ANALISI MATEMATICA B
LEZIONE 10 — 11.10.2021

Calcolare $\sum_{k=1}^{2n} 2k = 2+4+\dots+4n$

① $= \frac{4n+2}{2} \cdot 2n = 4n^2+2n$

② $= 2\sum_{k=1}^{2n} k = 2 \cdot \frac{2n(2n+1)}{2} = 4n^2+2n$

ES 2

$$\begin{cases} f(0)=1 \\ f(n+1) = g(n, f(n)) \end{cases}$$

ES

$f(n)=n!$

$$\begin{cases} 0! = 1 \\ (n+1)! = n! \cdot (n+1) \end{cases}$$

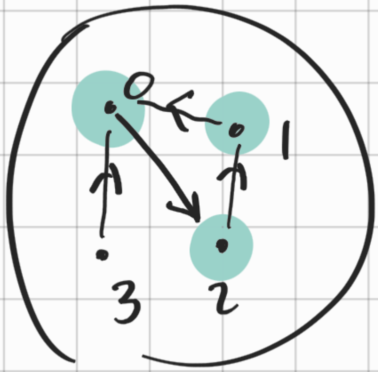
ES

$f(n) = 2^n \cdot n!$

$$\begin{aligned} f(n+1) &= 2^{n+1} (n+1)! \\ &= 2 \cdot (n+1) \cdot 2^n n! \\ &= 2 \cdot (n+1) \cdot f(n) \\ &= g(n, f(n)) \end{aligned}$$

$g(n,k) = 2(n+1) \cdot k$

Es 3



$$\begin{cases} f(0) = 0 \\ f(n+1) = g(f(n)) \end{cases}$$

$$f(13) = \underbrace{f(f(f \dots f}_{13}(0) \dots) = 2$$

$$2021 = 3 \cdot k + 2$$

$$f(n+3) = f(n) \qquad f(2021) = f(2) = 1$$

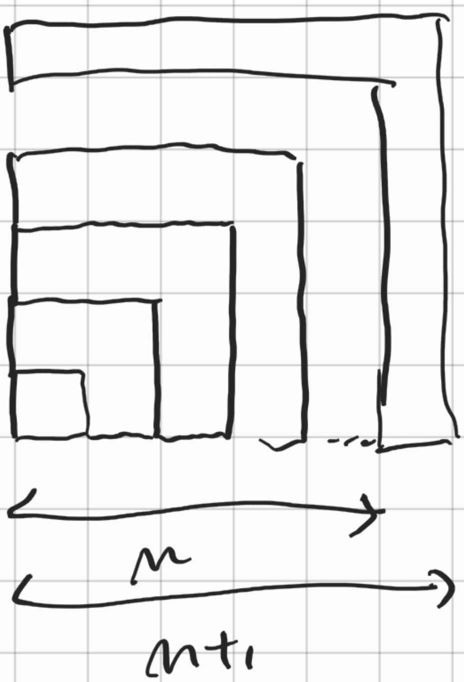
SOMMATORIE

progressione aritmetica $\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$

progressione geometrica di ragione q $\quad \sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}$ se $q \neq 1$

$$\sum_{k=1}^{n} k^q = ?$$

$$\sum_{k=1}^{n} k^2 = ?$$



Torniamo al caso lineare

$$1 + 3 + 5 + 7 = 4^2$$

$$(n+1)^2 - n^2 = \cancel{n^2} + 2n + 1 - \cancel{n^2} = \underline{2n+1}$$

$$\sum_{k=0}^{n} 2k+1 \stackrel{?}{=} (n+1)^2$$

$$\parallel$$

$$\sum_{k=0}^{n} \left[(k+1)^2 - k^2 \right]$$

$$= (\cancel{1^2} - 0^2) + (\cancel{2^2} - \cancel{1^2}) + (\cancel{3^2} - \cancel{2^2}) + \ldots + (\cancel{n^2} - \cancel{(n-1)^2}) + ((n+1)^2 - \cancel{n^2})$$

somma telescopica

$$= (n+1)^2 - 0 = (n+1)^2$$

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sommatoria

$$S_n = \sum_{k=1}^{n} a_k \,, \quad \begin{cases} S_0 = 0 \quad (\text{oppure } S_1 = a_1) \\ S_{n+1} = S_n + a_{n+1} \end{cases}$$

somme parziali

$$S_{n+1} - S_n = a_{n+1}$$

differenza finita

$$S_n = \sum_{k=1}^{n} (S_k - S_{k-1})$$

$$\sum k^2 = ?$$

$$\boxed{\begin{aligned} (n+1)^3 &= (n+1)(n^2+2n+1) \\ &= n^3 + 2n^2 + n \\ &\quad + n^2 + 2n + 1 \\ &= n^3 + 3n^2 + 3n + 1 \end{aligned}}$$

$$(n+1)^3 - n^3 = \cancel{n^3} + 3n^2 + 3n + 1 - \cancel{n^3} = \underline{3n^2 + 3n + 1}$$

