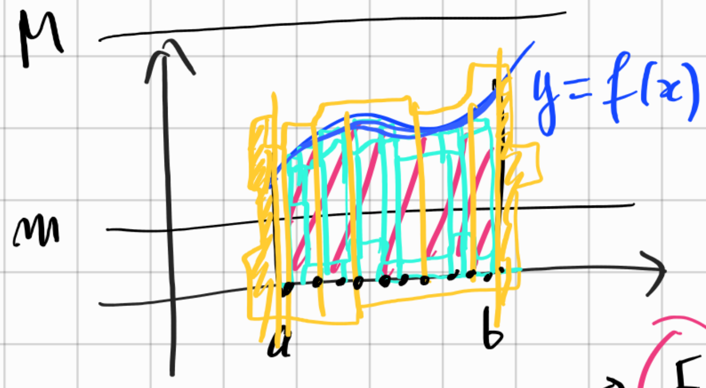
ANALISI MATEMATICA B
LEZIONE 58 – 21.2.2022

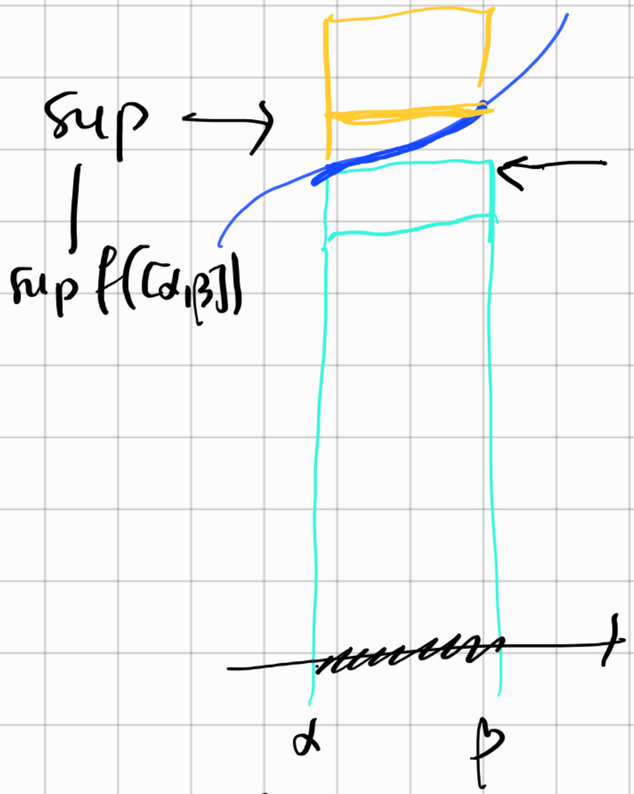
INTEGRALE di RIEMANN



$f:[a,b]\to\mathbb{R}$

f limitata $m\le f(x)\le M$

$\longrightarrow E=\{(x,y) : x\in[a,b], 0\le y\le f(x)\}$



$\inf\{f(x) : x\in[\alpha,\beta]\}$

$=\inf f([\alpha,\beta])$

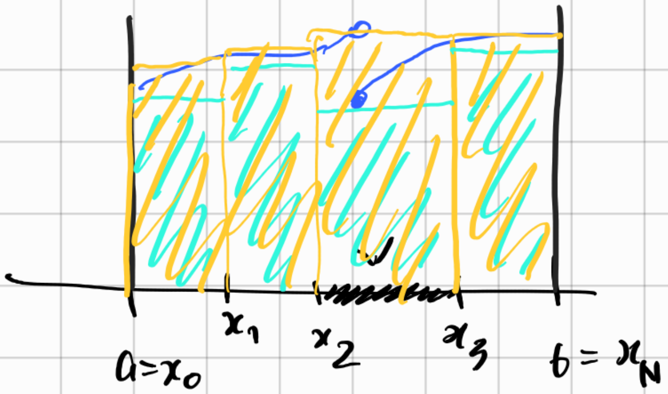
def (integrale di Riemann)

$f:[a,b]\to\mathbb{R}$, f limitata. $a<b$.

