ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 1 - 26.9.2024

MAR 16-18 aula G

GIO 16-18 aula N1

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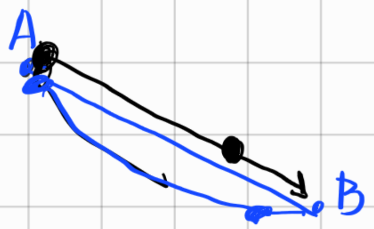
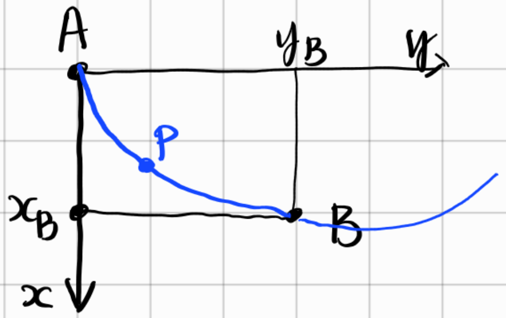


PROBLEMA della BRACHISTOCRONA

Trovare la curva che congiunge A e B per la quale un grave impiega il tempo minimo partendo da A per arrivare a B.



$$\frac{1}{2}mv^2 + mgh = \text{costante}$$

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad y = u(x) \qquad u(x(t)) = y(t)$$

$$\frac{1}{2}\left(\dot{x}^2(t) + \dot{y}^2(t)\right) - g\,x(t) = 0 \qquad \xleftarrow{\text{per } t=0} \qquad \begin{cases} x(0) = 0 \\ y(0) = 0 \\ \dot{x}(0) = 0 \\ \dot{y}(0) = 0 \end{cases}$$

$$\dot{x}^2 + \dot{y}^2 = 2gx$$

$$T = \int_0^T 1\,dt = \int_0^{x_B} \frac{1}{\dot{x}}\,dx \qquad dx = \dot{x}(t)\,dt$$

$$\dot{x}^2 + \dot{y}^2 = 2gx \qquad \dot{x}^2\left(1 + \frac{\dot{y}^2}{\dot{x}^2}\right) = 2gx$$

$$\left[\begin{array}{l} y(t) = u(x(t)) \\ \dot{y}(t) = u'(x(t))\,\dot{x}(t) \end{array}\right]$$

$$\dot{x}^2\left(1 + \left(u'(x)\right)^2\right) = 2gx \qquad |\dot{x}|^2 = \frac{2gx}{1+u'}$$

$$T = \int_0^{x_B} \sqrt{\frac{1+(u'(x))^2}{2g\,x}}\,dx$$

PROBLEMA

Trovare una funzione u tale che $u(0)=0$, $u(x_B)=y_B$,

$u\in C^1$ che rende minimo:

$$T(u) = \int_0^{x_B} \frac{\sqrt{1+(u')^2}}{\sqrt{2gx}}\,dx$$

$a=0$

$b=x_B$

$L(x,y,z)=\frac{\sqrt{1+z^2}}{\sqrt{2gx}}$

Più in generale possiamo considerare:

$$\boxed{\mathcal{L}(u) = \int_a^b L(x,u(x),u'(x))\,dx}$$

funzionale classico del CdV.

Lagrangiana: $L = L(x,y,z)$ $\qquad x\in\mathbb{R},\ y\in\mathbb{R}\ (\mathbb{R}^n),\ z\in\mathbb{R}\ (\mathbb{R}^n)$

$a<b$ $\qquad V=\{u\in C^1([a,b]) : u(a)=y_a,\ u(b)=y_b\}$

V è uno spazio affine

$$TV = V-V = \{\varphi\in C^1([a,b]) : \varphi(a)=0,\ \varphi(b)=0\}$$

è uno spazio vettoriale.

$$= C^1_0([a,b])$$

Data $\varphi\in C^1_0([a,b])$ dato $\varepsilon\in\mathbb{R}$

$\mathcal{L}(u)\le\mathcal{L}(u+\varepsilon\varphi)$ $\qquad$ se u è un minimo di $\mathcal{L}$

$\forall\varphi\ \forall\varepsilon$

$$F(\varepsilon) = \mathcal{L}(u+\varepsilon\varphi) \qquad F(0) \leq F(\varepsilon) \quad \forall \varepsilon \in \mathbb{R}$$

