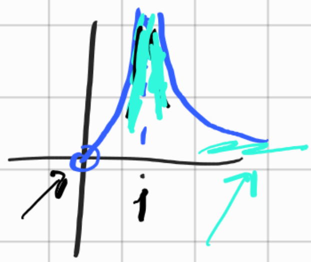
ANALISI MATEMATICA B
LEZIONE 69 — 21.3.2022

Es 3 test settimanale

$$\int_a^b \frac{1}{\sqrt{|\ln x|}}\,dx$$



1^- (1^+) $\frac{1}{\sqrt{|\ln x|}} \sim \boxed{\frac{1}{\sqrt{|x-1|}}}$

per $x \to 1^+$

e per $x \to 1^-$

$$\int_1^{1+\varepsilon} \frac{1}{(x-1)^p} < +\infty \iff p < 1.$$

$+\infty$ $\sqrt{|\ln x|} << x$ per $x \to +\infty$

$$\frac{1}{\sqrt{|\ln x|}} >> \frac{1}{x} \qquad \int_1^{+\infty} \frac{1}{x} = +\infty$$

$\boxed{\int_0^b f < +\infty}$ $\forall\ b < +\infty$

a) $(0,2]$

b) $(0,+\infty)$

c) $(1,+\infty)$

d) $(0,\frac{1}{2}]$

$$\int_0^{1/2} f(x)\,dx$$

$$\|$$

$$\lim_{\delta\to 0} \int_\delta^{1/2} f(x)\,dx$$

Integrali impropri

Sia f localmente R-int.

Se $\int_a^b |f(x)|\,dx$ è convergente

diremo che l'integrale $\int_a^b f(x)\,dx$

è <u>assolutamente convergente</u>.

$\exists f:[0,1]\to\mathbb{R}$ limitata

$|f|$ è R-int.

ma f non è R-int.

$f(x) = \mathbb{1}_{\mathbb{Q}}(x) - \frac{1}{2}$

Teo Sia f loc. R-integrabile.

integrale improprio

Se $\int_a^b |f(x)|\,dx$ è convergente

allora $\int_a^b f(x)\,dx$ è convergente.

Inoltre

$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f|$

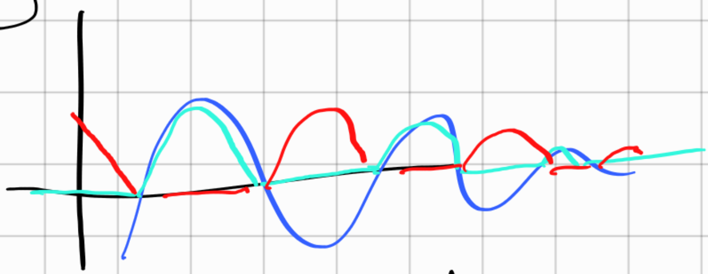
$a^+ = \begin{cases} a & \text{se } a \ge 0 \\ 0 & \text{se } a < 0 \end{cases}$

$a^- = \begin{cases} -a & \text{se } a \le 0 \\ 0 & \text{se } a > 0 \end{cases}$

dim $f = f^+ - f^-$

$f(x) = f(x)^+ - f(x)^-$

$|f| = f^+ + f^-$



$0 \le f^+ \le |f|$

$0 \le f^- \le |f|$

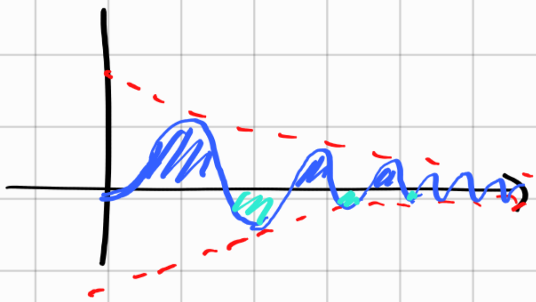
Se $|f|$ è integrabile $\left(\int_a^b |f| < +\infty\right)$

Allora f^+ e f^- lo sono.

$\Rightarrow f^+ - f^-$ lo è. $\square$

Es

$$\int_0^{+\infty} \sin(x^2)\cdot e^{-x}\,dx$$



$$\int_0^{+\infty} \left|\sin(x^2)\cdot e^{-x}\right|\,dx$$

$$\le \int_0^{+\infty} e^{-x}\,dx = \left[-e^{-x}\right]_0^{+\infty} = 1$$

$$\Rightarrow \int_0^{+\infty} \sin(x^2)\, e^{-x} \quad \text{è convergente.}$$

Es

$$\int_0^1 \frac{\sin\left(\frac{1}{x}\right)}{\sqrt{x}}\,dx \quad \text{è convergente}$$

Può darsi che una funzione con integrale convergente non sia assolutamente convergente.

