = \boxed{\int_a^c f} + \boxed{\int_c^b f}$$

anche se $c \notin [a,b]$

(assumendo che f sia R-integrabile su tutti gli intervalli coinvolti.

<u>Es</u>



$$\int_a^c f = \int_a^b f + \int_b^c f$$

$$\int_a^b f = \int_a^c f - \int_b^c f$$

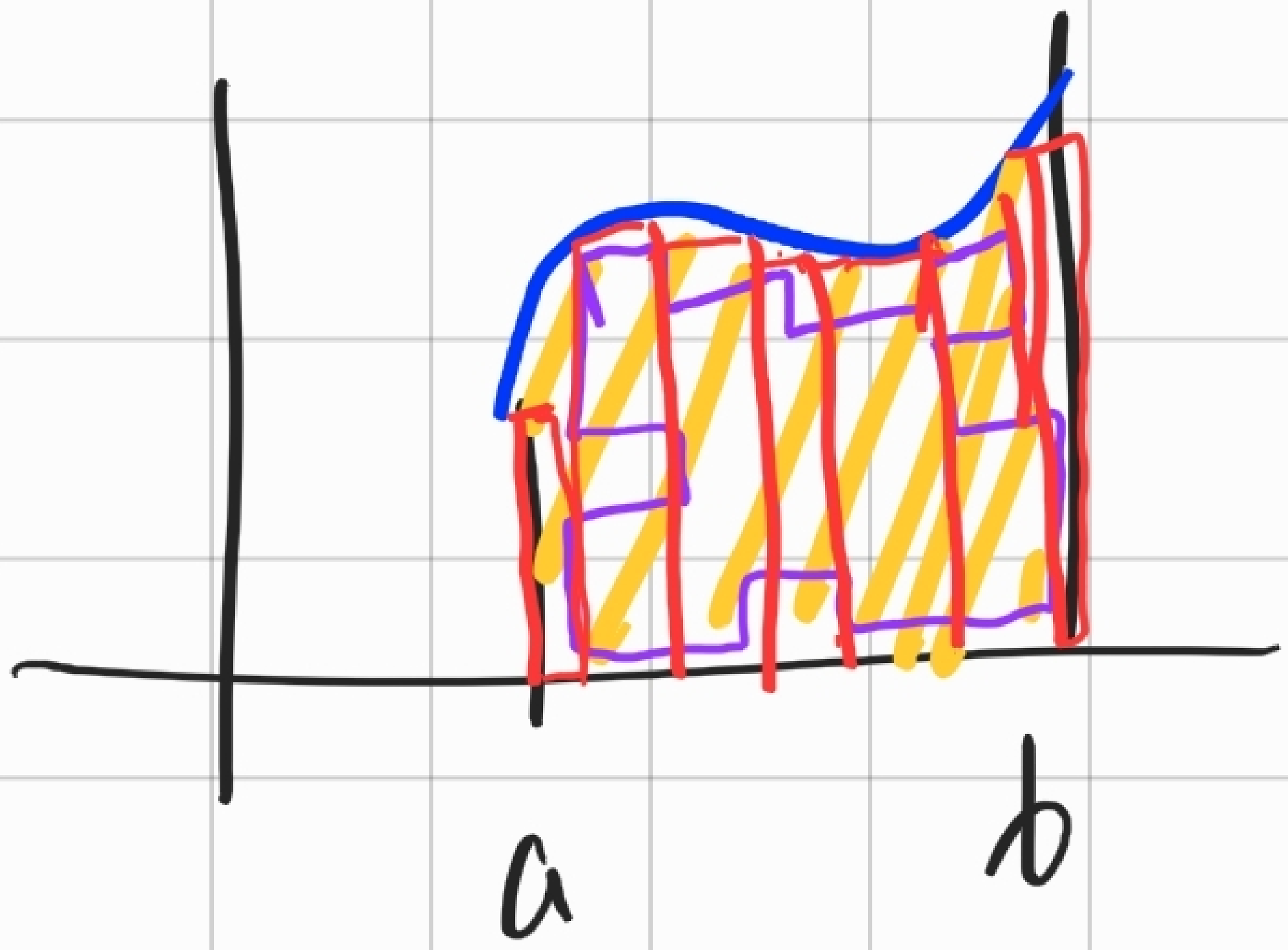
$$= \int_a^c f + \int_c^b f$$

convenzione

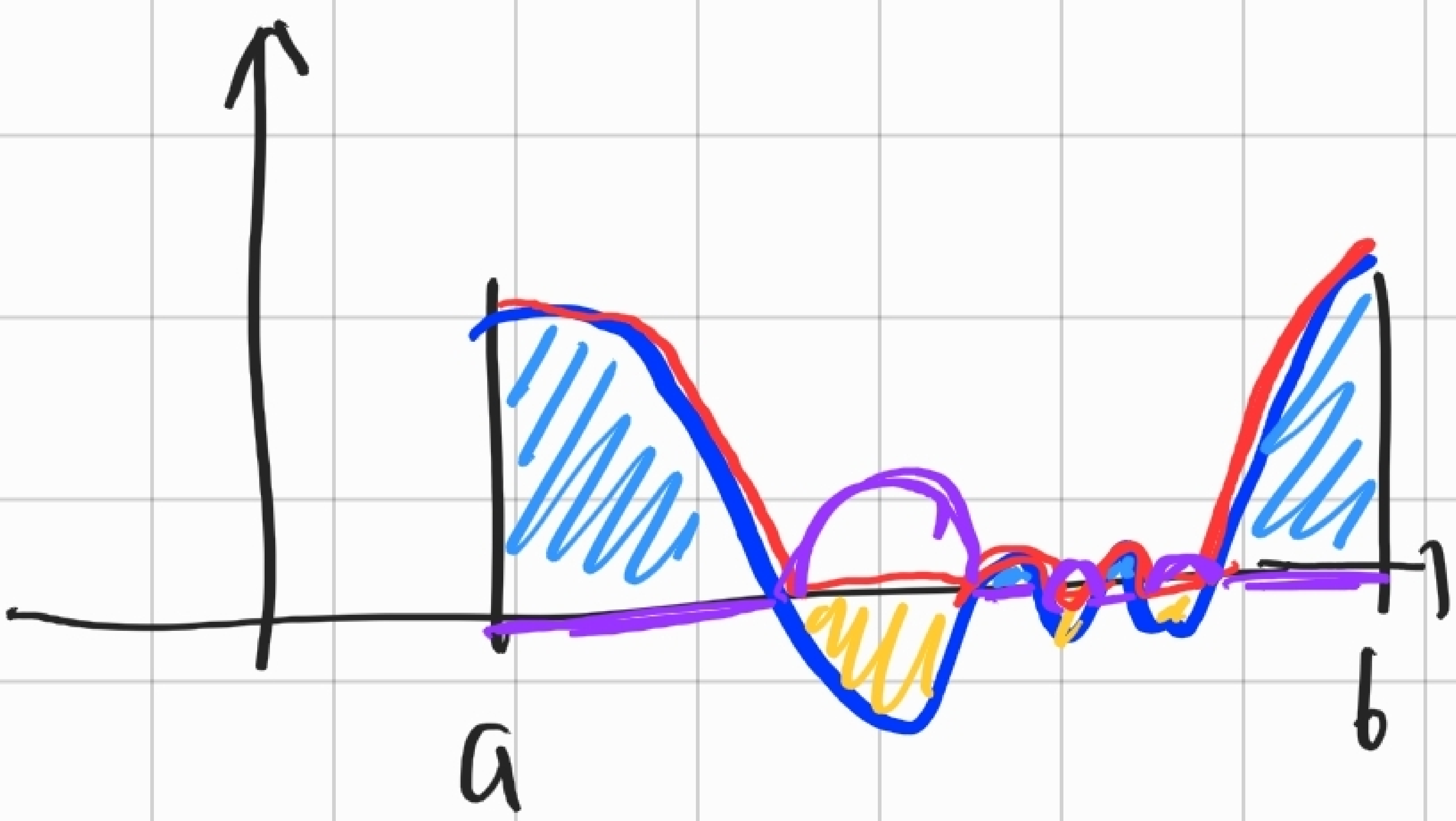
Teorema Se $f \geq 0$

$$\int_a^b f = m\left(\{(x,y) : x \in [a,b], 0 \leq y \leq f(x)\}\right)$$

misura di P-J



Se f ha segno qualunque



$$\int_a^b f = m(E^+) - m(E^-)$$

$$E^+ = \{(x,y) : x \in [a,b],\ 0 \le y \le f(x)\}$$

$$E^- = \{(x,y) : x \in [a,b],\ f(x) \le y \le 0\}$$

$$f = f^+ - f^-$$

$$\to f^+ = f \vee 0 = \begin{cases} f(x) & \text{se } x \ge 0 \\ 0 & \text{se } x < 0 \end{cases}$$

