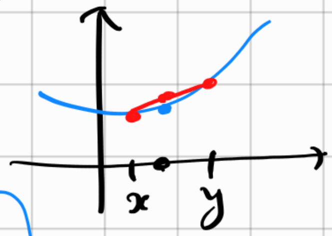
ANALISI MATEMATICA B
LEZIONE 54 - 20.2.2023

funzioni convesse (concava) $f: I \to \mathbb{R}$



$$\boxed{f((1-t)x+ty) \le (1-t)f(x) + t f(y)} \quad (\ge)$$

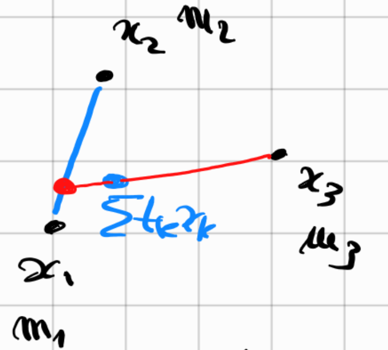
Se $f'' \ge 0$ (≤ 0) $\Rightarrow$ f è convessa (concava).

ES $f(x)=\ln(x)$ è concava $f'(x)=\frac{1}{x}$ $f''(x)=-\frac{1}{x^2}<0$

$$\ln\left(\frac{1}{2}x+\frac{1}{2}y\right) \ge \frac{1}{2}\ln x + \frac{1}{2}\ln y = \ln\sqrt{x} + \ln\sqrt{y}$$

$$\boxed{\frac{x+y}{2} \ge \sqrt{xy}}$$

Media aritmetica — Media geometrica



Teorema (diseguaglianza di Jensen)

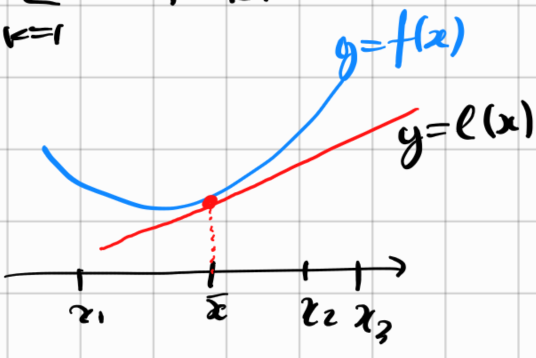
Se f convessa, $\sum_{k=1}^{N} t_k = 1$, $t_k \ge 0$

$$f(\bar{x}) = f\Big(\underbrace{\sum_{k=1}^{N} t_k x_k}\Big) \le \sum_{k=1}^{N} t_k f(x_k)$$

$$t_k = \frac{m_k}{m_1+m_2+m_3}$$

dim

$$\bar{x} = \sum_{k=1}^{N} t_k x_k$$



Sia $\ell(x)$ una retta di supporto

cioè $\ell(\bar{x}) = f(\bar{x})$ $\quad f(x) \ge \ell(x)$

$$f(\bar{x}) = \ell(\bar{x}) = \sum_{k=1}^{N} t_k\, \ell(x_k) \le \sum_{k=1}^{N} t_k\, f(x_k) \quad \square$$

ℓ è lineare $\qquad$ $f \ge \ell$

<u>ES</u> $\qquad \ln \dfrac{x_1+\dots+x_N}{N} \geq \sum_{k=1}^{N} \frac{1}{N} \ln x_k = \sum \ln \sqrt[N]{x_k}$

$$\frac{x_1+\dots+x_N}{N} \geq \sqrt[N]{x_1 \cdots x_N} \qquad AM \geq GM$$

<u>Disuguaglianza di Young.</u> se $p>1$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $x>0$, $y>0$

$$xy \le \frac{x^p}{p} + \frac{x^q}{q}.$$

$\left[\begin{array}{l} \text{ES } p=2, q=2 \\ 2xy \le x^2+y^2 \\ (x-y)^2 \geq 0 \end{array}\right]$

<u>dim</u>

$$xy = e^{\ln xy} = e^{\ln x + \ln y}$$

$$= e^{\frac{1}{p}\ln x^p + \frac{1}{q}\ln x^q} \le \frac{1}{p} e^{\ln x^p} + \frac{1}{q} e^{\ln x^q}$$

e^x convessa

$$= \frac{1}{p} x^p + \frac{1}{q} x^q \quad \square$$

<u>Disuguaglianza di Hölder</u> $\quad \frac{1}{p}+\frac{1}{q}=1 \qquad a_k, b_k \geq 0$

$$\to \sum_{k=1}^{N} a_k b_k \le \left(\sum_{k=1}^{N} a_k^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^{N} b_k^q\right)^{\frac{1}{q}}$$

