ANALISI MATEMATICA B
LEZIONE 13 — 18.10.2021

$\forall x>0 \; \exists y>0 : \quad \underbrace{\left(\underbrace{y^3+2y^2+3y}\right)^2} \leq \underbrace{x}$

$$\left(y^3+2y^2+3\underline{y}\right)^2 \leq \left(y+2y+3y\right)^2 = (6y)^2 = 36y^2$$

$0<y\leq 1$

$\Downarrow$

$0<y^2\leq 1$

$y^3 = y\cdot y^2 \leq y$

$2y^2 = 2y\cdot y \leq 2y$

$\leq 36y \overset{?}{\leq} x$

$\Uparrow$

$y = \frac{x}{36}$

$x \mapsto x^2 \qquad 0\leq x_1 \leq x_2 \qquad x_1^2 \leq x_2^2$

$$\#\mathbb{N} = \#\mathbb{Z} = \#\mathbb{Q} < \#\mathbb{R} = \#\mathbb{C}$$

$$\#(\mathbb{N}\times\mathbb{Z}) = \#\mathbb{N}$$

In base 3

~~0,357232...~~

~~0,791227~~

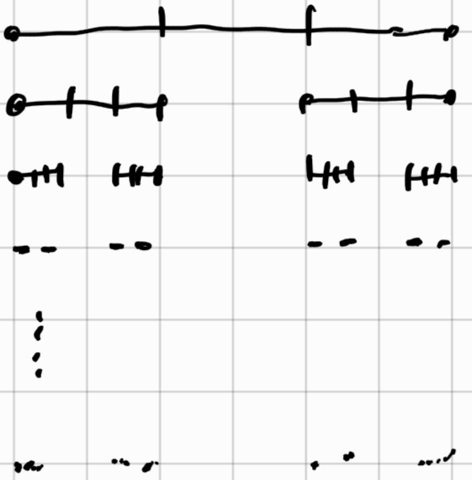
~~0,100333~~

$0,\bar{9} = 1$

$0,1221002\ldots \notin C$

$C = \{ x\in[0,1] : x$ scritto in base 3 non contiene mai la cifra 1 $\}$

$\underbrace{0,0220022\ldots}$



polvere di Cantor

$$\# C = \# \mathcal{P}(\mathbb{N}) > \# \mathbb{N}$$

<u>Teo</u> (Cantor) A insieme $\quad \# \mathcal{P}(A) > \# A$.

<u>dim</u> (i) $\# A \leq \# \mathcal{P}(A)$ ovvio

$$\exists f : A \to \mathcal{P}(A)$$
$$a \mapsto \{a\} \quad \text{iniettiva}$$

(ii) $\quad \# A \neq \# \mathcal{P}(A)$

Se fossero uguali esisterebbe $f : A \to \mathcal{P}(A)$ suriettiva.

$$C = \{ x \in A : \underline{x \notin f(x)} \}$$

Se f fosse suriettiva $\exists\ c \in A : f(c) = C$

$$c \in f(c) = C \iff c \notin f(c) = C \quad \text{assurdo}$$

$\square$

$$\# \mathcal{P}(\mathbb{N}) \leq \# \mathbb{R} \leq \# \mathcal{P}(\mathbb{Q}) = \# \overset{\text{continuo}}{\mathcal{P}(\mathbb{N})}$$

$\# \mathbb{N} = \aleph_0 \qquad \aleph_1$? IPOTESI DEL CONTINUO

NUMERI REALI

1. $\mathbb{R}$ è un campo, totalmente ordinato, continuo.

2. $\mathbb{R}$ è un gruppo, totalmente ordinato, denso e continuo.
   $+$ (diverso da $\{0\}$)

Queste proprietà caratterizzano $\mathbb{R}$.

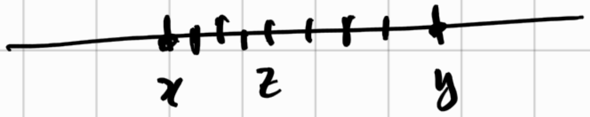
A è un gruppo: $*$ operatore su A
e elemento neutro
associativo $(x*y)*z = x*(y*z)$
inverso $\forall x \exists y: x*y = y*x = e$

A è ordinato $\leq$
(i) $x \leq x$
(ii) $x \leq y \wedge y \leq z \Rightarrow x \leq z$
(iii) $x \leq y \wedge y \leq x \Rightarrow x = y$
totalmente (iv) $x \leq y \vee y \leq x$.

