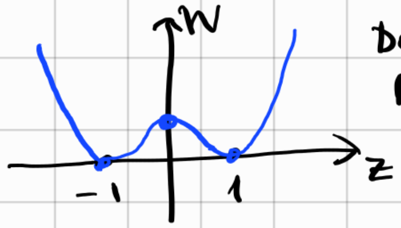
ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 5 - 11.3.2024

Esempi di non esistenza.

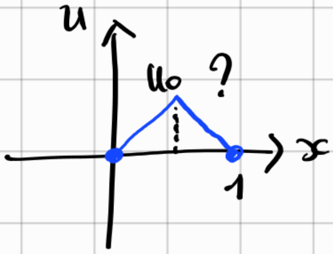
① $\mathcal{L}(u) = \int_0^1 W(u'(x))\,dx \qquad W(z) = (1-z^2)^2$



DOPPIO POZZO

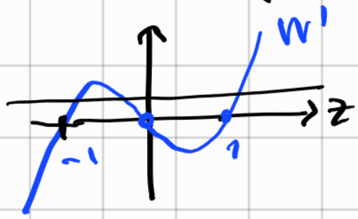
$\mathcal{L}(u) \to \min$

$u(0) = u(1) = 0$



$L(x,y,z) = W(z)$

$W'(z) = -4z(1-z^2)$



$\frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial z} = W'(z)$

E.L. : $\frac{d}{dx} W'(u'(x)) = 0$

$W'(u'(x)) = \text{cost}$

u' non può saltare se $u \in C^1$

$\Rightarrow$ u' è costante

$u(x) = mx + q \qquad u(0) = u(1) \Rightarrow m = 0$

$u(x) = q \qquad \Rightarrow q = 0 \qquad \Rightarrow u(x) = 0.$

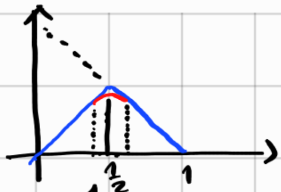
Se $u_0 \equiv 0$, $u_0' = 0$: $\quad \mathcal{L}(u_0) = \int_0^1 W(0) = \int_0^1 1 = 1.$

Ma se prendo $u_1(x) = \begin{cases} x & \text{se } x \in [0, \frac{1}{2}] \\ 1-x & \text{se } x \in [\frac{1}{2}, 1] \end{cases}$

$u_1'(x) = \begin{cases} 1 & \text{se } x \in [0, 1/2) \\ \nexists & \text{se } x = \frac{1}{2} \\ -1 & \text{se } x \in (\frac{1}{2}, 1] \end{cases}$

$\mathcal{L}(u_1) = \int_0^{1/2} W(1) + \int_{\frac{1}{2}}^1 W(-1) = 0 + 0 = 0$

Nota $u_1 \in C^1_p \setminus C^1$



ma posso approssimare u_1 con $u_\varepsilon \in C^1$:

$$u_\varepsilon(x) = \begin{cases} x & \text{se } x \le \frac{1}{2}-\varepsilon \\ 1-x & \text{se } x \geq \frac{1}{2}+\varepsilon \\ a\left(x-\frac{1}{2}\right)^2+b & \text{se } \left|x-\frac{1}{2}\right| \le \varepsilon \end{cases}$$

$u_\varepsilon \in C^0$ $\quad x=\frac{1}{2}\pm\varepsilon$ es: $x=\frac{1}{2}-\varepsilon \qquad \frac{1}{2}-\varepsilon = a\varepsilon^2+b$

$u_\varepsilon \in C^1$

$$u_\varepsilon'(x) = \begin{cases} 1 & \text{se } x \le \frac{1}{2}-\varepsilon \\ -1 & \text{se } x \geq \frac{1}{2}+\varepsilon \\ \boxed{2a\left(x-\frac{1}{2}\right)} & \text{se } \left|x-\frac{1}{2}\right| \le \varepsilon \end{cases}$$

