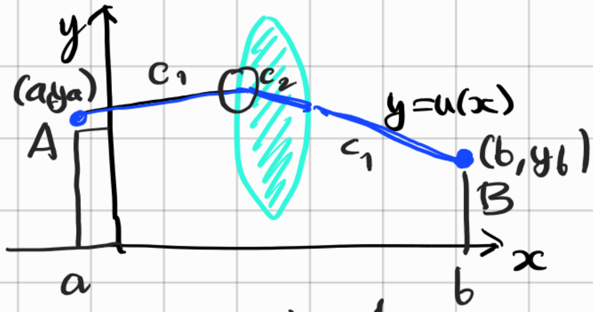
ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 2 - 1.10.2024

$$\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))\, dx \to \min$$

$$\text{E-L}: \quad \frac{\partial L}{\partial y}(x, u(x), u'(x)) = \frac{d}{dx} \frac{\partial L}{\partial z}(x, u(x), u'(x))$$

Esempio il problema di Fermat



$$\mathcal{L}(u) = \int_a^b g(x, u(x)) \cdot \sqrt{1 + (u'(x))^2}\, dx$$

$$g(x,y) = \frac{1}{c(x,y)}$$

$$\begin{cases} \mathcal{L}(u) \to \min \\ u(a) = y_a, \; u(b) = y_b \end{cases}$$

$$\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))\, dx \quad \text{con} \quad L(x,y,z) = g(x,y) \cdot \sqrt{1+z^2}$$

$$\frac{\partial L}{\partial y} = \frac{\partial g}{\partial y} \cdot \sqrt{1+z^2} \qquad \frac{\partial L}{\partial z} = g \cdot \frac{z}{\sqrt{1+z^2}}$$

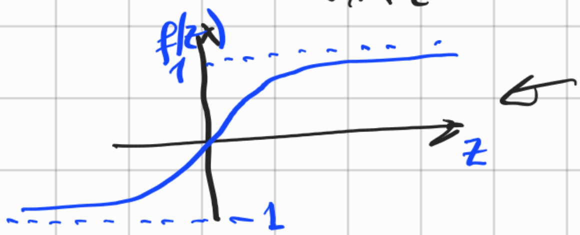
$$\text{E-L}: \quad \frac{\partial L}{\partial y} = \frac{d}{dx} \frac{\partial L}{\partial z}$$

$$\frac{\partial g}{\partial y}(x, u(x)) \cdot \sqrt{1 + (u'(x))^2} = \frac{d}{dx}\left(g(x, u(x)) \cdot \frac{u'(x)}{\sqrt{1 + (u'(x))^2}} \right)$$

① Dove g é costante:
$$0 = g(x, u(x)) \cdot \frac{d}{dx} \frac{u'(x)}{\sqrt{1 + (u'(x))^2}}$$

$$g \neq 0 \Rightarrow \underbrace{\frac{d}{dx} \frac{u'(x)}{\sqrt{1 + (u'(x))^2}}}_{} = 0 \quad \leftarrow \text{curvatura}$$

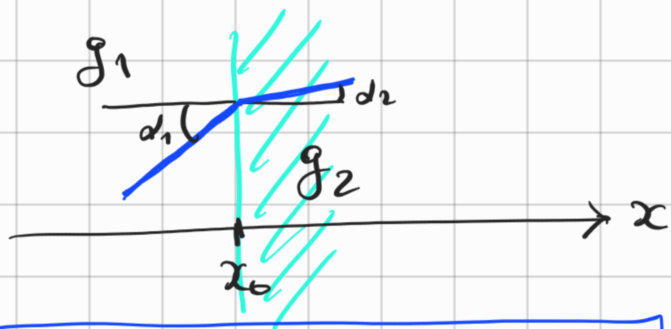
$$f(z) = \frac{u'(x)}{\sqrt{1+(u'(x))^2}} = \text{costante} \qquad f(z) = \frac{z}{\sqrt{1+z^2}}$$



$$\left[\; f'(z) = \frac{\sqrt{1+z^2} - z\frac{z}{\sqrt{1+z^2}}}{1+z^2} = \frac{1+z^2 - z^2}{\sqrt{1+z^2}\,(1+z^2)} > 0\right.$$

