LEZIONE 39 7.1.2022

Le serie di potenze $f(z) = \sum_{k=0}^{+\infty} a_k \cdot z^k$

$z \in \mathbb{C}$

$a_k \in \mathbb{C}$

$$\exp(z) = \boxed{\sum_{k=0}^{+\infty} \frac{z^k}{k!}}$$ Se $x \in \mathbb{R}$ $\boxed{\exp(x) = e^x}$

ha $R = +\infty \Rightarrow$ è definita $\forall z \in \mathbb{C}$

$\exp : \mathbb{C} \to \mathbb{C}$

- $\exp(0) = 1$, $\exp(1) = e$
- $\exp(z+w) = \exp(z) \cdot \exp(w)$
- $\exp(\bar{z}) = \overline{\exp(z)}$
- $\exp(-z) = \dfrac{1}{\exp(z)}$

- $\lim_{z \to 0} \dfrac{\exp(z) - 1}{z} = 1$
- $\exp z$ è continua.

Teorema $$\lim_{n \to +\infty} \left(1 + \frac{z}{n}\right)^n = \exp(z)$$

$$\underbrace{e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!}}\quad \Leftrightarrow \quad e^x = \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots}$$

Teorema Se $|z| \le 1$ si ha:

$$\left|\exp(z) - \sum_{k=0}^{n} \frac{z^k}{k!}\right| \le \frac{|z|^{n+1}}{n\cdot n!}$$

dim

$$\left|\sum_{k=0}^{+\infty}\frac{z^k}{k!} - \sum_{k=0}^{n}\frac{z^k}{k!}\right|$$

$$= \left|\sum_{k=n+1}^{+\infty}\frac{z^k}{k!}\right| \le \sum_{k=n+1}^{+\infty}\frac{|z|^k}{k!}$$

$$= |z|^{n+1}\sum_{k=n+1}^{+\infty}\frac{|z|^{k-(n+1)}}{k!}$$

$$= |z|^{n+1}\sum_{j=0}^{+\infty}\frac{|z|^j}{(n+1+j)!} \le |z|^{n+1}\sum_{j=0}^{+\infty}\frac{|z|^j}{n!\,(n+1)^{j+1}}$$

$$(n+1+j)! = 1\cdot 2\cdot 3\cdots n\cdot(n+1)\cdots(n+1+j)$$

$$= \frac{|z|^{n+1}}{n!\cdot(n+1)} \sum_{j=0}^{+\infty} \left(\frac{|z|}{(n+1)}\right)^{j} = \frac{|z|^{n+1}}{n!\,(n+1)} \cdot \frac{1}{1-\frac{|z|}{n+1}}$$

$|z| \le 1$

$$= \boxed{\frac{|z|^{n+1}}{n!\,\cancel{(n+1)}} \cdot \frac{\cancel{n+1}}{n+1-|z|}} \le \frac{|z|^{n+1}}{n!} \frac{1}{n+\cancel{1}-\cancel{1}}$$

$$= \frac{|z|^{n+1}}{n!\cdot n} \qquad \square$$

Se $|z| = 1$ $\qquad e = \exp(1)$

$$\left| e - \sum_{k=0}^{n} \frac{1}{k!} \right| \le \frac{1}{n!\cdot n}$$

$$e = \underbrace{1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!}} + \frac{1}{5!} + \varepsilon, \quad |\varepsilon| \le \frac{1}{600}$$

$$= 1{,}71\bar{6} + \varepsilon.$$

Teorema $\quad e$ é irrazionale

dim per assurdo $\quad \boxed{e = \frac{p}{q}} \quad p \in \mathbb{N},\ q \in \mathbb{N}$

$$e\cdot n! = \sum_{k=0}^{+\infty} \frac{n!}{k!} = \underbrace{\sum_{k=0}^{n} \frac{n!}{k!}}_{M\in\mathbb{N}} + n!\cdot\boxed{\sum_{k=n+1}^{+\infty} \frac{1}{k!}} \in \mathbb{N}$$

$$n!\cdot\sum_{k=n+1}^{+\infty} \frac{1}{k!} \le \frac{1}{n!\cdot n}$$

$$0 < n!\cdot\sum_{k=n+1}^{+\infty} \frac{1}{k!} \le \frac{1}{n}$$

$$M < \underbrace{e\cdot n!}_{\cancel{\mathbb{N}}\ \mathbb{Z}} \le M + \frac{1}{n} < M+1 \qquad \text{se } n > 1$$