Dato $P\subseteq[a,b]$, P finito, $a\in P$, $b\in P$,

$P=\{x_0,x_1,\dots,x_N\}$ $\quad a=x_0<x_1<\dots<x_N=b$

(P la chiamiamo suddivisione di $[a,b]$



$$S^*(f,P) = \sum_{k=0}^{N-1} \overbrace{(x_{k+1}-x_k)}^{\Delta x} \cdot \underbrace{\sup f([x_k,x_{k+1}])}$$

$$S_*(f,P) = \sum_{k=0}^{N-1} (x_{k+1}-x_k) \cdot \underbrace{\inf f([x_k,x_{k+1}])}$$

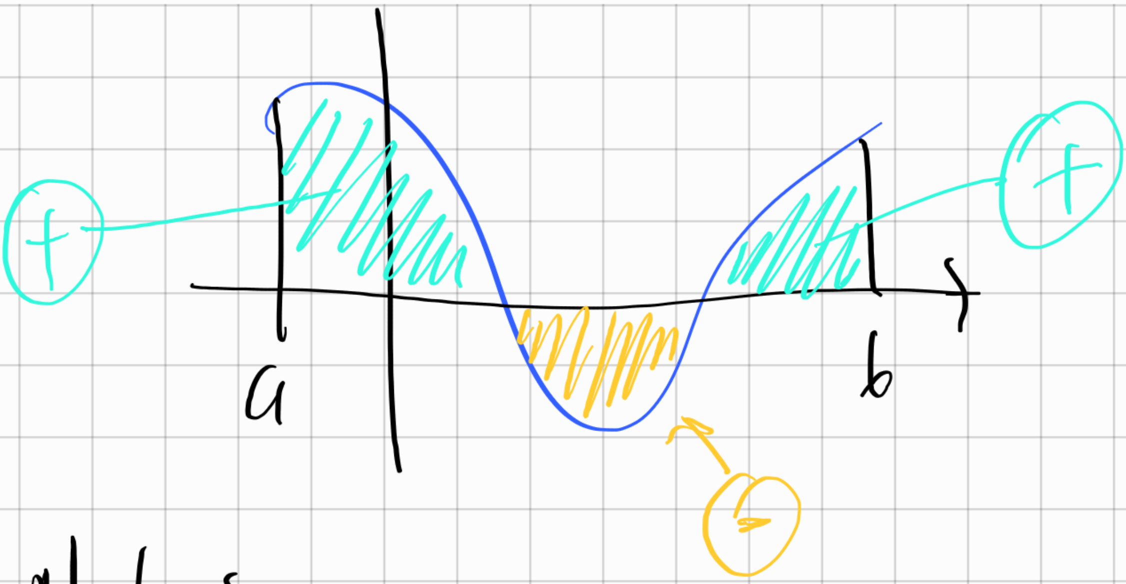
$$S^*(f) = \inf_P S^*(f,P)$$

$$S_*(f) = \sup_P S_*(f,P)$$

Se $S^*(f) = S_*(f)$ diremo che f è Riemann-integrabile su $[a,b]$ e scriveremo:

$$\int_a^b f = S^*(f) = S_*(f).$$

Notazione equivalente: $\int_a^b f = \int_a^b f(x)\,dx$.

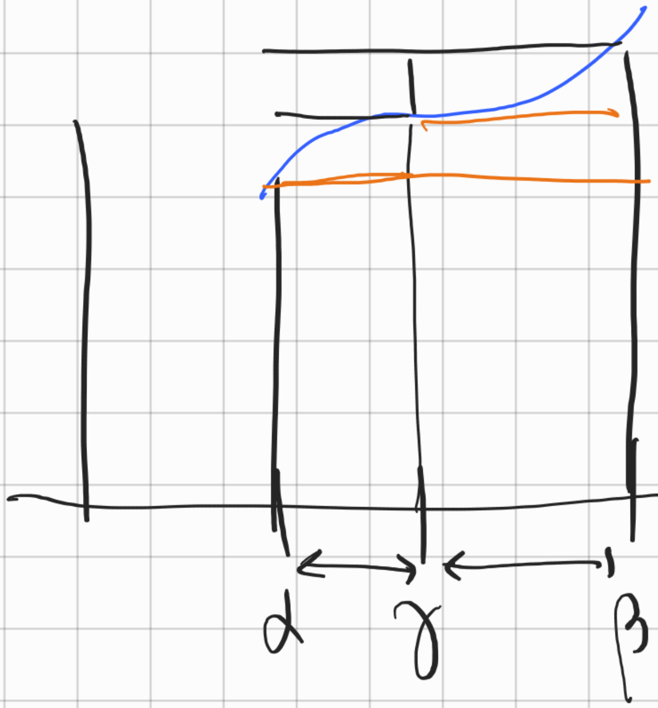


Notazione Se $a < b$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Osservazioni Se $P \subseteq Q$