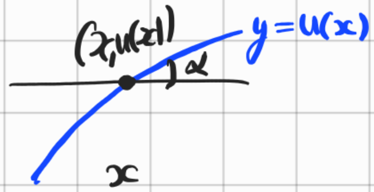
$$\Rightarrow \quad u'(x) = m \quad \text{costante} \qquad u(x) = mx + q.$$

② Sulla superficie della lente



E.L.

$$\boxed{\cancel{\frac{\partial g}{\partial y}} \cdot \sqrt{1+u'^2} = \frac{d}{dx}\left(g \cdot \frac{u'}{\sqrt{1+u'^2}}\right)}$$



$$g(x,y) = g(x)$$

↑

IPOTESI

$$\frac{u'}{\sqrt{1+u'^2}} = \sin\alpha_u(x)$$

$$u'(x) = \operatorname{tg}\alpha$$

$$\sqrt{1+u'^2} = \sqrt{1+\operatorname{tg}^2\alpha} = \frac{1}{\cos\alpha}$$

$$1+\operatorname{tg}^2\alpha = 1+\frac{\sin^2\alpha}{\cos^2\alpha} = \frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha}$$

$$\cos^2\alpha = \frac{1}{1+\operatorname{tg}^2\alpha}$$

$$\sin^2\alpha = \frac{1+\operatorname{tg}^2\alpha - 1}{1+\operatorname{tg}^2\alpha} = \frac{\operatorname{tg}^2\alpha}{1+\operatorname{tg}^2\alpha}$$

$$\sin\alpha = \frac{\operatorname{tg}\alpha}{\sqrt{1+\operatorname{tg}^2\alpha}}$$

$$g(x)\cdot\sin\alpha(x) = \text{costante}$$

$$g_1 \sin\alpha_1 = g_2 \sin\alpha_2$$

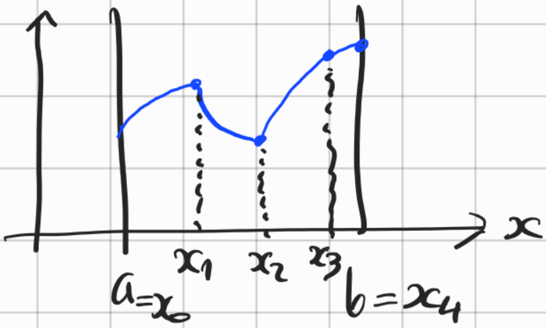
$$\boxed{\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{g_2}{g_1} = \frac{c_1}{c_2}}$$

LEGGE DI SNELL

$\alpha_1 =$ angolo di incidenza

$\alpha_2 =$ angolo di rifrazione.

<u>Def</u> u è C^1 a pezzi $u \in C^1_p([a,b])$ se



(1) $u \in C^0([a,b])$

(2) esistono $n-1$ punti:

$a = x_0 < x_1 < \dots < x_n = b$

$u \in C^1([x_k, x_{k+1}])$ $\quad k = 0, \dots, n-1$.

$$\mathcal{L}(u) = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} L(x, u(x), u'(x))\, dx$$

ha senso!

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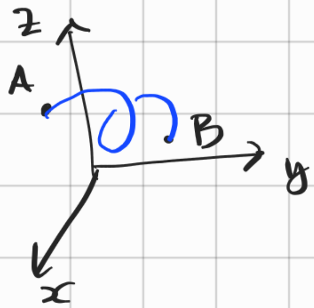
Sempre Fermat ma con le curve

$$\underline{u} : [a,b] \longrightarrow \mathbb{R}^n$$
$$x \longmapsto \underline{u}(x)$$



$$\mathcal{L}(\underline{u}) = \int_a^b g(\underline{u}(x)) \cdot \underbrace{|\underline{u}'(x)|\, dx}_{ds}$$

$$\begin{cases} \mathcal{L}(\underline{u}) \longrightarrow \min \\ \underline{u}(a) = A \\ \underline{u}(b) = B \end{cases}$$

$$L(x, \underline{y}, \underline{z}) = g(\underline{y}) \cdot |\underline{z}|$$

IN GENERALE

$$\boxed{\mathcal{L}(\underline{u}) = \int_a^b L(x, \underline{u}(x), \underline{u}'(x))\, dx}$$

$$L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(x, \underline{y}, \underline{z}) \longmapsto L(x, \underline{y}, \underline{z})$$