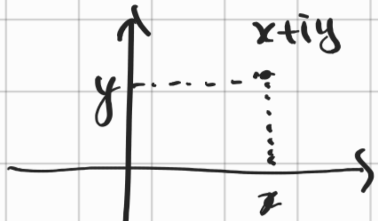
se $n \geq q$ $\quad e\cdot n! = \frac{p}{q}\cdot \underbrace{n!}\ \in \mathbb{N}$

assurdo.

FUNZIONI TRIGONOMETRICHE

$z\in\mathbb{C}$ $\quad e^{z} = \exp(z) \in \mathbb{C}$ $\qquad z = x+iy \qquad x,y\in\mathbb{R}$

$$e^{x+iy} = e^{x}\cdot \boxed{e^{iy}}$$



$$\boxed{e^{iy} = \cos y + i \operatorname{sen} y}$$ Formula Eulero

$$\begin{cases} \cos y \overset{\text{def.}}{=} \operatorname{Re} e^{iy} \\ \sin y \overset{\text{def.}}{=} \operatorname{Im} e^{iy} \end{cases}$$

$\sin : \mathbb{R} \to \mathbb{R}$

$\cos : \mathbb{R} \to \mathbb{R}$

Teorema proprietà di sin e cos.

$z = x+iy$

$\bar{z} = x-iy$

(1) $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$

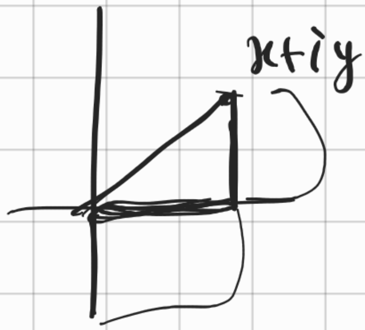
$\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

$$\left[\begin{array}{l} \operatorname{Re} z = \dfrac{z+\bar{z}}{2} \\ \operatorname{Im} z = \dfrac{z-\bar{z}}{2i} \end{array}\right]$$

(2) $\sin(-x) = -\sin x$

$$\frac{e^{-ix} - e^{ix}}{2i} = -\frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos(-x) = \frac{e^{-ix} + e^{ix}}{2} = \cos(x)$$



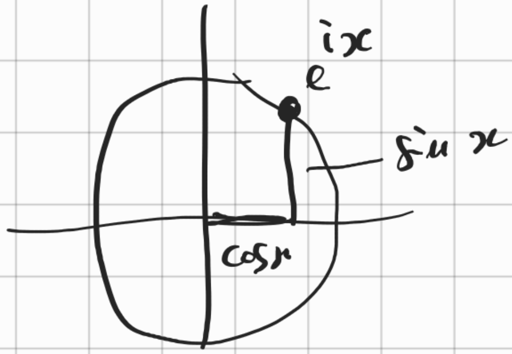
(3) $\boxed{\cos^2 x + \sin^2 x = 1}$

$$\cos^2 x + \sin^2 x = \left(\mathrm{Re}\, e^{ix}\right)^2 + \left(\mathrm{Im}\, e^{ix}\right)^2 = \left|e^{ix}\right|^2$$

$$|z|^2 = z \cdot \bar{z}$$

$$= e^{ix} \cdot \overline{e^{ix}} = e^{ix} \cdot e^{-ix}$$

$$= e^{ix} \cdot \frac{1}{e^{ix}} = 1.$$



(4) Formule di addizione

$$e^{i\alpha} = \cos\alpha + i \,\mathrm{sen}\,\alpha \qquad e^{i\beta} = \cos\beta + i\,\mathrm{sen}\,\beta$$

