ANALISI MATEMATICA B
LEZIONE 62 - 10.3.2023

$$\int \frac{P(x)}{Q(x)}\,dx \qquad P, Q \text{ polinomi.}$$

integrali che si riconducono a funzioni razionali

1. $\displaystyle\int R(e^{\lambda x})\,dx$ $\qquad R$ razionale $\qquad R(x) = \dfrac{P(x)}{Q(x)}$

$$\begin{cases} y = e^{\lambda x} \\ x = \dfrac{\ln y}{\lambda} \\ dx = \dfrac{1}{\lambda y} \end{cases} \qquad \int R(e^{\lambda x})\,dx = \int \frac{R(y)}{\lambda y}\,dy$$

Esempio

$$\int \frac{2\sqrt{e^x} + e^{2x}}{e^x - 4}\,dx \qquad \lambda = \frac{1}{2}$$

$$y = e^{\frac{x}{2}} = \sqrt{e^x} \qquad x = 2\ln y \qquad dx = \frac{2}{y}\,dy$$

$$= \int \frac{2y + y^4}{y^2 - 4}\cdot\frac{2}{y}\,dy$$

$$= \int \frac{2y^3 + 4}{y^2 - 4}\,dy = \int \frac{2y(y^2-4) + 8y + 4}{y^2 - 4}\,dy$$

$$= \int 2y\,dy + 4\int \frac{2y+1}{y^2-4}\,dy$$

$$= y^2 + \int\left[\frac{3}{y+2} + \frac{5}{y-2}\right]dy$$

$$Ay - 2A + By + 2B = 2y + 1 \qquad \begin{cases} A + B = 2 \\ 2(B-A) = 1 \end{cases}$$

$$\left[\; B-A=\frac{1}{2} \qquad \begin{cases} B=A+\frac{1}{2} \\ 2A+\frac{1}{2}=2 \end{cases} \qquad \begin{cases} A=\frac{3}{4} \\ B=\frac{5}{4} \end{cases} \;\right]$$

$$= y^2 + 3\ln|y+2| + 5\ln(y-2)$$

$$= y^2 + \ln|y+2|^3(y-2)^5 \qquad y=\sqrt{e^x}$$

$$= e^x + \ln\left(\sqrt{e^x}+2\right)^3\cdot\left(\sqrt{e^x}-2\right)^5$$

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funzioni razionali in $\sin^2$, $\cos^2$, $\sin\cdot\cos$

$$\int R(\sin^2 x, \cos^2 x, \sin x\cdot\cos x)\,dx$$

$$R(x,y,z) = \frac{P(x,y,z)}{Q(x,y,z)}$$

P, Q polinomi di 3 variabili.

$$\begin{cases} \cos^2 x = \dfrac{1}{1+tg^2 x} = \dfrac{1}{1+t^2} \\ \sin^2 x = \dfrac{tg^2 x}{1+tg^2 x} = \dfrac{t^2}{1+t^2} \\ \sin x\cdot\cos x = \dfrac{tg\, x}{1+tg^2 x} = \dfrac{t}{1+t^2} \end{cases}$$

$$tg = \frac{\sin}{\cos}$$

$$\begin{cases} t = tg\, x \\ x = arctg\, t \\ dx = \dfrac{1}{1+t^2}\,dt \end{cases}$$

$$\int R(\sin^2 x, \cos^2 x, \sin x\cos x)\,dx = \int \underbrace{R\left(\frac{t^2}{1+t^2}, \frac{1}{1+t^2}, \frac{t}{1+t^2}\right)\cdot\frac{1}{1+t^2}\,dt}$$