<u>ES</u> $\quad p=q=2$

$$(\underline{a}, \underline{b}) = \sum_{k=1}^{N} a_k b_k = \left(\sum_{k=1}^{N} a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{N} b_k^2\right)^{\frac{1}{2}}$$

$$= |\underline{a}| \cdot |\underline{b}|$$

<u>Cauchy-Schwarz</u>

dim (Hölder) $\quad A = \left(\sum a_k^p\right)^{\frac{1}{p}} \quad B = \left(\sum b_k^q\right)^{\frac{1}{q}}$

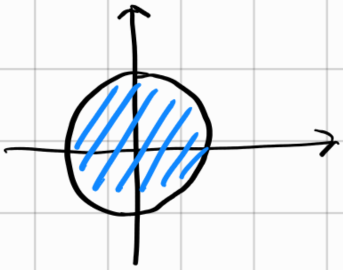
Tesi: $\quad \dfrac{\sum a_k b_k}{A \cdot B} \leq 1$

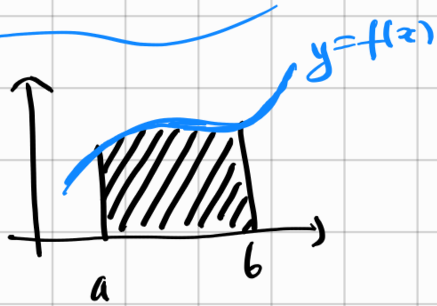
cioè $\quad \boxed{\sum \dfrac{a_k}{A} \cdot \dfrac{b_k}{B} \leq 1.}$

$$\sum_{k=1}^{N} \underbrace{\frac{a_k}{A} \frac{b_k}{B}} \overset{\text{Young}}{\leq} \frac{1}{p}\sum_{k=1}^{N}\left|\frac{a_k}{A}\right|^p + \frac{1}{q}\sum_{k=1}^{N}\left(\frac{b_k}{B}\right)^q$$

$$= \frac{1}{p}\,\cancel{\frac{\sum a_k^p}{A^p}} + \frac{1}{q}\,\cancel{\frac{\sum b_k^q}{B^q}} = 1.$$

INTEGRALI

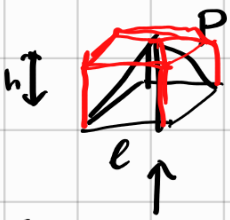




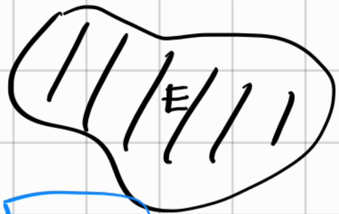
$\int_a^b f(x)dx$

MISURA PEANO-JORDAN

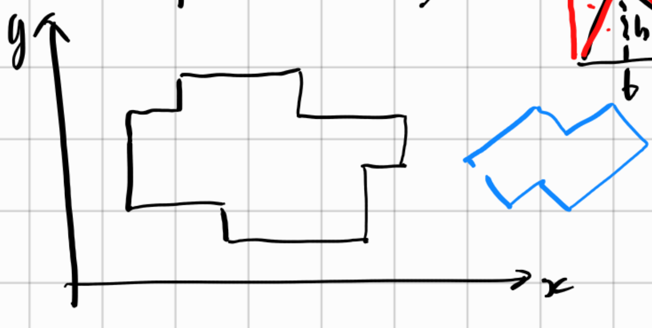
Idea: voglio definire l'area di un sottoinsieme $E \subseteq \mathbb{R}^2$.



$V(P) = \boxed{\frac{1}{3} l^2 \cdot h}$



$A(T) = \boxed{\frac{1}{2} b \cdot h}$



Assiomatizzazione dell'area

$m(E) =$ area di E

$E \subseteq \mathbb{R}^2$

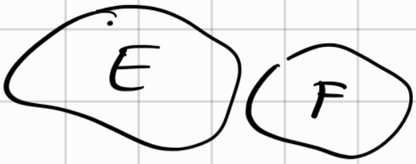
$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

~~(0) $m(E) \geq 0$ (positività)~~

(1) $\boxed{m(E \cup F) = m(E) + m(F)}$ (additività) se $E \cap F = \emptyset$

(2) $m\left(\boxed{[a,b] \times [c,d]}\right) = (b-a) \cdot (d-c)$ (normalizzazione)

$\boxed{a \leq b}$
$c \leq d$



(3) $E \subseteq F \Rightarrow m(E) \leq m(F)$ (monotonia)

Es $m(\emptyset) = 0$

$\emptyset \subseteq [0,\varepsilon] \times [0,\varepsilon]$

$m(\emptyset) \leq \varepsilon^2$

$m(\emptyset) \leq 0$

oppure

$\emptyset = \emptyset \cup \emptyset$

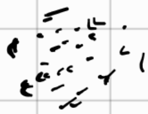
$m(\emptyset) = 2\, m(\emptyset)$

$\Downarrow$

$m(\emptyset) = 0.$ ☐

(1),(2),(3) determinano m. Lo vedremo.