Se $\underline{u}$ è un minimo: $\forall \underline{\varphi} \in C_0^1([a,b], \mathbb{R}^n)$

$$0 = \frac{d}{d\varepsilon} \mathcal{L}(\underline{u} + \varepsilon \underline{\varphi}) \Big|_{\varepsilon = 0} = \int_a^b \frac{d}{d\varepsilon} L(x, \underbrace{\underline{u} + \varepsilon \underline{\varphi}}_{y_1 \dots y_n}, \underbrace{\underline{u}' + \varepsilon \underline{\varphi}'}_{z_1 \dots z_n}) \Big|_{\varepsilon=0} dx$$

$$= \int_a^b \Big[\sum_{k=1}^n \underbrace{\frac{\partial L}{\partial y_k}(x, \underline{u}, \underline{u}')}\, \varphi_k(x) + \sum_{k=1}^n \frac{\partial L}{\partial z_k}(x, \underline{u}, \underline{u}')\, \varphi_k'(x) \Big] dx$$

$$= \int_a^b \Big[\nabla_{\underline{y}} L(x, \underline{u}, \underline{u}') \cdot \underline{\varphi} + \nabla_{\underline{z}} L(x, u, u') \cdot \underline{\varphi}'(x) \Big] dx$$

$$= \int_a^b \Big[\nabla_{\underline{y}} L(x, u, u') - \frac{d}{dx} \nabla_{\underline{z}} L(x, u, u') \Big] \cdot \underline{\varphi}(x)\, dx$$

$$+ \cancel{\Big[\nabla_{\underline{z}} L(x, u, u')\, \underline{\varphi}(x) \Big]_a^b}$$

$$(\varphi(a) = 0, \quad \varphi(b) = 0)$$

Se prendo $\underline{\varphi} = \begin{pmatrix} 0 \\ \vdots \\ \varphi_k \\ \vdots \\ 0 \end{pmatrix}$ $\qquad 0 = \int \left[\nabla_y L - \frac{d}{dx} \nabla_z L \right]_k \cdot \varphi_k$

$\forall \varphi_k$

$$\Rightarrow \quad \nabla_y L - \frac{d}{dx} \nabla_z L = 0$$

$$\boxed{\text{E-L: VETTORIALE} \qquad \nabla_{\underline{y}} L(x, \underline{u}, \underline{u}') = \frac{d}{dx} \nabla_{\underline{z}} L(x, \underline{u}, \underline{u}')}$$

Torniamo a Fermat: $\qquad L(x, \underline{y}, \underline{z}) = g(\underline{y}) \cdot |\underline{z}|$

$$\nabla_{\underline{y}} L = \nabla g(\underline{y}) \cdot |\underline{z}| \qquad \nabla_{\underline{z}} L = g(\underline{y}) \cdot \frac{\underline{z}}{|\underline{z}|}$$

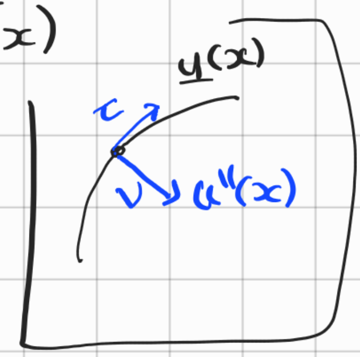
E-L: $\quad \nabla g(\underline{u}(x)) \cdot |u'(x)| = \frac{d}{dx} \left(g(\underline{u}(x)) \cdot \frac{\underline{u}'(x)}{|\underline{u}'(x)|} \right)$

$$= \left(\nabla g(\underline{u}) \cdot \underline{u}'(x) \right) \frac{\underline{u}'(x)}{|u'(x)|} + g(u) \cdot \frac{d}{dx} \frac{\underline{u}'(x)}{|\underline{u}'(x)|}$$

Supponiamo: $\quad |u'(x)| = 1$ $\quad$ (riparametrizzo per lunghezza d'arco)

$u'(x) = \tau$

$$\nabla g(\underline{u}) = (\nabla g \cdot \tau)\, \tau + g(u) \cdot \underline{u}''(x)$$