Se $F'(0)$ esiste allora $F'(0) = 0$

$$\frac{d}{d\varepsilon} F(\varepsilon)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_a^b L\left(x, u(x)+\varepsilon\varphi(x), u'(x)+\varepsilon\varphi'(x)\right) dx \Big|_{\varepsilon=0}$$

$$= \int_a^b \frac{d}{d\varepsilon} L\big(x, \underbrace{u(x)+\varepsilon\varphi(x)}_{y}, \underbrace{u'(x)+\varepsilon\varphi'(x)}_{z}\big) dx \Big|_{\varepsilon=0}$$

$$= \int_a^b \left[\frac{\partial L}{\partial y}\left(x, u+\varepsilon\varphi, u'+\varepsilon\varphi'\right) \cdot \varphi(x) + \frac{\partial L}{\partial z}\left(x, u+\varepsilon\varphi, u'+\varepsilon\varphi'\right) \varphi'(x)\right] dx \Big|_{\varepsilon=0}$$

$$= \int_a^b \left[\frac{\partial L}{\partial y}\left(x, u(x), u'(x)\right) \varphi(x) + \underbrace{\frac{\partial L}{\partial z}\left(x, u(x), u'(x)\right) \varphi'(x)\right] dx}_{\text{Per parti}}$$

$$= 0 \qquad \forall \varphi \in C_0^1([a,b])$$

integrando per parti:

$$= \int_a^b \left[\frac{\partial L}{\partial y}(x, u, u') - \frac{d}{dx}\left(\frac{\partial L}{\partial z}(x, u(x), u'(x))\right)\right] \varphi(x)\, dx$$

$$+ \cancel{\left[\frac{\partial L}{\partial z}(x, u(x), u'(x)) \cdot \varphi(x)\right]_a^b} \qquad \varphi(a) = \varphi(b) = 0$$

$F(0)$ minimo

$\Uparrow$

$\mathcal{L}(u)$ minimo

$$= \int_a^b g(x)\, \varphi(x)\, dx = 0$$

$$g(x) = \underbrace{\frac{\partial L}{\partial y}\left(x, u(x), u'(x)\right) - \frac{d}{dx} \frac{\partial L}{\partial z}\left(x, u(x), u'(x)\right)}$$

Lemma fondamentale del CdV

Sia $g \in C^0([a,b])$ Se

$$\int_a^b g(x)\varphi(x)\,dx = 0 \qquad \forall \varphi \in C_c^\infty((a,b))$$

$$\left[\begin{array}{l} C_c^\infty(\Omega) = \{\varphi \in C^\infty(\Omega) : \operatorname{spt}\varphi \subset\subset \Omega\} \\ \operatorname{spt}\varphi = \overline{\{x \in \Omega : \varphi(x) \neq 0\}} \end{array}\right]$$

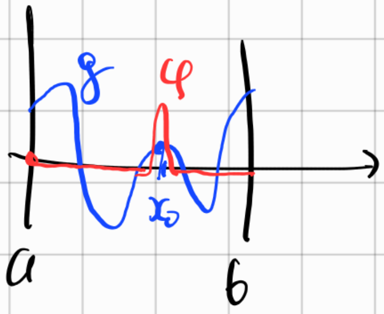
$$C^1_0([a,b]) \supseteq C_c^\infty((a,b))$$

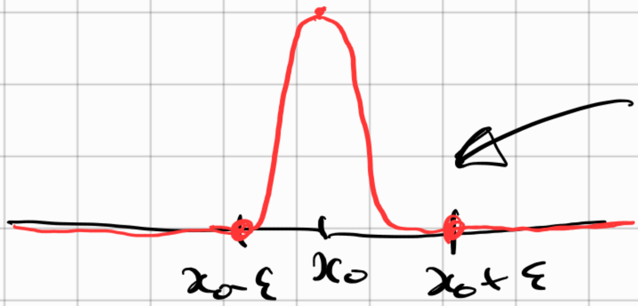
Allora $g \equiv 0$.

dim prendiamo $x_0 \in (a,b)$. Per assurdo supponiamo $g(x_0) \neq 0$. SPDG $g(x_0) > 0$.

$\exists \varepsilon > 0$ $\quad g(x) > \dfrac{g(x_0)}{2} > 0$

$\forall x \in [x_0-\varepsilon, x_0+\varepsilon] \subseteq (a,b)$





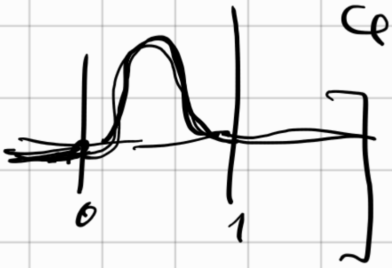
$\exists\, \varphi \in C_c^\infty(a,b)$ che mi fa questo lavoro:

$\operatorname{spt}\varphi \subseteq [x_0-\varepsilon, x_0+\varepsilon]$

$\varphi \geq 0$

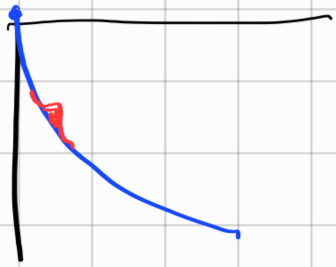
$\varphi(x_0) > 0$.