Esercizio

$$\int \frac{1}{\cos x \cdot (\sin x + \cos x)}\, dx =$$

$$= \int \frac{1}{\sin x \cos x + \cos^2 x}\, dx \qquad t = \tg x$$

$$= \int \frac{1}{\frac{t}{\cancel{1+t^2}} + \frac{1}{\cancel{1+t^2}}} \cdot \frac{1}{\cancel{1+t^2}}\, dt$$

$$= \int \frac{1}{t+1}\, dt = \ln|t+1| = \ln|1+\tg x|$$

Funzione razionale in $\sin x$, $\cos x$

$$\int R(\sin x, \cos x)\, dx \qquad R(x,y) \text{ razionale}$$

$$\begin{cases} \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2} \\ \sin x = 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2} = \frac{2t}{1+t^2} \end{cases}$$

$$\left[\begin{array}{l} t = \tg \frac{x}{2} \\ x = 2 \arctg t \\ dx = \frac{2}{1+t^2}\, dt \end{array}\right.$$

Esercizio

$$\int \frac{1}{\sin x}\, dx \qquad t = \tg \frac{x}{2} \qquad \sin x = \frac{2t}{1+t^2}$$

$$= \int \frac{\cancel{1+t^2}}{2t} \cdot \frac{2}{\cancel{1+t^2}}\, dt = \int \frac{1}{t} = \ln|t|$$

$$= \ln\left|\tg \frac{x}{2}\right|$$

Funzioni razionali in $\sqrt[n]{x}$

$$\int R(\sqrt[n]{x})\,dx \qquad\qquad y = \sqrt[n]{x}$$

$$= \int R(y)\cdot n y^{n-1}\cdot dy \qquad\qquad x = y^n \qquad dx = n\,y^{n-1}$$

Esercizio

$$= \int \frac{\sqrt[4]{x}}{\sqrt{x}+\sqrt[3]{x}}\,dx \qquad\qquad n=12 \qquad y = \sqrt[12]{x}$$

$$= \int \frac{y^3}{y^6+y^4}\cdot 12\cdot y^{11}\,dy$$

$$= 12\int \frac{y^{14}}{y^6+y^4}\,dy = 12\int \frac{y^{10}}{y^2+1}\,dy =$$

$$\begin{array}{l|l}
y^{10} & y^2+1 \\
\hline
y^{10}+y^8 & y^8-y^6+y^4-y^2+1 \\
-y^8 & \\
-y^8-y^6 & \\
y^6 & \\
y^6+y^4 & \\
-y^4 & \\
-y^4-y^2 & \\
y^2 & \\
y^2+1 & \\
-1 &
\end{array}$$

$$= 12\int\left[y^8-y^6+y^4-y^2+1\right]dy - 12\int\frac{1}{y^2+1}\,dy$$

$$= \frac{4}{3}y^9 - \frac{12}{7}y^7 + \frac{12}{5}y^5 - 4y^3 + 12y - 12\cdot\operatorname{arctg} y$$

$$= \frac{4}{3}\cdot\sqrt[4]{x^3} - \frac{12}{7}\sqrt[12]{x^7} + \frac{12}{5}\sqrt[12]{x^5} - 4\sqrt[4]{x} + 12\sqrt[12]{x} - 12\operatorname{arctg}\sqrt[12]{x}$$

PAGANI - SALSA: INTEGRALI DA 15''

I

$$\int \frac{e^x + \cos x}{e^x + \sin x}$$

$$= \ln\left(e^x + \sin x\right)$$

II

$$\int \frac{2}{\sqrt{x} - \sqrt{x+2}} = -\int \frac{2\left[\sqrt{x} + \sqrt{x+2}\right]}{2}$$

$$= -\frac{3}{2}\left[x^{3/2} + (x+2)^{3/2}\right]$$

III

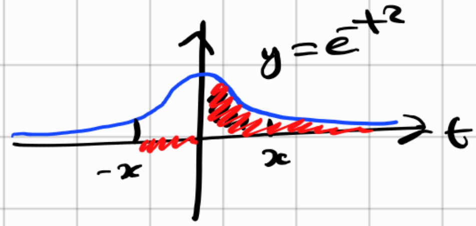
$$\int \frac{\ln \ln x}{x} = \ln x\left(\ln \ln x - \ln x\right)$$