$$f^- = (-f) \vee 0$$

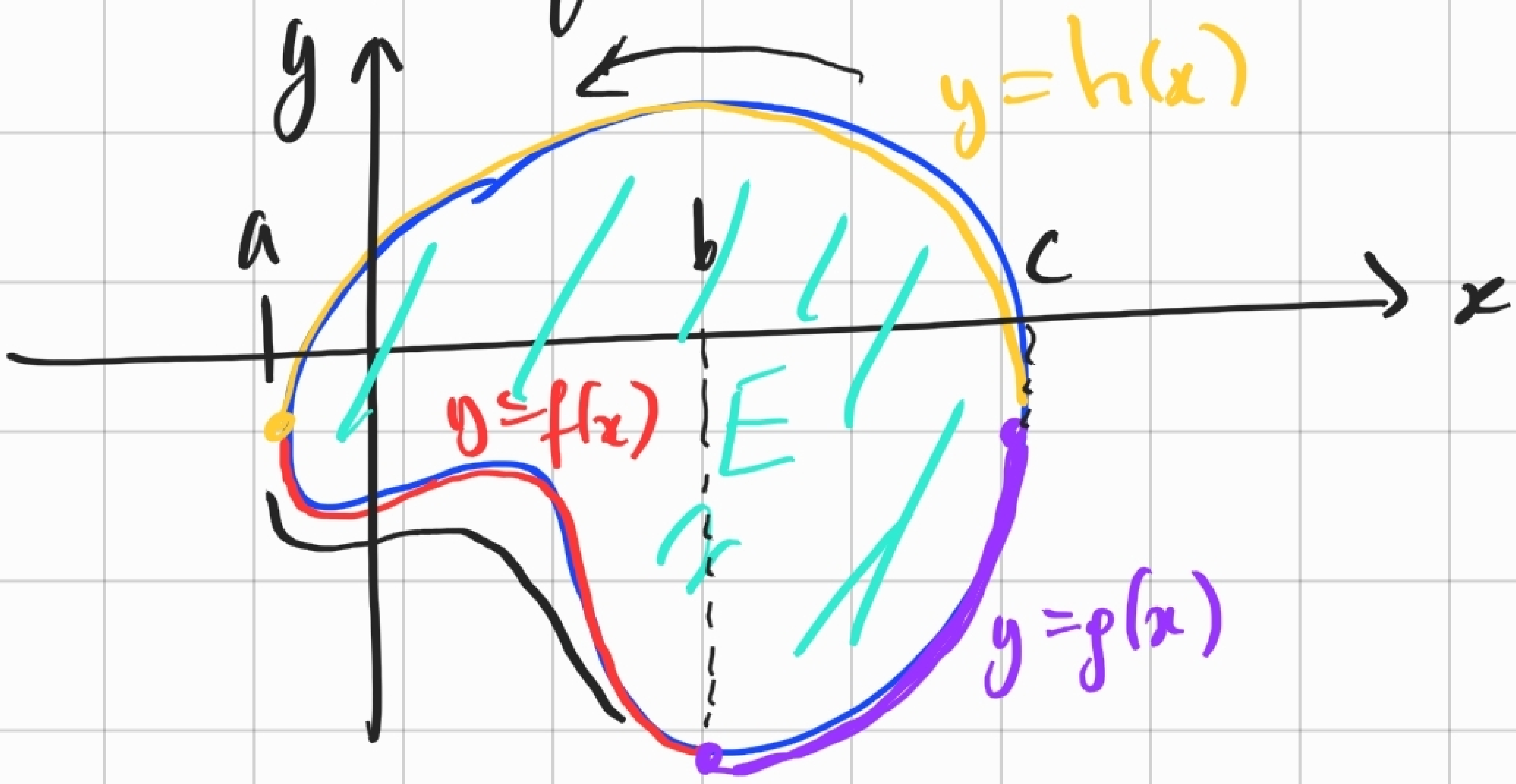
$$\int_a^b f^+ = m(E^+), \quad \int_a^b f^- = m(E^-)$$

