ANALISI MATEMATICA B
LEZIONE 9 – 8.10.2021

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

$f: X \to X$

proprietà della iterazione

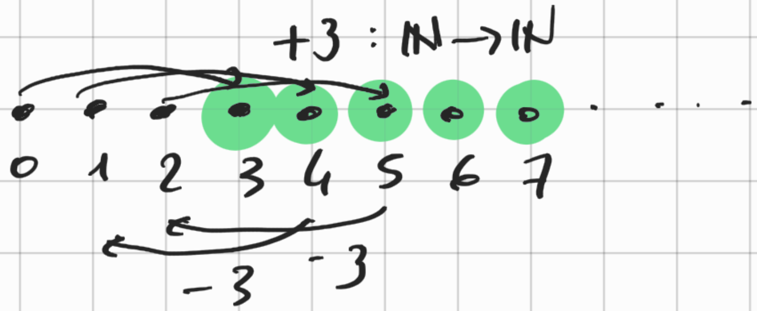
$f^n \circ f^m = f^{n+m}$

$(f^n)^m = f^{n \cdot m}$

iterata

$f^n = \underbrace{f \circ f \circ \dots \circ f}_{n\text{-volte}}$

$\mathbb{Z} = \{\dots, -2, -1, +0, +1, +2, +3, \dots\}$



Immaginiamo che $\mathbb{N} \subseteq \mathbb{Z}$

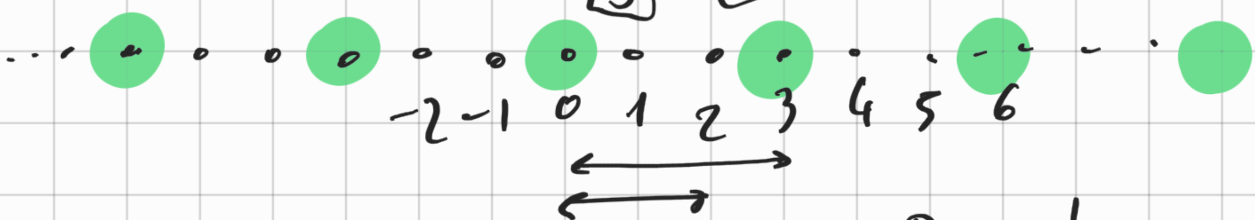
Se $f: X \to X$ è invertibile (bigettiva)

$f^{-1}: X \to X$ $\qquad f^{-n} = (f^{-1})^n$

$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{Z} \setminus \{0\} \right\} / \sim$

$\frac{p}{-q} \sim \frac{-p}{q}$

$\boxed{\frac{2}{3}} \sim \boxed{\frac{4}{6}}$



$\frac{p}{q} \sim \frac{p'}{q'} \iff pq' = p'q$

$$\frac{p}{q} \cdot \frac{q}{p} = 1$$

$\mathbb{Z}$, $\mathbb{Q}$ è un gruppo additivo
→ { elemento neutro
operazione associativa
c'è l'opposto

$\mathbb{Q} \setminus \{0\}$ è un gruppo moltiplicativo

+ Vale proprietà distributiva: $a \cdot (b+c) = a \cdot b + a \cdot c$

Relazione d'ordine $\leq$

$$\begin{cases} a \leq b \iff a+c \leq b+c \\ a \leq b \underset{c>0}{\iff} a \cdot c \leq b \cdot c \end{cases}$$

<u>Successione</u>

$$f : \mathbb{N} \to X$$

$f(0), f(1), f(2) \ldots\ldots$

<u>Analogia</u> $\quad \underline{f} : \{0,1,2\} \to X$

$$\underline{f} = (\underline{f}(0), \underline{f}(1), \underline{f}(2))$$

$$\underline{v} \in X^3 \qquad \underline{v} = (v_0, v_1, v_2)$$

$$X^3 = \{0,1,2\} \to X$$

$$A^B = \{ f : \ f : B \to A \}$$

$$\underline{a} \in X^{\mathbb{N}} \qquad \underline{a} : \mathbb{N} \to X$$

$$\underline{a} = (a_0, a_1, a_2, \dots \dots)$$

$$a_n = \underline{a}(n)$$

Es $\quad a_n = \frac{1}{2^n} \qquad \underline{a}(n) = \frac{1}{2^n}$

$$\underline{a} : \mathbb{N} \to \mathbb{Q}$$

$$\underline{a} = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \dots \right)$$

SOMMATORIA

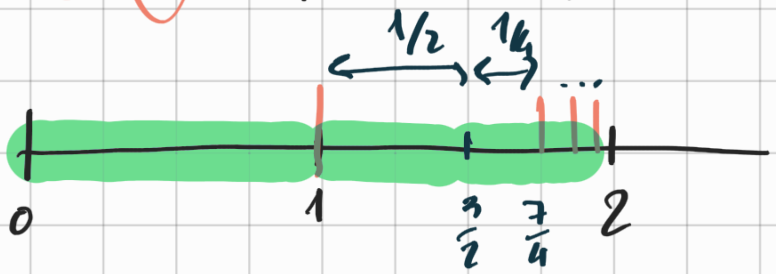
$$\sum_{k=0}^{n-1} a_k = a_0 + a_1 + a_2 + \dots + a_{n-1}$$