$|u'(x)|^2 = 1$

$$0 = \frac{d}{dx} (u'(x))^2 = 2\, u'(x) \cdot u''(x)$$

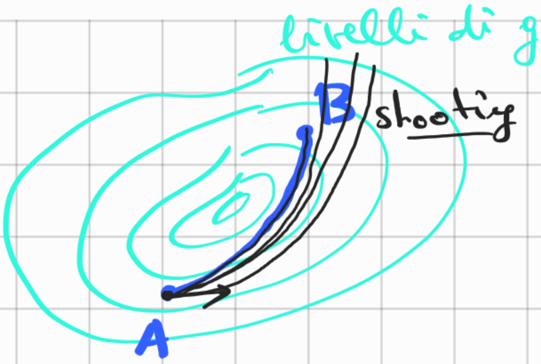
$$u''(x) \perp u'(x)$$

$$\underline{u}''(x) = k \cdot \underline{\nu}$$

$$\nabla g(\underline{u}) = (\nabla g \cdot \tau)\, \underline{\tau} + k\, g(u) \cdot \underline{\nu}$$

Prodotto scalare con $\underline{v}$:

$$\boxed{\frac{\partial g}{\partial \overline{v}} = k \cdot g(u)}$$



EQUAZIONE DI BELTRAMI

$$L(x,y,z) = L(y,z)$$

$$\mathcal{I}(u) = \int_a^b L(u(x), u'(x))\, dx$$

E-L :
$$\frac{\partial L}{\partial y}(u(x), u'(x)) = \frac{d}{dx} \frac{\partial L}{\partial z}(u(x), u'(x))$$

EQ. differenziale del secondo ordine, autonoma

non dipende da x

Moltiplico tutto per $u'(x)$

$$u' \cdot \frac{\partial L}{\partial y}(u,u') = u' \frac{d}{dx} \frac{\partial L}{\partial z}(u,u')$$

$$\frac{d}{dx} L(u,u') = \frac{\partial L}{\partial y} \cdot u' + \frac{\partial L}{\partial z} \cdot u''$$

$$= u' \frac{d}{dx} \frac{\partial L}{\partial z} + \frac{\partial L}{\partial z} u''$$

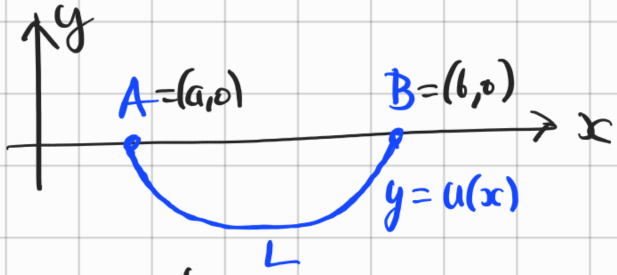
$$= \frac{d}{dx}\left(u' \cdot \frac{\partial L}{\partial z} \right)$$

$$\frac{d}{dx}\left(L - u' \frac{\partial L}{\partial z} \right) = 0$$

$$\boxed{L - u' \frac{\partial L}{\partial z} = \text{cost.}}$$

EQ. di BELTRAMI.

CATENARIA



La catena minimizza l'energia potenziale:

$$\mathcal{I}(u) = \int_a^b u(x)\sqrt{1+(u'(x))^2}\,dx$$

$u: [a,b] \to \mathbb{R}$

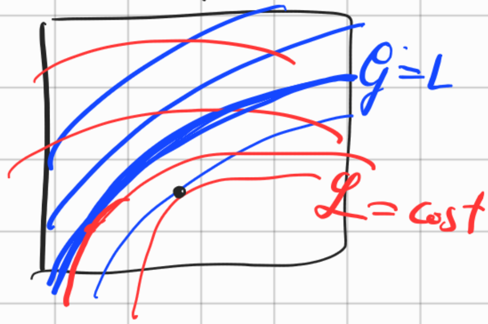
$u(a) = 0$

$u(b) = 0$

è un problema vincolato:

$$\mathcal{G}(u) = \int_a^b \sqrt{1+(u'(x))^2}\,dx$$

$$\begin{cases} \mathcal{I}(u) \longrightarrow \min \\ \mathcal{G}(u) = L \\ u(a) = u(b) = 0 \end{cases}$$



Moltiplicatori di Lagrange: "$\nabla \mathcal{L} = \lambda \nabla \mathcal{G}$"

CONTINUA NELLA PROSSIMA LEZIONE