ANALISI MATEMATICA B
LEZIONE 75 – 16.4.2021

Metodi risolutivi

- eq. lineari del I ordine
- eq. variabili separabili I ordine
- eq. lineari a coefficienti costanti di ordine qualunque

$$u^{(n)}(x) + \sum_{k=0}^{n-1} a_k \cdot u^{(k)}(x) = b(x)$$

$b=0$ omogenea

$b\neq 0$ non omogenea.

$$Lu = P(D)u = b(x)$$

$$P(\lambda) = \lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k$$

$$P(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_n)$$

$$u_k(x) = e^{\lambda_k x}$$

$$\text{span}\{u_1, u_2, \dots, u_k\} = \ker L = \ker P(D)$$

$$\| $$

$$\{\text{soluzioni dell'equazione omogenea } Lu=0\}$$

- radici reali distinte
- radici con molteplicità

- radici complesse
- eq. non omogenea
  - metodo di somiglianza
  - metodo della variazione delle costanti

ES $\boxed{u'' + 2u' + u = 0}$

$P(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$

$\lambda_1 = -1, \quad m_1 = 2$

molteplicità

$(D+1)^2 u = 0$

$\underbrace{(D+1)\boxed{(D+1)u}}$ $= 0$

$(D+1)u = u' + u$

$u' + u = 0 \leftarrow \qquad u' - \lambda u = 0$

$u_1(x) = e^{\lambda x}$.

una sol. è $\quad u(x) = e^{-x}$

$? \overset{D+1}{\longmapsto} e^{-x} \overset{D+1}{\longmapsto} 0 \qquad (D+1)u = e^{-x}$

$u' + u = e^{-x}$

$u' e^x + u e^x = 1$

$(u \cdot e^x)' = x'$

$u e^x = x \qquad u_2 = x e^{-x}$.

u_2 è indipendente da u_1 $\qquad \frac{x e^{-x}}{e^{-x}} = x$ non è costante.

$$\ker (D+1)^2 = \{\text{soluzioni di } Lu=0\} = \operatorname{span}\{u_1, u_2\}$$

$$u(x) = A \cdot e^{-x} + B \cdot x e^{-x} = (A+Bx)e^{-x} \quad \square$$

In generale quanto fa:

(q polinomio)

$$(D-\lambda)\left[q(x) \cdot e^{\lambda x}\right] =$$

$$= \left(q(x)e^{\lambda x}\right)' - \lambda q(x) e^{\lambda x}$$

$$= q'(x) e^{\lambda x} + \cancel{q(x) \lambda e^{\lambda x}} - \cancel{\lambda q(x) e^{\lambda x}}$$

$$= q'(x) e^{\lambda x}$$

$$(D-\lambda)^m \left[q(x) \cdot e^{\lambda x}\right] = q^{(m)}(x) \cdot e^{\lambda x}$$

$Lu = \lambda u$

autovettore

$(L-\lambda)u = 0$

$(L-\lambda)^m u = 0$

Se $\deg q < m$ $\quad (D-\lambda)^m \left[q(x) e^{\lambda x}\right] = 0.$

una base di sol. è: $1 \cdot e^{\lambda x}, \; x e^{\lambda x}, \; x^2 e^{\lambda x}, \ldots, x^{m-1} e^{\lambda x}$

Es

$$u^{IV} - 6u''' + 9u'' = 0$$

$$P(\lambda) = \lambda^4 - 6\lambda^3 + 9\lambda^2$$
$$= \lambda^2(\lambda^2 - 6\lambda + 9) = \lambda^2(\lambda - 3)^2$$

$\lambda_1 = 0$, $m_1 = 2$

$\lambda_2 = 3$, $m_2 = 2$.

$u_1 = e^{0\cdot x} = 1$, $\quad u_2 = x \cdot e^{0\cdot x} = x$

$u_3 = e^{3\cdot x}$, $\quad u_4 = x \cdot e^{3x}$

$$\begin{aligned} u(x) &= A \cdot u_1 + B u_2 + C u_3 + D u_4 \\ &= A + B\cdot x + C e^{3x} + D \cdot x \cdot e^{3x} \\ &= A + Bx + (C + Dx)\, e^{3x} \end{aligned}$$

Se le radici sono complesse?