$A=\mathbb{R}$
$+$
0
$-x$
$\leq$
$x \leq y \Rightarrow x+z \leq y+z$

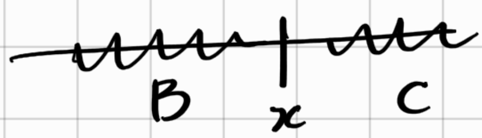
A gruppo ordinato: monotonia
$x \leq y \Rightarrow x*z \leq y*z$
$z*x \leq z*y$

A denso: $x<y \Rightarrow \exists z: x<z<y$



A continuo: $\forall B,C \subseteq A$, $B \neq \emptyset$, $C \neq \emptyset$ $\quad B \leq C$

$\exists x: \quad B \leq x \leq C$



Teorema (isomorfismo)

Se A e B sono gruppi, totalmente ordinati, densi e continui

Allora fissato $u \in A$ ($u \neq e$), $m \in B$ ($m \neq e$) esiste una unica funzione bijettiva

$$\varphi : A \to B$$
$$x \mapsto \varphi(x)$$

$\varphi(u) = m$,

$\varphi(x *_A y) = \varphi(x) *_B \varphi(y)$ (omomorfismo)

$(\varphi(e_A) = e_B)$

$x \leq_A y \Rightarrow \varphi(x) \leq_B \varphi(y)$ (crescente)

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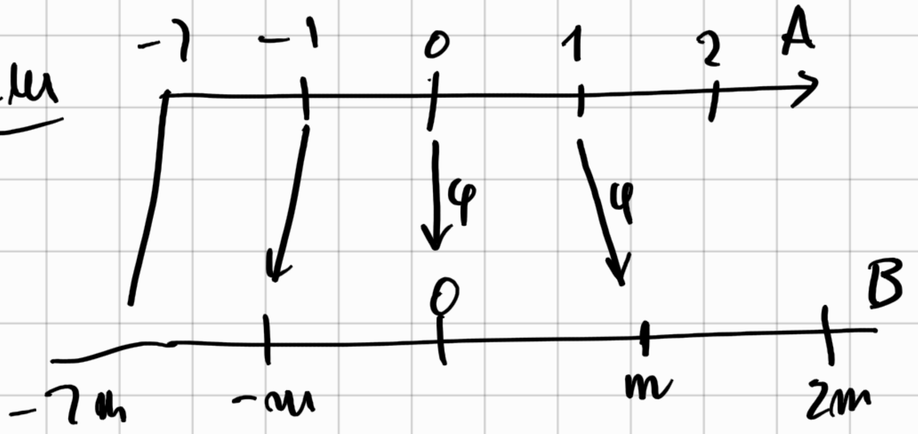
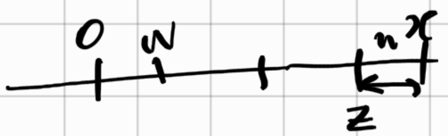
dim



$e = 0$
$* = +$
$u = 1$

densità



$2z \overset{?}{<} x$

se $2z > x \overset{?}{\Rightarrow} (x-z) < x$

Dato $x > 0$ $\forall n \in \mathbb{N}$ $\exists\, y > 0$: $ny < x$. ✓

Definisco $\frac{x}{n} : \sup \{ y : ny \le x \}$ se $x > 0$, $n \in \mathbb{N}$

(Osservazione $\quad \mathbb{R} \quad \mathbb{R}_+ = \{ x \in \mathbb{R} : x > 0 \}$)

$$\varphi\left(\frac{p}{q}\right) = \frac{\varphi(p)}{\varphi(q)}$$

$$\frac{p_1}{q_1} \le x \le \frac{p_2}{q_2}$$

$$\varphi\left(\frac{p_1}{q_1}\right) \le \varphi(x) \le \varphi\left(\frac{p_2}{q_2}\right)$$

(Osservazione

$$a^{p/q} = \sqrt[q]{a^p}$$

a^x si definisce per approssimazione)

$\square$

$\mathbb{R}, +$ $\qquad 1 \in \mathbb{R}$ $\qquad 1 > 0.$

Dato $\underline{\underline{m > 0}}$ $\quad \exists!\ f : \mathbb{R} \to \mathbb{R}$

$$f(1) = m$$

$$f(x+y) = f(x) + f(y)$$

$f(x) =: m \cdot x$ $\qquad m \cdot (x+y) = m \cdot x + m \cdot y$

$(\mathbb{R}, +)$ $\qquad (\mathbb{R}_+, \cdot)$

1 $\qquad a > 1$

$\exists!$ $\quad f(1) = a$ $\qquad a^1 = a$

$$\boxed{f(x+y) = f(x) \cdot f(y)}$$

$$a^{x+y} = a^x \cdot a^y$$

$$f(x) = a^x$$

$(a^x)^y = \boxed{a^{x \cdot y}}$ $\qquad y \mapsto a^{x \cdot y}$

$(ab)^x = a^x \cdot b^x$ $\qquad 1 \mapsto a^x$

$x \mapsto a^x \cdot b^x$

$$x * y = x(y-1) - y + 2$$

$$\parallel$$

$$\underbrace{(x-1)(y-1)+1}$$

$$\varphi(x) = x+1$$

$$\mathbb{R}_+ \xrightarrow[\varphi]{+1} G$$

$$*$$

$$g * h = \varphi\left(\varphi^{-1}(g) \cdot \varphi^{-1}(h)\right)$$