$$\left[\; u(x) = \begin{cases} e^{-\frac{1}{x(1-x)}} & x>0 \\ 0 & \text{se } x \leq 0 \\ & \text{se } x \geq 0 \end{cases} \;\right]$$



$$\int_a^b g(x)\cdot\varphi(x)\,dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} g(x)\varphi(x)\,dx \geq \frac{g(x_0)}{2}\int_{x_0-\varepsilon}^{x_0+\varepsilon}\varphi > 0$$

$\square$



$g(x)=0$ è l'Equazione di EULERO-LAGRANGE:

E-L
$$\boxed{\frac{\partial L}{\partial y}(x,u(x),u'(x)) = \frac{d}{dx}\frac{\partial L}{\partial z}(x,u(x),u'(x))}$$

$$\frac{\partial L}{\partial y}(x,u,u') = \frac{d}{dx}\frac{\partial L}{\partial z}(x,u,u')$$

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial z} \qquad \begin{array}{l} y=u(x) \\ z=u'(x) \end{array}$$

$$\left(\frac{\partial L}{\partial u} = \frac{d}{dx}\frac{\partial L}{\partial u'}\right)$$

Torniamo alla BRACHISTOCRONA:

$$L(x,y,z) = \frac{\sqrt{1+z^2}}{\sqrt{2gx}} \qquad \frac{\partial L}{\partial y}=0,\ \frac{\partial L}{\partial z} = \underbrace{\frac{z}{\sqrt{1+z^2}}\cdot\frac{1}{\sqrt{2gx}}}$$

E-L: $$0 = \frac{d}{dx}\,\frac{u'(x)}{\sqrt{1+(u'(x))^2}}\,\frac{1}{\sqrt{2gx}}$$

quindi: $$\frac{u'(x)}{\sqrt{1+(u'(x))^2}}\,\frac{1}{\sqrt{\cancel{2g}\,x}} = \frac{c}{\sqrt{\cancel{2g}}}$$

$$u' = c\sqrt{1+(u')^2}\,\sqrt{x} \qquad (u')^2 = c^2\cdot(1+(u')^2)\,x$$

$$(u')^2\cdot(1-c^2x) = c^2\cdot x$$

$$u' = \sqrt{\frac{c^2x}{1-c^2x}} \qquad u = \int\frac{c\sqrt{x}}{\sqrt{1-c^2x}}\,dx$$

$$u = \int\frac{c\,x}{\sqrt{x-c^2x^2}}\,dx = \frac{1}{c}\int\frac{c^2x}{\sqrt{x-c^2x^2}}\,dx =$$

$$\left[D\sqrt{x-c^2x^2} = \frac{1}{2}\,\frac{1-2c^2x}{\sqrt{x-c^2x^2}} \right]$$

$$= \frac{1}{c}\left[\int\frac{1}{2\sqrt{x-c^2x^2}}\,dx - \sqrt{x-c^2x^2}\right] =$$

completamento del quadrato

$$\frac{1}{\sqrt{1-s^2}}$$

$$\left[\sqrt{\underbrace{x}-\underbrace{c^2x^2}} = \sqrt{\frac{1}{4c^2}-\left(\frac{1}{2c}-cx\right)^2} = \frac{1}{2c}\sqrt{1-(1-2c^2x)^2}\right]$$

$$= \int\frac{1}{\sqrt{1-(1-2c^2x)^2}}\,dx - \frac{1}{c}\sqrt{x-c^2x^2} \qquad u(0)=0$$

$$\downarrow$$

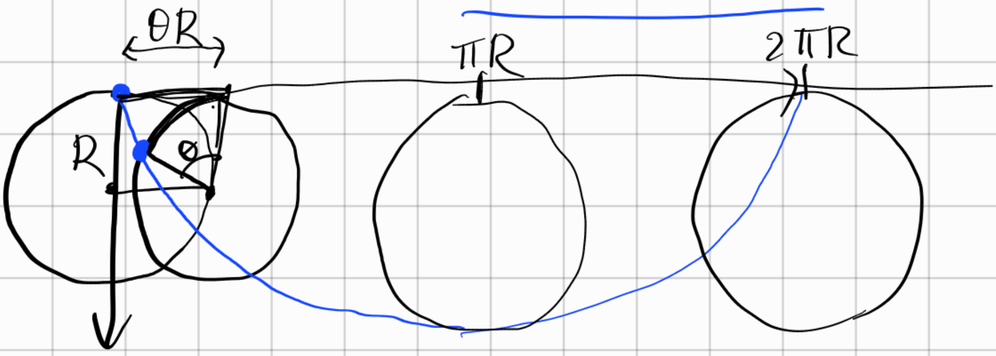
$$= \frac{\arcsin(1-2c^2x)}{-2c^2} - \frac{1}{c}\,\frac{1}{2c}\sqrt{1-(1-2c^2x)^2} + k$$