Es

$u' = u$, $\quad \lambda = 1$

$u'' = u$, $\quad \lambda^2 - 1 = 0 \quad (\lambda+1)(\lambda-1) = 0$

$u^{iv} = u$ $\quad$ sin, cos $\quad \lambda^4 - 1 = 0$

$(\lambda^2 - 1)(\lambda^2 + 1) = 0$

$(\lambda-1)(\lambda+1)(\lambda+i)(\lambda-i)$

$e^{x} \quad e^{-x} \quad e^{-ix} \quad e^{ix}$

$$\text{span}_{\mathbb{C}}\left\{ e^{ix}, e^{-ix} \right\} = \text{span}_{\mathbb{C}} \left\{ \sin x, \cos x \right\}$$

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

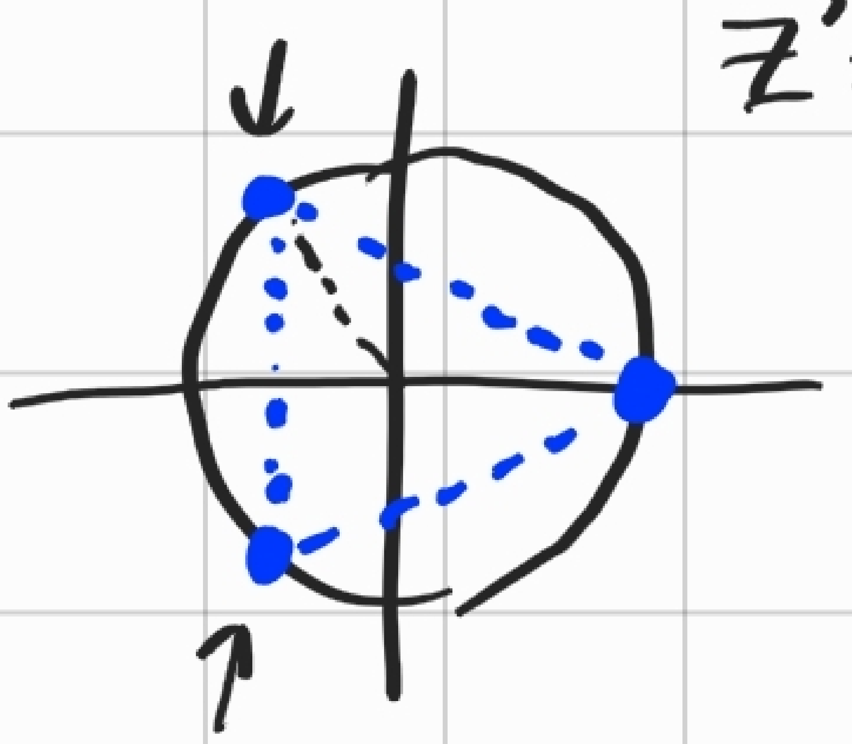
Es

$$u''' = u$$

$$u''' - u = 0 \qquad P(\lambda) = \lambda^3 - 1$$

$$= (\lambda - 1)(\lambda^2 + \lambda + 1)$$

$$= \dots\dots$$



$$z^3 = 1$$

$$\lambda_1 = 1$$

$$\lambda_{2,3} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$u_1 = e^{1\cdot x} = e^{x}$$

$$u_2 = e^{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x} = e^{-\frac{x}{2}} e^{i\frac{\sqrt{3}}{2}x} = e^{-\frac{x}{2}}\left(\cos\frac{\sqrt{3}}{2}x + i\,\mathrm{sen}\,\frac{\sqrt{3}}{2}x\right)$$

$$u_3 = e^{\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)x} = e^{-\frac{x}{2}} e^{-i\frac{\sqrt{3}}{2}x} = e^{-\frac{x}{2}}\left(\cos\frac{\sqrt{3}}{2}x - i\,\mathrm{sen}\,\frac{\sqrt{3}}{2}x\right)$$

$$v_2 = e^{-\frac{x}{2}}\cos\frac{\sqrt{3}}{2}x \qquad v_3 = e^{-\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x$$

$$u(x) = A u_1 + B v_2 + C v_3$$

$$= A\cdot e^{x} + B\,e^{-\frac{x}{2}}\cos\frac{\sqrt{3}}{2}x + C\,e^{-\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x$$

$$= A\cdot e^{x} + e^{-\frac{x}{2}}\left(B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right)$$

$$\rightarrow \rho\cdot\cos\left(\frac{\sqrt{3}}{2}x + \varphi\right)$$

$$\rho = \sqrt{B^2 + C^2} \qquad \frac{B}{\sqrt{B^2+C^2}} = \cos\varphi, \quad \frac{C}{\sqrt{B^2+C^2}} = \sin\varphi$$

In campo complesso:

$$u(x) = Ae^{x} + Be^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x} + Ce^{\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x}.$$

Esempio $\quad u^{IV} + 2u'' + u = 0$

$$P(\lambda) = \lambda^4 + 2\lambda^2 + 1 = 0$$
$$= (\lambda^2+1)^2$$
$$= (\lambda+i)^2(\lambda-i)^2$$

$$\left.\begin{array}{l} \to \lambda_1 = i, \ m_1 = 2 \\ \to \lambda_2 = -i, \ m_2 = 2 \end{array}\right\} \to \begin{cases} e^{0x}\cdot\cos x = \cos x \\ e^{0x}\cdot\sin = \sin x \end{cases}$$

$$u(x) = A\cos x + B\sin x + Cx\cos x + Dx\sin x$$

Eq. NON omogenee.

$$Lu = b \qquad L = P(D)$$

Le soluzioni sono:

$$u(x) = \underbrace{u_0(x)}_{} + u_*(x)$$

$\boxed{\text{tutte le sol. della omogenea } Lu=0}$ $\qquad$ $\boxed{\text{Una sol. particolare della non omogenea}}$

Devo trovare una soluzione particolare.
I° metodo: somiglianza.