$\underline{a} \in X^n$ oppure $\underline{a} \in X^{\mathbb{N}}$

$\uparrow$ c'è la somma: $+$

$$\underline{a} = (a_0, a_1, \dots, a_{n-1})$$

$$\sum_{k=0}^{4} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{31}{16}$$



Definizione formale:

$$\sum_{i=0}^{n-1} : X^n \to X$$

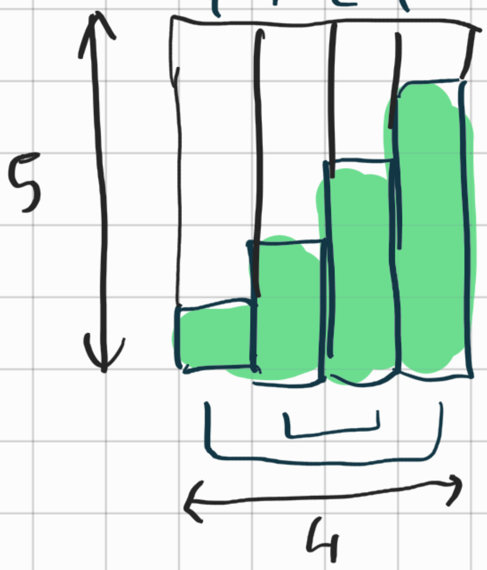
$$\begin{cases} \sum_{k=0}^{-1} a_k = 0 \\ \sum_{k=0}^{n} a_k = \sum_{k=0}^{n-1} a_k + a_n \end{cases}$$

Esercizio calcolare $\sum_{k=0}^{n-1} k = \frac{(n-1) \cdot n}{2}$

$$\sum_{k=0}^{3} k = 0 + 1 + 2 + 3 = 6$$

$$\sum_{k=0}^{n-1} k = 0 + 1 + 2 + \ldots \ldots + n-1$$

$$1 + 2 + \ldots + 100 = \frac{101}{2} \cdot 100 = 5050$$



$m \cdot k + q$

$$\begin{array}{l} 0 + 1 + \ldots\ldots + n-1 \\ (n-1) + (n-2) \ldots + 1 + 0 \\ \hline (n-1) + (n-1) + \ldots - + (n-1) \end{array} = (n-1) \cdot n$$

Esercizio dimostriamo per induzione che

$$P(n): \sum_{k=0}^{n-1} k = \frac{(n-1)\cdot n}{2} \qquad n \in \mathbb{N}$$

dim (i) $n=0$ $\quad 0=0$

$n=1$ $\quad 0=0$

(ii) $\underbrace{P(n)}_{} \overset{?}{\Rightarrow} \underbrace{P(n+1)}_{}$

Assumo $\displaystyle\sum_{k=0}^{n-1} k \overset{!}{=} \frac{(n-1)\cdot n}{2}$ (ipotesi induttiva)

$$\sum_{k=0}^{n} k = \sum_{k=0}^{n-1} k + n = \frac{(n-1)\cdot n}{2} + n \overset{?}{=} \frac{n\cdot(n+1)}{2}$$

per definizione

$$\frac{(n-1)\cdot n}{2} + n = \frac{(n-1)\cdot n + 2n}{2} = \frac{n(n-1+2)}{2} = \frac{n(n+1)}{2}$$

Più in generale $b \geq a$ $\displaystyle\sum_{k=m}^{n} a_k$

$$\begin{cases} \displaystyle\sum_{k=m}^{m} a_k = a_m \\ \displaystyle\sum_{k=m}^{n+1} a_k = \sum_{k=m}^{n} a_k + a_{n+1} \end{cases}$$

Propietà:

$$\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$

$$\sum_{k=m}^{n} c \cdot a_k = c \cdot \sum_{k=m}^{n} a_k$$

[$\sum_{k=m}^{n}$ è un funzionale lineare]

$$F(\underline{a} + \underline{b}) = F(\underline{a}) + F(\underline{b})$$

$$F(c \cdot \underline{a}) = c \cdot F(\underline{a})$$

$$\sum_{k=m}^{n} a_k = \sum_{j=0}^{n-m} a_{m+j}$$

cambio di variabile

$k = m + j$

Esercizio

$$\sum_{k=7}^{12} (2k+1) = \frac{2 \cdot 12 + 1 + 2 \cdot 7 + 1}{2} \cdot 6$$

$$= 120$$

$k = 7 + j$

$$\sum_{j=0}^{5} (2(7+j)+1) = \sum_{j=0}^{5} (2j + 15) =$$

$$= 2 \sum_{j=0}^{5} j + \sum_{j=0}^{5} 15 = 2 \cdot \frac{5 \cdot 6}{2} + 6 \cdot 15$$

$$= 6 \cdot (5 + 15) = 120.$$

Oss $\quad n \cdot m = \sum_{k=0}^{n-1} m$

PROGRESSIONE GEOMETRICA

q fissato. $\quad a_k = q^k$

$$x = \sum_{k=0}^{n-1} q^k = ?$$

$$q \cdot x = q \cdot \sum_{k=0}^{n-1} q^k = \sum_{k=0}^{n-1} q^{k+1} = \sum_{j=1}^{n} q^j$$

$j = k+1$

$$= \sum_{j=0}^{n-1} q^k - q^0 + q^n = x - 1 + q^n$$

$$q \cdot x = x - 1 + q^n$$

$$1 - q^n = (1-q)\,x$$

$q \neq 1$

$$x = \frac{1-q^n}{1-q} \qquad \square$$

$q = \frac{1}{2}, \; n = 4$

$$x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

$$q \cdot x = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

$$= x - 1 + \frac{1}{16}$$