$$S^*(f, P) \geq S^*(f, Q)$$
$$S_*(f, P) \leq S_*(f, Q)$$

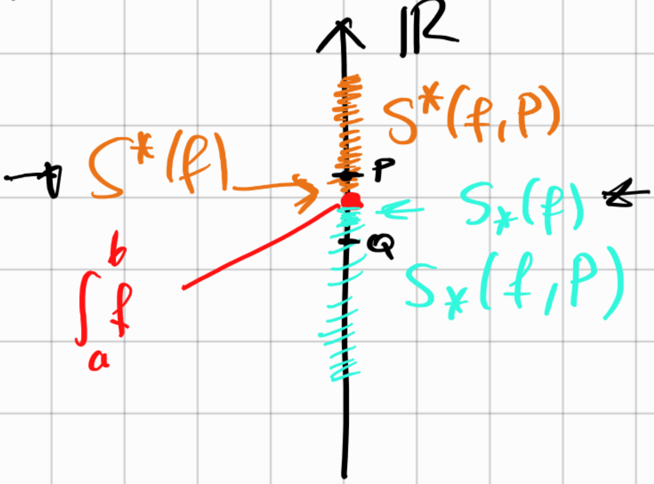


$$\sup f([\alpha,\beta]) \geq \sup f([\alpha,\gamma])$$
$$\quad '' \qquad \geq \sup f([\gamma,\beta])$$

$$(\beta-\alpha)\sup f([\alpha,\beta]) =$$
$$[(\gamma-\alpha)+(\beta-\gamma)]\cdot \sup(f[\alpha,\beta]) \geq$$
$$(\gamma-\alpha)\sup f([\alpha,\gamma]) + (\beta-\gamma)\sup f([\gamma,\beta])$$

$$\begin{array}{ccc} S^*(f,P) & \overset{?}{\geq} & S_*(f,Q) \\ \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\leq$}} \\ S^*(f,P\cup Q) & \overset{!}{\geq} & S_*(f,P\cup Q) \end{array}$$

perché



I caratterizzazione:

- f è R-integrabile su $[a,b]$ $\Longleftrightarrow$ $\forall \varepsilon>0\ \exists P : S^*(f,P)-S_*(f,P)<\varepsilon$

$$S^*(f) = \inf_P S^*(f,P)$$

$$\left[\begin{array}{l} S^*(f)+\varepsilon \text{ non è il minimo} \\ \Rightarrow \exists P \text{ t.c. } S^*(f,P) < S^*(f)+\varepsilon \end{array}\right]$$

$$\left[\begin{array}{l} \forall\varepsilon>0 \quad \exists \overset{P\cup Q}{P} : S^*(f,P) - S^*(f) < \varepsilon/2 \\ \qquad\qquad \exists \underset{P\cup Q}{Q} : S_*(f) - S_*(f,Q) < \varepsilon/2 \end{array}\right.$$

II caratterizzazione: se $f:[a,b]\to\mathbb{R}$ limitata

se $\exists\, P_n$ t.c. $\boxed{\lim_{n\to+\infty} S^*(f,P_n) - S_*(f,P_n) = 0}$

allora f è R-integrabile su $[a,b]$

e $\int_a^b f = \lim_{n\to+\infty} S^*(f,P_n) = \lim_{n\to+\infty} S_*(f,P_n)$. $(*)$

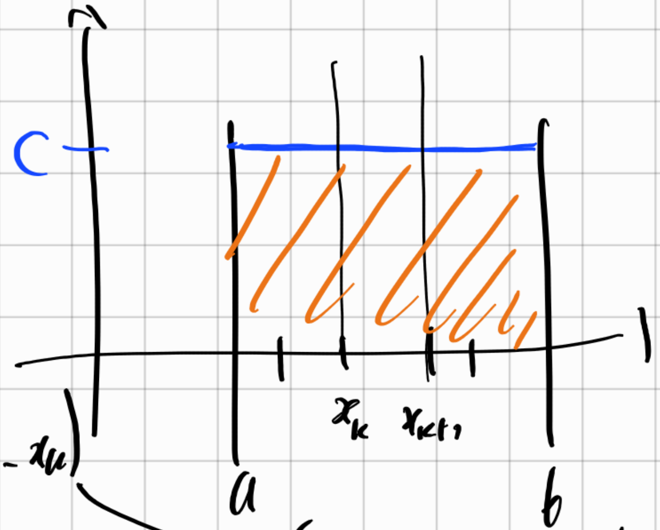
Viceversa se f è R-integrabile su $[a,b]$