$$u(0) = -\frac{\arcsin 1}{2c^2} + k = 0 \qquad k = \frac{\pi}{4c^2}$$

$$u(x) = \frac{1}{2c^2}\left[\underbrace{-\arcsin(1-2c^2x) + \frac{\pi}{2}} - \sqrt{1-(1-2c^2x)^2}\right]$$

$$= \frac{1}{2c^2}\left[\arccos(1-2c^2x) - \sqrt{1-(1-2c^2x)^2}\right]$$

CICLOIDE



$$\begin{cases} x(\theta) = R - R\cos\theta \\ y(\theta) = \theta R - R\sin\theta \end{cases}$$

$y = u_R(x)$

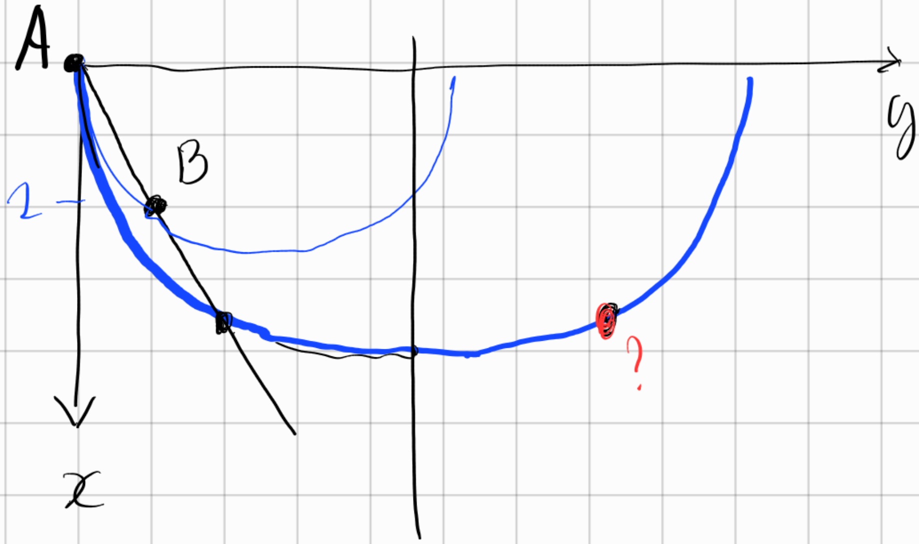
$$\begin{cases} \cos\theta = 1 - \frac{x}{R} \\ \% \end{cases} \qquad \theta = \arccos\left(1-\frac{x}{R}\right)$$

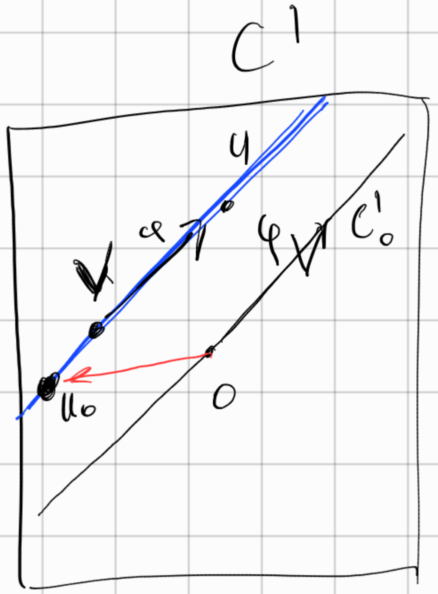
$$y = u_R(x) = R\left[\arccos\left(1-\frac{x}{R}\right) - \sqrt{1-\left(1-\frac{x}{R}\right)^2}\right]$$

$$u(x) = u_R(x) \qquad R = \frac{1}{2c^2}$$

$y = u(x)$ è una cicloide $\qquad u(x) = u_R(x)$

$$u_R(x) = R \cdot u_1\left(\frac{x}{R}\right)$$





$V = \{u \in C^1 : u(a) = y_a,\ u(b) = y_b\}$

$u, v \in V$

$\varphi = u - v$

$$C^1_0 = \begin{cases} \varphi \in C^1 \\ \varphi(a) = 0 \\ \varphi(b) = 0 \end{cases}$$

$\boxed{V = u_0 + C^1_0}$ $\quad u_0 \in V$ fissata

ogni $u \in V$ $\quad u = u_0 + (u - u_0)$ $\quad u - u_0 \in C^1_0$