$$e^{i(\alpha+\beta)} = e^{i\alpha + i\beta} = e^{i\alpha} \cdot e^{i\beta} =$$

$$= (\cos\alpha + i\,\mathrm{sen}\,\alpha) \cdot (\cos\beta + i\,\mathrm{sen}\,\beta)$$

$$= \cos\alpha\cos\beta - \operatorname{sen}\alpha\,\operatorname{sen}\beta + i(\operatorname{sen}\alpha\cos\beta + \cos\alpha\,\sin\beta)$$

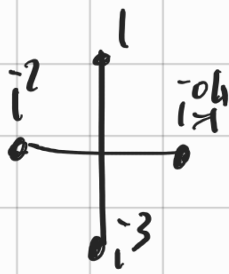
$$= \cos(\alpha+\beta) + i\,\operatorname{sen}(\alpha+\beta)$$

(5) sin e cos : $\mathbb{R} \to \mathbb{R}$ sono continue

in quanto $e^{ix} : \mathbb{R} \to \mathbb{C}$

(6)
$$e^{ix} = \sum_{k=0}^{+\infty} \frac{(ix)^k}{k!} = \sum_{k=0}^{+\infty} i^k \frac{x^k}{k!} \qquad x \in \mathbb{R}$$

$i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$



$$e^{ix} = \sum_{k=0}^{+\infty} i^{2k} \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} i^{2k+1} \frac{x^{2k+1}}{(2k+1)!}$$

$$= \underbrace{\sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(2k)!}}_{\cos x} + i \cdot \underbrace{\sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}_{\sin x}$$

$$\boxed{\begin{cases} \cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots \\[2ex] \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots \end{cases}}$$

(7) $\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}$ $\quad$ $\boxed{\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2} \quad (*)}$

$\Big(\quad \lim_{z\to 0}\frac{e^z-1}{z}=1 \qquad z=ix \qquad \lim_{x\to 0}\frac{e^{ix}-1}{ix}=1$

$\frac{1}{i}=-i$ $\qquad \lim_{x\to 0}\boxed{\frac{\cos x+i\,\mathrm{sen}\,x-1}{ix}}=1$

$$\frac{\cos x-1}{ix}+\frac{\mathrm{sen}\,x}{x}\longrightarrow 1$$

$$-\frac{\cos x-1}{x}\cdot i+\frac{\mathrm{sen}\,x}{x}\longrightarrow 1$$

$\boxed{\frac{\mathrm{sen}\,x}{x}\to 1}=\mathrm{Re}\,1.$ $\qquad \boxed{\frac{1-\cos x}{x}}\to 0=\mathrm{Im}\,1.$

ok

$$f(x)=\frac{1-\cos x}{x^2}=\frac{1-\sum_{k=0}^{+\infty}(-1)^k\frac{x^{2k}}{(2k)!}}{x^2}=\frac{-\sum_{k=1}^{+\infty}(-1)^k\frac{x^{2k}}{(2k)!}}{x^2}$$

$$=-\sum_{k=1}^{+\infty}(-1)^k\frac{x^{2k-2}}{(2k)!}$$ $\leftarrow$ serie di potenze $R=+\infty$

(si applichi il criterio del rapporto

$$a_k = (-1)^k \frac{x^{2k-2}}{(2k)!}$$

$$\frac{|a_{k+1}|}{|a_k|} = \frac{|x|^{2k}}{(2k+2)!} \Big/ \frac{|x|^{2k-2}}{(2k)!}$$

$$= \frac{|x|^2}{(2k+2)(2k+1)} \longrightarrow 0 \quad \forall x.$$

$g(x)$ è continua (visto la volta scorsa)

$g(x) \to g(0)$ per $x \to 0$

$$g(0) = \sum_{k=1}^{+\infty} (-1)^k \frac{0^{2k-2}}{(2k)!}$$

$$= \frac{(-1)}{2!} = -\frac{1}{2}$$