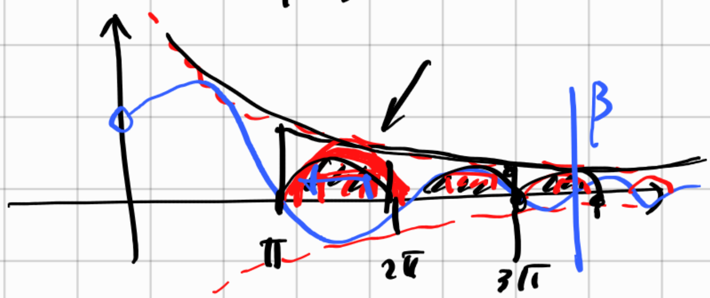
Per le serie:

$$\sum \frac{(-1)^k}{k}$$

Esempio

$$\int_1^{+\infty} \frac{\sin x}{x}\,dx$$

Per casa dimostrate che $\int_1^{+\infty} \left|\frac{\sin x}{x}\right| dx = +\infty$



$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx \ge \frac{1}{(k+1)\pi}\int_{k\pi}^{(k+1)\pi} |\sin x|\,dx = \frac{\int_0^{\pi} \sin x\,dx}{(k+1)\pi} = \frac{2}{\pi}\cdot\frac{1}{k+1}$$

$$\int_0^{+\infty} \frac{|\sin x|}{x}\,dx = \lim_{\beta\to+\infty}\int_0^{\beta} \frac{|\sin x|}{x}\,dx = \lim_{n\to+\infty}\int_0^{n\pi} \frac{|\sin x|}{x}\,dx$$

$$= \sum_{k=0}^{+\infty} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx$$

$$+\infty = \frac{2}{\pi}\sum_{k=0}^{+\infty} \frac{1}{k+1} \le \sum_{k=0}^{+\infty} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx$$

$\square$

MA

$$\boxed{\int_\pi^{+\infty} \frac{\sin x}{x}\,dx} \quad \text{è convergente.}$$

💡 integro per parti: $\sin x \cdot x^{-1}$

per parti

$$\int_\pi^{+\infty} \frac{\sin x}{x}\,dx = \left[\frac{-\cos x}{x}\right]_\pi^{+\infty} - \int_\pi^{+\infty} \frac{-\cos x}{-x^2}\,dx$$

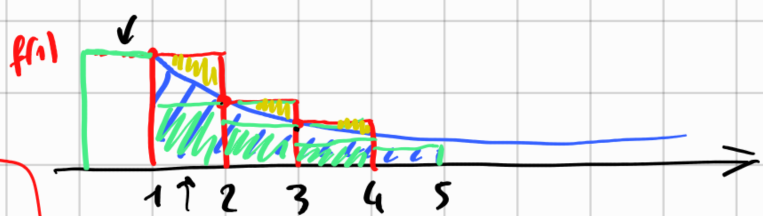
$$= 0 - \frac{1}{\pi} - \int_\pi^{+\infty} \frac{\cos x}{x^2}\,dx$$

è assolutamente convergente!

$$\int_\pi^{+\infty} \left|\frac{\cos x}{x^2}\right| dx \le \int_\pi^{+\infty} \frac{1}{x^2}\,dx < +\infty$$

Legame tra $\sum$ e $\int$

<u>Teorema</u> Sia $f:[1,+\infty) \to \mathbb{R}$ positiva, decrescente, localmente R-integrabile.



Allora

$$\int_1^{+\infty} f(x)\,dx \quad \text{e} \quad \sum_{k\geq 1}^{+\infty} f(k) \quad \text{hanno lo stesso carattere.}$$

dim

$$\sum_{k=2}^{+\infty} f(k) \le \int_1^{+\infty} f(x)\,dx \le \sum_{k=1}^{+\infty} f(k)$$

Es $\boxed{\sum_{k\geq 1}^{+\infty} \frac{1}{k} = +\infty}$ in quanto $\int_1^{+\infty} \frac{1}{x}\,dx = \left[\ln x\right]_1^{+\infty} = +\infty$

$p>1$ $\quad \sum \frac{1}{k^p}$ converge $\iff$ $\int \frac{1}{x^p}\,dx$ converge

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Studio di funzioni integrali

$$F(x) = \int_0^{x^2} e^{-t^2}\,dt \qquad\qquad F:\mathbb{R}\to\mathbb{R}$$

f

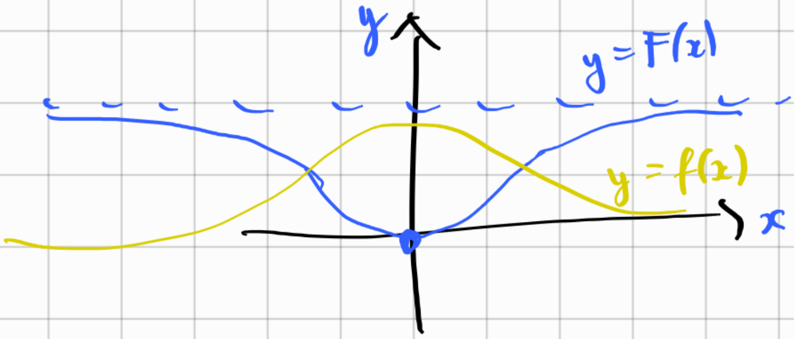
- Teorema fondamentale del calcolo
- Integrali impropri

$F'(x) = ?$

$$G(x) = \int_0^{x} e^{-t^2}\,dt$$
$$G'(x) = e^{-x^2}$$

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left[G(x^2)\right] = G'(x^2)\cdot 2x$$
$$= e^{-x^4}\cdot 2x$$

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty}\int_0^{x^2} e^{-t^2}\,dt = \lim_{y\to+\infty}\int_0^{y} e^{-t^2}\,dt$$
$$= \int_0^{+\infty} e^{-t^2}\,dt \in \mathbb{R}$$



$$\begin{array}{c|ccc} & & 0 & \\ \hline F' & - & 0 & + \end{array}$$

min

$$F(-x) = F(x)$$

$$F(0) = 0$$

$$F'(x) = e^{-x^4}\cdot 2x$$

$$\lim_{x \to 0} \frac{F(x)}{x}$$

uso Hospital o Taylor

Sugli appunti:

$$\int_0^x o(t^n)\, dt = o(t^{n+1})$$

ci servono delle ipotesi