$x=\frac{1}{2}\pm\varepsilon$ es: $x=\frac{1}{2}-\varepsilon \qquad 1 = 2a(-\varepsilon)$

$$\begin{cases} a = -\frac{1}{2\varepsilon} \\ \frac{1}{2}-\varepsilon = -\frac{\varepsilon}{2}+b \end{cases}$$

$$\begin{cases} a = -\frac{1}{2\varepsilon} \\ b = \frac{1}{2}-\frac{\varepsilon}{2} \end{cases}$$

$$u_\varepsilon(x) = \begin{cases} -\frac{\left(x-\frac{1}{2}\right)^2}{2\varepsilon} + \frac{1-\varepsilon}{2} & \text{se } \left|\frac{1}{2}-x\right| \le \varepsilon \\ x & \text{se } x \le \frac{1}{2}-\varepsilon \\ 1-x & \text{se } x \geq \frac{1}{2}+\varepsilon \end{cases}$$

$$\mathcal{I}(u_\varepsilon) = \cancel{\int_0^{\frac{1}{2}-\varepsilon} W(1)} + \int_{\frac{1}{2}-\varepsilon}^{1/2+\varepsilon} W\left(-\frac{x-\frac{1}{2}}{\varepsilon}\right) dx + \cancel{\int_{\frac{1}{2}+\varepsilon}^{1} W(-1)}$$

$$= \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} \left(1-\left(\frac{x-\frac{1}{2}}{\varepsilon}\right)^2\right)^2 dx = \int_{-\varepsilon}^{\varepsilon} \left(1-\frac{s^2}{\varepsilon^2}\right)^2 ds =$$

$s = x-\frac{1}{2}$

$ds = dx$

$$= \int_{-\varepsilon}^{\varepsilon} \left[1 - \frac{2s^2}{\varepsilon^2} + \frac{s^4}{\varepsilon^4}\right] ds = \left[s - \frac{2s^3}{3\varepsilon^2} + \frac{s^5}{5\varepsilon^4}\right]_{-\varepsilon}^{\varepsilon}$$

$$= 2\varepsilon - \frac{4\varepsilon^3}{3\varepsilon^2} + \frac{2\varepsilon^5}{5\varepsilon^4} = 2\varepsilon - \frac{4}{3}\varepsilon + \frac{2}{5}\varepsilon$$

$$= \frac{16}{15}\varepsilon \longrightarrow 0 \qquad \text{se } \varepsilon \to 0.$$

$u_\varepsilon \in C^1$, $\quad u_\varepsilon(0)=0$, $\quad u_\varepsilon(1)=0$

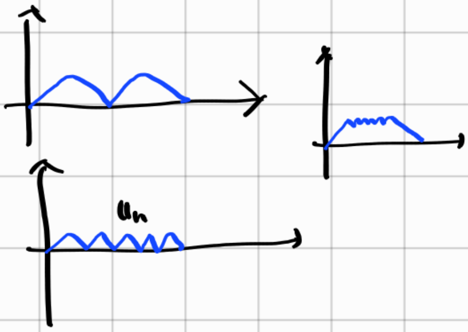
$\mathcal{L}(u_\varepsilon) \longrightarrow 0$. $\qquad \mathcal{L}(u) \geq 0 \quad \forall u$.

$$\inf_{u\in C^1_0} \mathcal{L} = 0. \qquad \Big(\text{anche } \inf_{u\in C^\infty} \mathcal{L} = 0\Big)$$

$\exists\, u_0 \in C^1_p : \mathcal{L}(u_0) = 0$. $\qquad \mathcal{L}(u_0) = \min_{C^1_p} \mathcal{L}$

ma $\min_{C^1} \mathcal{L}$ non esiste.

Su C^1_p il minimo non è unico

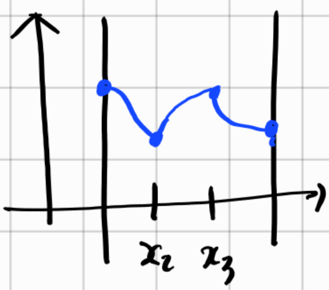


$u_n \rightrightarrows 0 \qquad \underset{\displaystyle 1}{\underset{\|}{\mathcal{L}(0)}} > \liminf_{n\to+\infty} \mathcal{L}(u_n) \longleftarrow$

$\mathcal{L}$ non è s.c.i (semicontinuo inferiormente)

<u>Come trovare i minimi su C^1_p.</u>

$$C^1_p([a,b]) = \Big\{ u \in C^0([a,b]) : \exists\, x_0, \dots, x_n \text{ tali che } a = x_0 < x_1 < \dots < x_n = b, \quad u \in C^1([x_{k-1}, x_k]) \Big\}$$



$$\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))\,dx$$

si intende: $$= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} L(x, u(x), u'(x))\,dx$$

$$\left[\frac{d}{d\varepsilon}\mathcal{I}(u+\varepsilon\varphi)\right]_{\varepsilon=0} = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \left[\frac{\partial L}{\partial y}\cdot\varphi + \frac{\partial L}{\partial z}\cdot\varphi'\right] dx$$

$$= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \boxed{\cancel{\left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial z}\right]}}\cdot\varphi + \left[\frac{\partial L}{\partial z}\cdot\varphi\right]_{x_{k-1}}^{x_k}$$