$y = \ln x$

$$\int \ln y = y \ln y - y$$

SE NON RIESCO A SCRIVERE LA PRIMITIVA POSSO STUDIARE LE FUNZIONI INTEGRALI

Es. $F(x) = \int_0^x e^{-t^2}\,dt$



$F(0) = 0$

$$F(-x) = \int_0^{-x} e^{-t^2}\,dt = -\int_0^{x} e^{-s^2}\cdot ds = -F(x)$$

$$\begin{cases} s = -t \\ ds = -dt \end{cases}$$

Segno di F: se $x>0$ $e^{-t^2} \geq 0 \Rightarrow F(x) > 0$

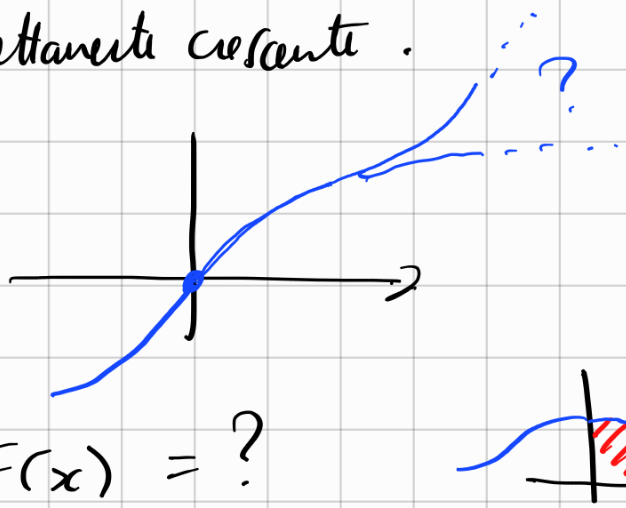
se $x<0$ $F(x) < 0$ (dispari)

x		0	
$F(x)$	$-$	0	$+$

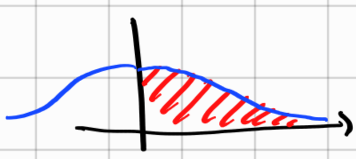
Segno di F':

$F'(x) = e^{-x^2} > 0$ per il teo. fondamentale del calcolo

F è strettamente crescente.



$\lim_{x\to+\infty} F(x) = ?$



↑ c'è sugli appunti, richiede altri strumenti dimostriamo solo che questo limite è finito

$$F(x) = \int_0^x e^{-t^2}\,dt \leq \int_0^x \frac{1}{1+t^2}\,dt = \operatorname{arctg} x \xrightarrow[x\to+\infty]{} \frac{\pi}{2}$$

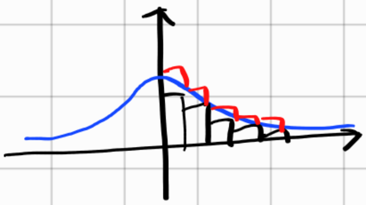
$$e^{-t^2} = \frac{1}{e^{t^2}} \qquad e^{t^2} \geq 1+t^2$$

$e^x \geq 1+x \quad \forall x$

$\lim_{x\to+\infty} F(x)$ esiste perché F monotona

$$\lim_{x\to+\infty} F(x) \leq \lim_{x\to+\infty} \operatorname{arctg} x = \frac{\pi}{2}.$$

Si potrebbe calcolare numericamente.



PROSSIMA SETTIMANA: INTEGRALI IMPROPRI

$$\int_0^{+\infty} e^{-x^2}\,dx := \lim_{x\to+\infty} \int_0^x e^{-t^2}\,dt \quad \text{è finito}$$

$$\int_0^1 \frac{1}{x}\,dx := \lim_{x\to 0} \int_x^1 \frac{1}{t}\,dt \quad \text{è infinito}$$