Osservazione
$$(D-\lambda)[q(x)e^{\lambda x}] = q'(x)\cdot e^{\lambda x}$$

$$(D-\lambda)[q(x)e^{\mu x}] = \qquad \mu \neq \lambda$$

$$= q'(x)e^{\mu x} + q(x)\mu e^{\mu x} - \lambda q(x)e^{\mu x} = ((\mu-\lambda)q(x) + q'(x))e^{\mu x}$$
$$= \tilde{q}(x)\, e^{\mu x}$$

$$\deg \tilde{q} = \deg q$$

$$V_\lambda^n = \{ q(x)e^{\lambda x} : q \text{ polinomio}, \deg q < n \}$$

$$(D-\lambda) : V_\lambda^n \to V_\lambda^n \qquad (D-\lambda)^m v = 0$$
se $m \geq \deg v$.

$$(D-\lambda) : V_\mu^n \to V_\mu^n$$
è iniettivo.

$$\ker(D-\lambda) = \{0\}$$
$\Rightarrow$ è un isomorfismo.

Quindi: $P(D)\, u = b$

$$(D-\lambda_1)^{m_1}\underbrace{(D-\lambda_2)^{m_2}}\cdots\underbrace{(D-\lambda_k)^{m_k}}\, u = r(x)\, e^{\mu x}$$

Se $\mu \neq \lambda_k \quad \forall k$.

$\exists\, q(x)$ t.c. $\quad u_*(x) = \underline{q(x)\, e^{\mu x}}$

$\deg q = \deg r$

$$P(D)[u_*] = \tilde{q}(x)\, e^{\mu x}$$

Es $\qquad u''(x) + u(x) = e^{-x} \qquad \mu = -1$

$$P(\lambda) = \lambda^2 + 1$$

$$(D+i)(D-i)\, u$$

$$P(\mu) = (-1)^2 + 1 = 2 \neq 0$$

$$u_*(x) = \underbrace{c \cdot e^{-x}}$$

$u_*'' + u^* = e^{-x} \quad \leftarrow$ impongo l'uguaglianza per trovare c

$$u_y'(x) = -c\,e^{-x}$$

$$u_y''(x) = c\,e^{-x}$$

$$L u_y = u_y'' + u_y = c e^{-x} + c e^{-x} = 2 c e^{-x}$$

$$2 c e^{-x} = e^{-x} \leftarrow b$$

$$c = \frac{1}{2} \qquad u_y(x) = \frac{1}{2} e^{-x}$$

Sol. generale:

$$u(x) = A\cos x + B\sin x + \frac{1}{2} e^{-x}.$$

Se μ è radice del pol?

ES $\quad u'' + \omega^2 u = \cos(\theta x) = \dfrac{e^{i\omega x} + e^{-i\omega x}}{2}$

$$P(\lambda) = \lambda^2 + \omega^2$$

$$P(i\omega) = 0 \qquad P(-i\omega) = 0$$

Principio di sovrapposizione.

$$L u = b_1 + b_2$$

$$L u_1^* = b_1$$

$$L u_2^* = b_2$$

$$L(u_1^* + u_2^*) = b_1 + b_2.$$

ES

$$\boxed{u'' + u = \cos(2x)}$$

$$u_* = A \cdot \cos(2x) + B \sin(2x)$$

$$u_*' = -2A \sin(2x) + 2B\cos(2x)$$

$$u_*'' = -4A\cos(2x) - 4B\sin(2x)$$

$$u_*'' + u_* = -3A\cos(2x) - 3B\sin(2x)$$

$$= \cos(2x)$$

$$\begin{cases} B = 0 \\ -3A = 1 \end{cases} \qquad \begin{cases} A = -\frac{1}{3} \\ B = 0 \end{cases}$$

$$u_*(x) = -\frac{1}{3}\cos(2x)$$

$$u(x) = a \cdot \cos x + b \sin x - \frac{1}{3}\cos 2x$$

Se $P(\mu) = 0$

$$(D-\lambda_1)^{m_1} \cdots (D-\lambda_k)^{m_k} u = r(x) e^{\mu x}$$

$$\lambda_k = \mu$$

$$u_* = q(x) \cdot e^{\mu x}$$

$$(D-\mu)^m u_* = q^{(m)}(x) e^{\mu x}$$

Basta scegliere $u_*(x) = q(x) \cdot x^m \cdot e^{\mu x}$

$\deg q = \deg r$, m è la molteplicità di μ

L

come radici di $P(\lambda)$.