$$f = f^+ - f^-$$

$$\int_a^b f = \int_a^b f^+ - \int_a^b f^-$$

$$= m(E^+) - m(E^-)$$

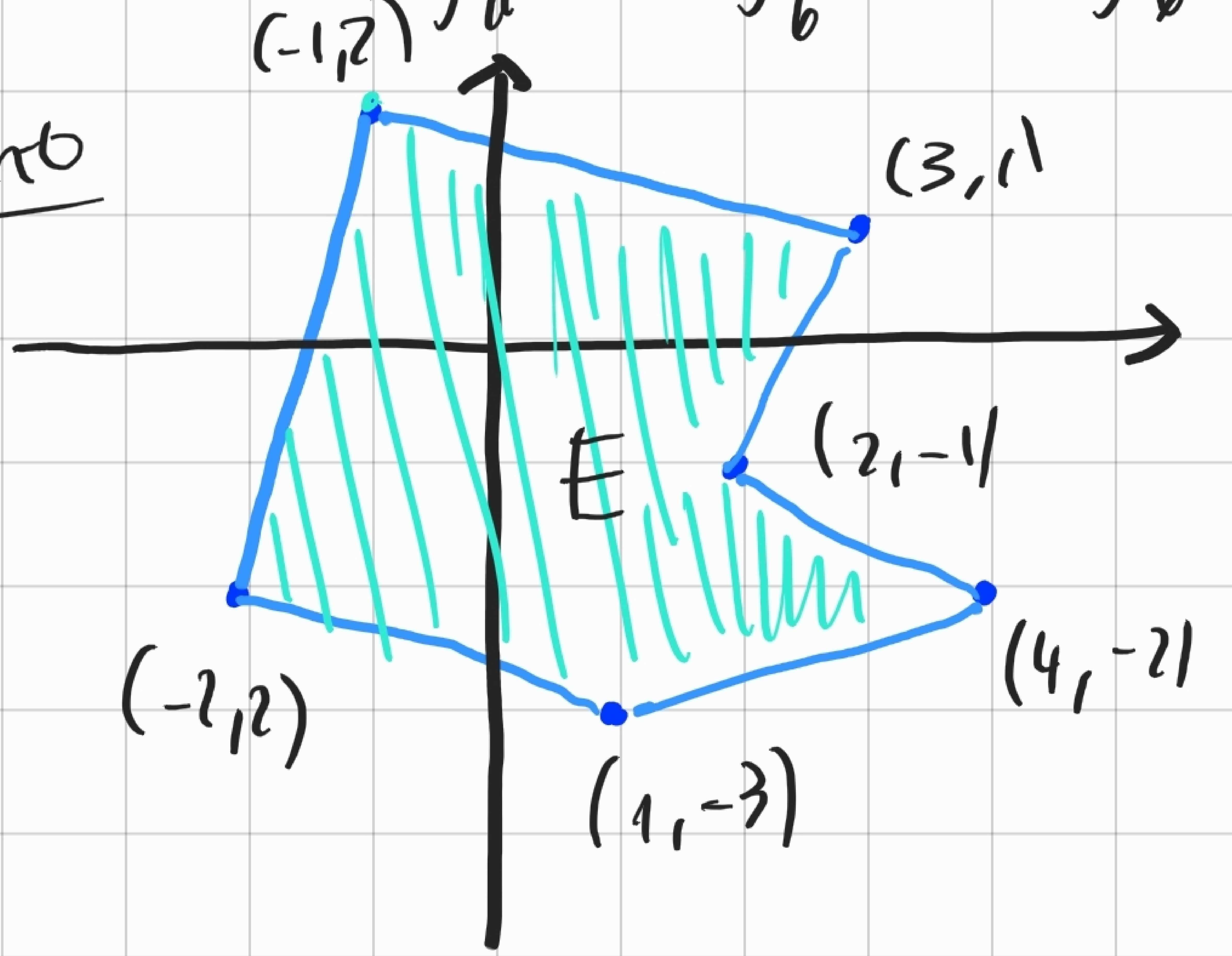
Come si calcola l'area di una superficie tramite integrali?



$$m(E) = \int_a^c h - \int_a^b f - \int_b^c g$$

$$= \int_a^c h + \int_b^a f + \int_b^c g$$

Esercizio



Quanto è l'area di E?