Se $\varphi \in C_c^\infty([x_{k-1}, x_k])$

vale EL $$\boxed{\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial z}}$$

$$= \sum_{k=1}^{n} \left[\frac{\partial L}{\partial z}\cdot\varphi\right]_{x_{k-1}}^{x_k} = \sum_{k=1}^{n} \left[\frac{\partial L}{\partial z}(x, u(x), u'(x))\varphi(x)\right]_{x_{k-1}}^{x_k}$$

$$= \sum_{k=1}^{n} \left[\frac{\partial L}{\partial z}(x_k, u(x_k), u'_-(x_k))\cdot\varphi(x_k) - \frac{\partial L}{\partial z}(x_{k-1}, u(x_{k-1}), u'_+(x_{k-1}))\cdot\varphi(x_{k-1})\right]$$

$\varphi(x_j) = 0$ se $j \neq k$, $\varphi(x_k) \neq 0$
$k = 1, \dots, n-1$

$$= \underbrace{\left[\frac{\partial L}{\partial z}(x_k, u(x_k), u'_-(x_k)) - \frac{\partial L}{\partial z}(x_k, u(x_k), u'_+(x_k))\right]\varphi(x_k)}_{=0}$$

ERDMANN-WEIERSTRASS:

$$\boxed{\frac{\partial L}{\partial z}\left(x,u(x),u'_-(x)\right)=\frac{\partial L}{\partial z}\left(x,u(x),u'_+(x)\right)}$$

* Nell'esempio precedente $(u_0)'_-\left(\frac{1}{2}\right)=1$ $\quad \frac{\partial L}{\partial z}=W'$

$(u_0)'_+\left(\frac{1}{2}\right)=-1$ $\quad W'(1)=W'(-1)$.

* Se $\frac{\partial L}{\partial z}$ è iniettiva (per x,y fissati, rispetto a z) non ci può essere un punto angoloso!

* Se fissati x,y $\quad z\mapsto L(x,y,z)$ è strettamente convessa allora $\frac{\partial L}{\partial z}$ è iniettiva.

* Vedi es. Fermat è condizione di Snell.

Esempio cattivo 2

$$W(z)=(1-z^2)^2$$

$$\mathcal{L}(u)=\int_0^1\left[W(u'(x))+\frac{1}{2}(u(x))^2\right]dx$$

$u(0)=0,\ u(1)=0,\qquad \mathcal{L}(u)\to\min.$

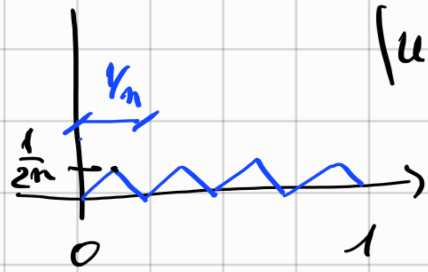
$\mathcal{L}(u)\geq 0\quad\forall u.$

$\mathcal{L}(u)=0 \Rightarrow W(u'(x))=0$ e $u(x)=0$ per quasi ogni $x\in(a,b)$

$u=0 \Rightarrow u'=0 \Rightarrow W(u'(x))=1$ per quasi ogni x

$\Rightarrow \mathcal{L}(u)\neq 0.$

Se $u_n \in C^1_p$



$|u_n'| = 1$

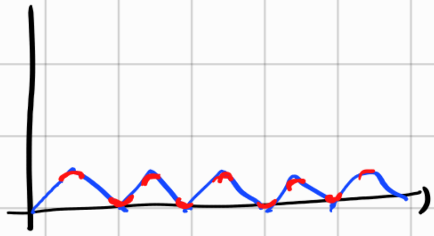
$W(u_n') = 0$

$$\mathcal{L}(u_n) = \int_0^1 \left(W(u_n') + \frac{1}{2} u_n^2 \right)$$

$$= \frac{1}{2}\int_0^1 u_n^2 \le \frac{1}{2} \int_0^1 \max u_n = \frac{1}{2}\int_0^1 \frac{1}{2n} = \frac{1}{4n} \longrightarrow 0$$