$$(A) = \sum_{k=0}^{n} \left(3k^2+3k+1\right) = \sum_{k=0}^{n}\left[(k+1)^3-k^3\right]$$

$$= (n+1)^3 - 0 = (n+1)^3$$

$$\| $$

$$\underbrace{\overset{(X)}{3\sum_{k=0}^{n} k^2} + \overset{(B)}{3\sum_{k=0}^{n} k} + \overset{(C)}{\sum_{k=0}^{n} 1}}$$

$$(A) = (X) + (B) + (C)$$

$$(X) = 3\sum_{k=0}^{n} k^2 = \overset{(A)}{(n+1)^3} - \overset{(B)}{3\,\frac{n(n+1)}{2}} - \overset{(C)}{(n+1)}$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}(n+1)\left[(n+1)^2 - \frac{3}{2}n - 1\right] = \dots$$

$$= \frac{1}{3}(n+1)\left[n^2+2n+\cancel{1}-\frac{3}{2}n-\cancel{1}\right]$$

$$= \frac{1}{3}(n+1)\,n\left[n+\frac{1}{2}\right]$$

$$= \frac{n(n+1)(2n+1)}{6}.$$

Per casa verificare la formula per induzione

Per casa $\displaystyle\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 \overset{?}{=} \left(\frac{n(n+1)}{2}\right)^2$

Somma di Mengoli

$$\frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n^2+n}$$

Problema di Basilea

$$\sum_{k=0}^{+\infty} \frac{1}{k^2} = ?$$

$$\boxed{\sum_{k=1}^{n} \frac{1}{k^2+k} = \sum_{k=1}^{n} \left[\frac{1}{k} - \frac{1}{k+1}\right] = 1 - \frac{1}{n+1}}$$

COEFFICIENTI BINOMIALI

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^k b^{n-k}}$$

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

$P(x) = (1+x)^n$ è un polinomio

- espressione polinomiale
- funzione polinomiale
- polinomio

Espressione polinomiale: espressione in cui compaiono solo le operazioni di addizione $+$ e moltiplicazione $\cdot$

e costanti e variabili

$\uparrow$ in un anello

ES $P(x) = (x+2) \cdot (2 + 3 \cdot (x+3))$

applicando le regole di anello (commutativo)

manca l'inverso nell'prodotto

+ $\cdot$
associative
distributive
ha neutro e opposto
+ commutative
($\cdot$ commutativo)

$$(x+2) \cdot (2 + 3 \cdot (x+3))$$

$$= (x+2) \cdot (2 + 3x + 9)$$

$$= (x+2)(3x+11)$$

$$= 3x^2 + 17x + 22$$

$x^2 = x \cdot x$

monomi $c \cdot x^k$

forma canonica

polinomi = espressioni polinomiali
equivalenza.

Principio di identità dei polinomi

due espressioni polinomiali sono equivalenti
$\Updownarrow$
hanno la stessa forma canonica.

grado di un polinomio: l'esponente più alto dei monomi nella forma canonica.

Funzioni polinomiali scelgo un dominio.

$$x \longmapsto P(x)$$

Def coefficiente binomiale

$$\begin{cases} x^0 = 1. \\ x^{n+1} = x \cdot x^n \end{cases}$$

$$0^0 = 1$$

$$\to (1+x)^n = \sum_{k=0}^{n} c_k \cdot x^k$$

$$\binom{n}{k} := c_k$$

Allora
$$(a+b)^n = a^n \cdot \left(1+\frac{b}{a}\right)^n$$
$$= a^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{b}{a}\right)^k$$
$$= \sum_{k=0}^{n} \binom{n}{k} b^k \cdot a^{n-k}$$

$$(a+b)^n = (b+a)^n \implies \binom{n}{k} = \binom{n}{n-k}$$

Triangolo di Tartaglia

	$k=0$	$k=1$	$k=2$	...			
$n=0$	1						
$n=1$	1	1					
$n=2$	1	2	1				
$n=3$	1	3	3	1			
$n=4$	1	4	6	4	1		
$n=5$	1	5	10	10	5	1	
$n=6$	1	6	15	20	15	6	1

$$(1+x)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1.$$

Teo

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

(Per convenzione $\binom{n}{k} = 0$ se $k > n$ o $k < 0$)

dim

$$(1+x)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k$$

$$(1+x)\cdot(1+x)^n = (1+x)\cdot \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{k=0}^{n} \binom{n}{k} x^{k+1} \qquad j = k+1$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{j=1}^{n+1} \binom{n}{j-1} x^j$$

$$= \sum_{k=0}^{n+1} \binom{n}{k} x^k + \sum_{k=0}^{n+1} \binom{n}{k-1} x^k$$

$$\binom{n}{n+1} = 0 \qquad \binom{n}{-1} = 0$$

per convenzione

$$= \sum_{k=0}^{n+1} \boxed{\left[\binom{n}{k} + \binom{n}{k-1} \right]} x^k$$

Teorema

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

dim $\triangle$ per induzione su n usando Tartaglia.