allora $\exists\, P_n$ t.c. $(*)$

dim $\varepsilon = \frac{1}{n}$

Esempio banale: Se $f(x) = c$

$$\int_a^b f = \int_a^b c$$



$$S^*(f,P) = \sum_{k=0}^{N-1} (x_{k+1} - x_k)\cdot c$$

$$\| \qquad\qquad \|$$

$$S_*(f,P) \qquad c\sum_{k=0}^{N-1}(x_{k+1} - x_k)$$

$$\| $$

$$c\cdot(b-a)$$

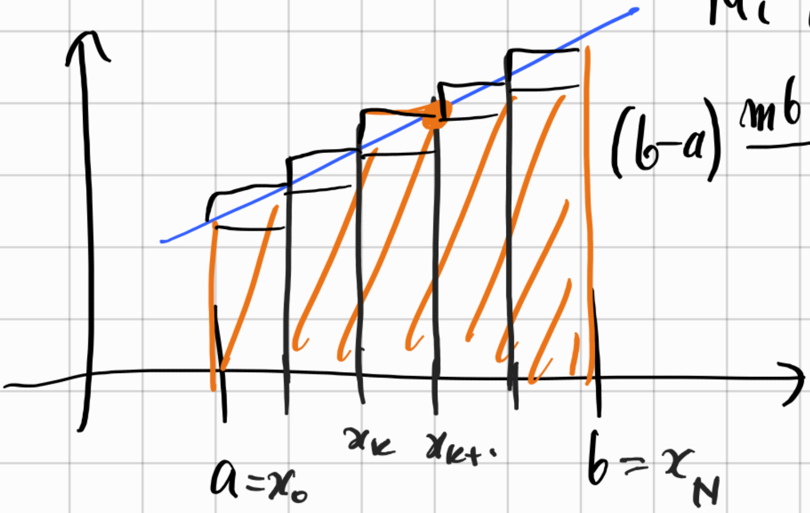
$$(\cancel{x_1} - \underset{\substack{\| \\ a}}{x_0}) + (x_2 - \cancel{x_1}) + \dots + (\overset{\substack{b \\ \|}}{x_N} - \cancel{x_{N-1}})$$

Esempio

$$f(x) = mx + q$$

$m \geq 0$

$(m < 0)$



Mi aspetto $(b-a)\dfrac{mb+q+ma+q}{2}$

$$x_k = a + k \cdot \frac{b-a}{N} \qquad x_0 = a \ , \quad x_N = b \, .$$

$$x_{k+1} - x_k = \frac{b-a}{N} \qquad P_N = \{ x_k : k = 0, \dots, N \}$$

$$S^*(f, P_N) = \sum_{k=0}^{N-1} \frac{b-a}{N} \cdot f(x_{k+1}) = \frac{b-a}{N} \sum_{k=0}^{N-1} (m \cdot x_{k+1} + q) = \dots$$

$$S_*(f, P_N) = \sum_{k=0}^{N-1} \frac{b-a}{N} \cdot f(x_k) = \frac{b-a}{N} \sum_{k=0}^{N-1} (m\, x_k + q) = \dots$$

$$S^*_*(f, P_N) = \frac{b-a}{N} \sum_{k=0}^{N-1} \left[m \cdot \left(a + \underset{k}{(k+1)} \cdot \left(\frac{b-a}{N} \right) \right) + q \right]$$

$$= \frac{b-a}{N} \sum_{k=0}^{N-1} \left[m \cdot a + m \underset{k}{(k+1)} \frac{b-a}{N} + q \right]$$

$$= \frac{b-a}{N} \left[N \cdot m \cdot a + m \cdot \frac{b-a}{N} \sum_{k=0}^{N-1} \underset{k}{(k+1)} + N \cdot q \right]$$

$$= (b-a)\left[m \cdot a + m \frac{(b-a)}{N^2} \cdot N \cdot \frac{1+N \overset{+N-1}{}}{2} + q \right]$$

$$\xrightarrow{N \to +\infty} (b-a)\left[m \cdot a + m \frac{(b-a)}{2} + q \right]$$

$$= (b-a)\left[m \frac{a}{2} + m \frac{b}{2} + q \right] =$$

$$= \frac{1}{2}(b-a)\left(m(a+b) + 2q \right) \qquad \square$$

Esercizio

$$\boxed{\int_a^b x^2 \, dx} \qquad 0 < a \leq b$$

$$\sum_{k=0}^{N} k^2 = \ldots\ldots$$