Es $$u'' + \omega^2 u = \cos(\omega x)$$

$$P(\lambda) = \lambda^2 + \omega^2 = (\lambda + i\omega)(\lambda - i\omega)$$

$$u_0(x) = A\cos(\omega x) + B\sin(\omega x)$$

$$\omega^2 \cdot \quad u_p(x) = a \cdot x \cdot \cos\omega x + b \cdot x \cdot \sin\omega x$$

$$0 \cdot \quad u_p'(x) = bx\omega\cos\omega x - ax\omega\sin\omega x + a\cos\omega x + b\sin\omega x$$

$$1 \cdot \quad u_p''(x) = -ax\omega^2\cos\omega x - bx\omega^2\sin\omega x + 2b\omega\cos\omega x - 2a\omega\sin\omega x$$

$$\underset{+}{u_p'' + \omega^2 u_p} = 2b\omega\cos\omega x - 2a\omega\sin\omega x \overset{!}{=} \cos\omega x$$

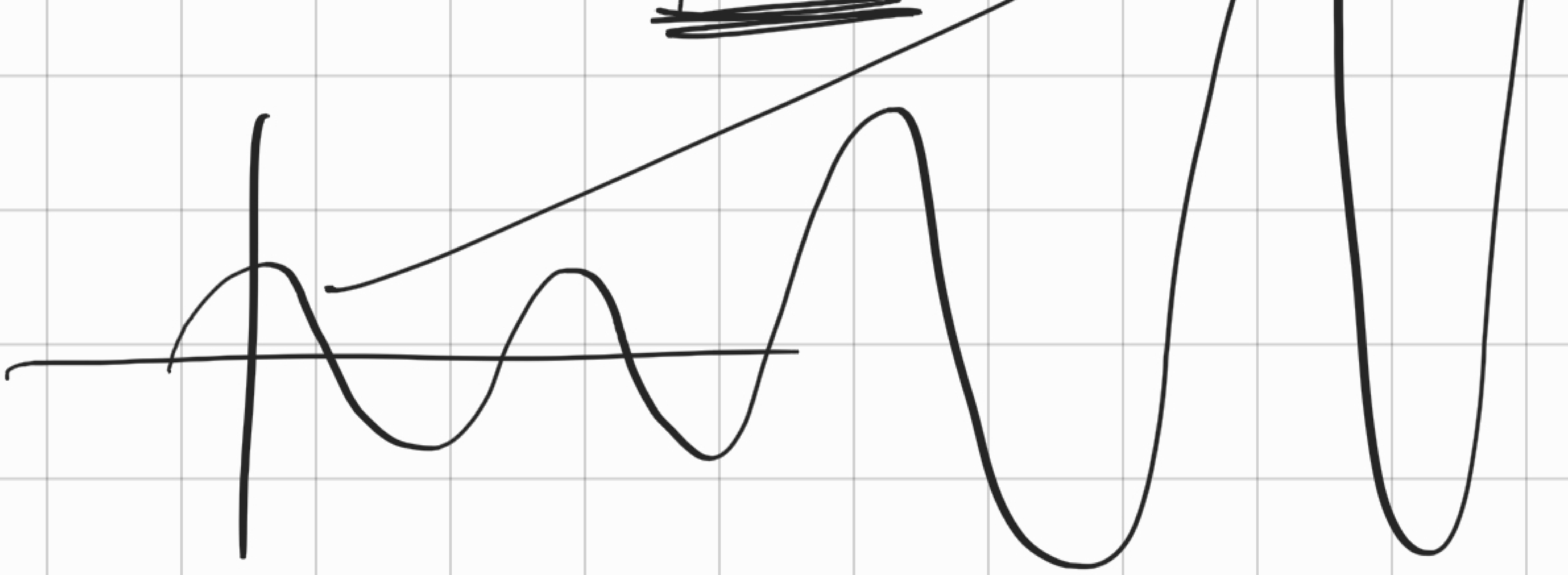
$$b = \frac{1}{2\omega} \qquad a = 0$$

CORREZIONE!

$$u_p(x) = \frac{x}{2\omega}\sin(\omega x)$$

$$u(x) = A\cos\omega x + B\sin\omega x + \frac{x}{2\omega}\sin(\omega x)$$

$$= A \cos \omega x + \left(B + \frac{x}{2\omega}\right) \sin \omega x$$



$\square$