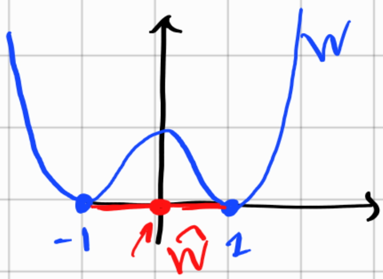
per $n \to \infty$

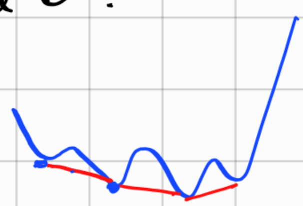
$\inf_{C^1_p} \mathcal{L} = 0$ non c'è minimo.



Regolarizzando opportunamente capisco che $\inf_{C_c^\infty([0,1])} \mathcal{L} = 0$.

* Tutte le successioni minimizzanti tendono a 0.





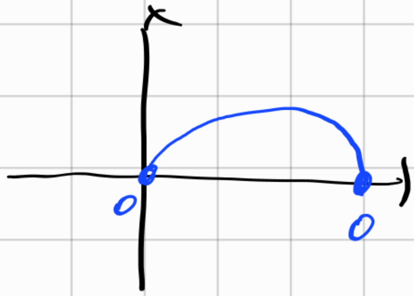
* $u=0$ si approssima con $u' = \pm 1$

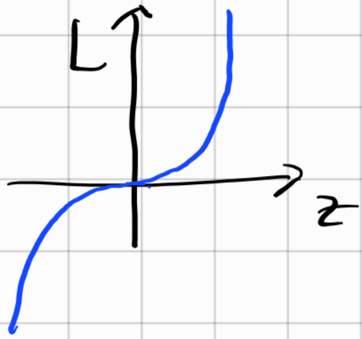
Fenomeno di "omogeneizzazione": microstrutture

* $u=0$ sarebbe minimo se al posto di W mettessi $\hat{W}$ = inviluppo convesso di W.

Esempio 3 $\quad \mathcal{L}(u) = \int_0^1 (u'(x))^3 \, dx \qquad L(x,y,z) = z^3$

$u(0)=0, \; u(1)=0, \; \mathcal{L}(u) \to \min.$

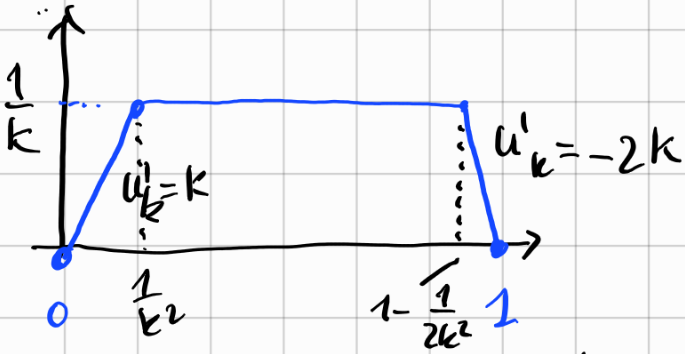




Per avere $\mathcal{L}$ piccolo (negativo) vorrei $u'<0$

ma $\quad \int_0^1 u'(x)\,dx = [u(x)]_0^1 = u(1)-u(0) = 0$

Idea:



$$u_k(x) = \begin{cases} kx & \text{se } x \le \frac{1}{k^2} \\ \frac{1}{k} & \text{se } \frac{1}{k^2} \le x \le 1 - \frac{1}{2k^2} \\ 2k - 2kx & \text{se } x \ge 1 - \frac{1}{2k^2} \end{cases}$$

$u_k \in C^0.$

$$u_k'(x) = \begin{cases} k & \text{se } x < \frac{1}{k^2} \\ 0 & \text{altrimenti} \\ -2k & \text{se } x > 1 - \frac{1}{2k^2} \end{cases}$$

$$\mathcal{L}(u_k) = \int_0^{\frac{1}{k^2}} (k)^3 + \int_{1-\frac{1}{2k^2}}^{1} (-2k)^3 = \frac{1}{k^2} k^3 - \frac{8k^3}{2k^2}$$

$$= -3k \longrightarrow -\infty$$

$\mathcal{L}(u_k) \to -\infty$ per $k \to +\infty$.

$\inf_{C^1_p} \mathcal{L} = -\infty$.

Allisciando : $\inf_{C^\infty_c} \mathcal{L} = -\infty$.