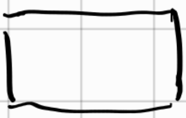
MA non esiste $m : \mathcal{P}(\mathbb{R}^2) \to \mathbb{R} \subseteq [0,+\infty]$



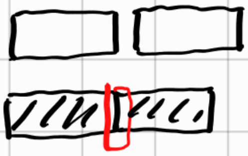
E può essere troppo brutto per essere misurato.

la misura si definisce su una classe di sottoinsiemi che chiameremo "insiemi misurabili".

① i rettangoli cartesiani sono misurabili



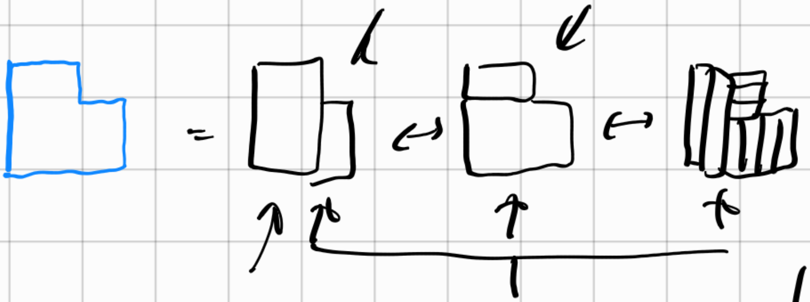
② i plurirettangoli cartesiani sono misurabili:

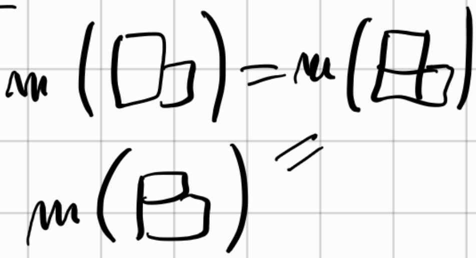


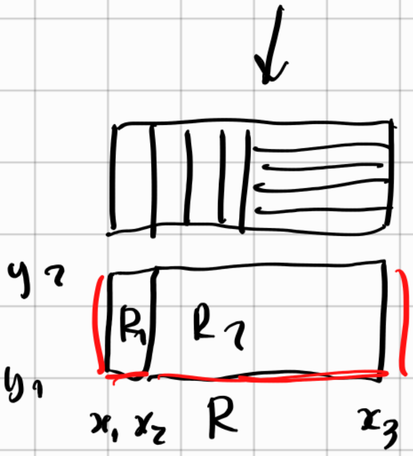
$$m(E\cup F) = m(E\setminus F) + m(E\cap F) + m(F\setminus E)$$

$$\boxed{m(E\cap F) + m(E\cup F) = m(E) + m(F)}$$

Dubbio

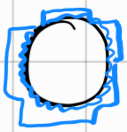






$$m(R) = (x_3-x_1)\cdot(y_2-y_1) =$$

$$m(R_1)+m(R_2) = (x_2-x_1)(y_2-y_1) + (x_3-x_2)(y_2-y_1)$$

$E =$  E limitato.

$$m^*(E) = \inf\{m(P) : P \text{ plurirettangolo cartesiano}, P \supseteq E\}$$

$$m_*(E) = \sup\{m(P) : P \text{ pl. cart. } P \subseteq E\}$$

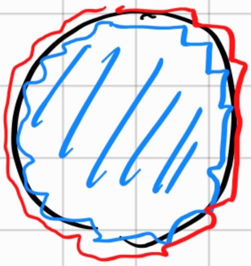
Diciamo che E è P-J. misurabile se $m^*(E) = m_*(E)$.

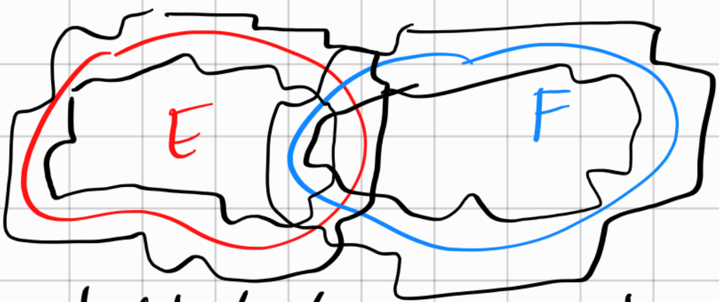
Poniamo $m(E) = m^*(E)$.

$$E = \{(x,y)\in[0,1]\times[0,1] : x\in\mathbb{Q},\ y\in\mathbb{Q}\}$$

$m^*(E)=1$ E non è P-J misurabile.

$m_*(E)=0$





Vale l'additività degli insiemi P-J misurabili.

Cosa manca. $L:\mathbb{R}^2\to\mathbb{R}^2$ lineare.

E P-J misurabile.

$$m(L(E)) = ?$$