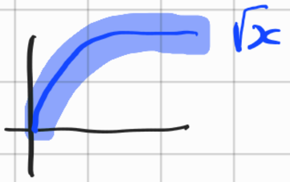
ANALISI MATEMATICA B
LEZIONE 13 – 16.10.2024

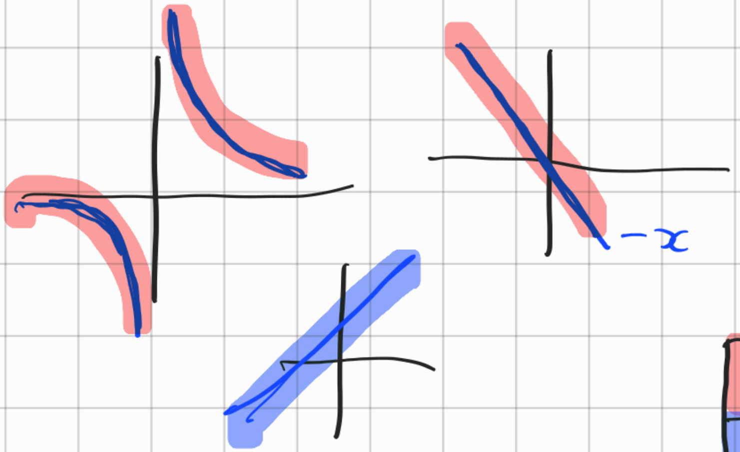
Esercizio test settimanale:

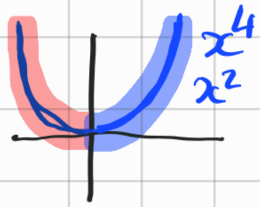
$$f(x) = \left(1 - \sqrt{\sqrt{x} + (x^2+1)^4}\right)^4 - \frac{1}{\sqrt{x+1}}$$

$x \geq 0$

f è strettamente crescente





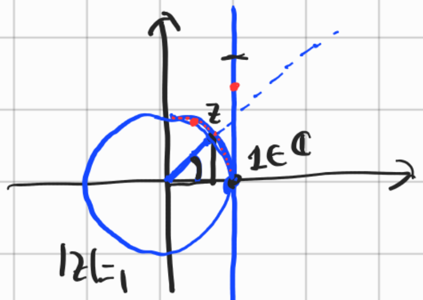


Numeri complessi

(Es $\tau = 360$)

Fissato $\tau > 0$ (misura dell'angolo giro)

$\exists!\ \varphi : \mathbb{R} \to U = \{ z \in \mathbb{C} : |z| = 1 \}$



$\varphi(x+y) = \varphi(x) \cdot \varphi(y) \; (\Rightarrow \varphi(0) = 1)$

$\varphi(\tau) = 1$ e $\varphi(t) \neq 1$ se $0 < t < \tau$

$\operatorname{Im} \varphi(t)$ crescente per $\underbrace{0 < t < \frac{\tau}{4}}$

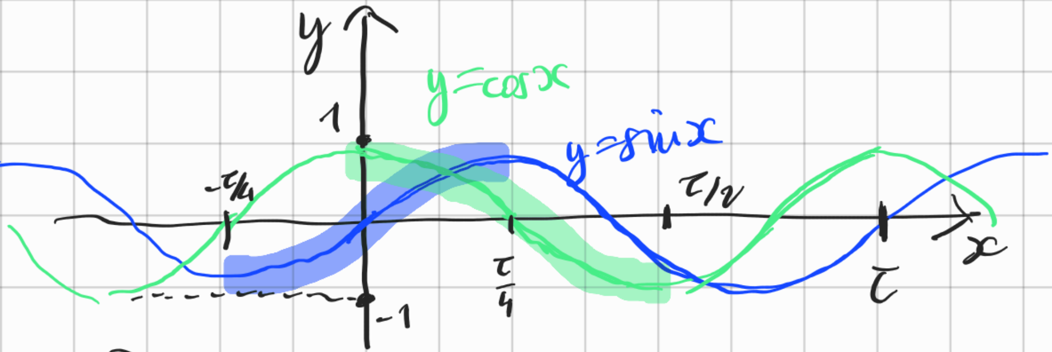
$\uparrow$

$z \in U \qquad |z| = 1$

$$\begin{cases} \cos \operatorname{Arg} z = \operatorname{Re} z \\ \sin \operatorname{Arg} z = \operatorname{Im} z \end{cases} \qquad \cos^2 + \sin^2 = 1$$

$\sin_\tau : \mathbb{R} \to \mathbb{R}$

$$\begin{cases} \sin_\tau(x) = \operatorname{Im} \varphi(x) \\ \cos_\tau(x) = \operatorname{Re} \varphi(x) \end{cases}$$



Proprietà delle funzioni trigonometriche:

(1) sin, cos sono τ-periodiche

(f è τ-periodica se $f(x+\tau)=f(x)$)

(e τ è il minimo periodo)

(2) $\sin^2 x + \cos^2 x = 1$

(3) $\sin(x+y) = \sin x \cos y + \sin y \cos x$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

(4) sin è dispari, cos è pari

($\sin(-x) = -\sin x$) ($\cos(-x) = \cos x$)

(5) $\sin : [-\frac{\tau}{4}, \frac{\tau}{4}] \to [-1,1]$ è strettamente crescente e bigettivo (vedi lemma di estensione monotona)

$\cos : [0, \frac{\tau}{2}] \to [-1,1]$ è strettamente decrescente e bigettivo

(6) $-1 \le \sin x \le 1$, $-1 \le \cos x \le 1$

dim (1) $\varphi(\tau) = 1$ φ è τ-periodica $\Rightarrow$ Re, Im sono τ-periodiche.

(2) $\varphi(x) \in U$ $|\varphi(x)| = 1 \Rightarrow \sqrt{Re^2 + Im^2} = 1$.

(3) $\varphi(x+y) = \varphi(x)\cdot\varphi(y)$

$$\cos(x+y) + i\sin(x+y) = (\cos x + i \operatorname{sen} x)\cdot(\cos y + i \operatorname{sen} y)$$

$$= \cos x \cos y - \operatorname{sen} x \operatorname{sen} y + i(\operatorname{sen} x \cos y + \cos x \operatorname{sen} y).$$

(4) $1 = \varphi(x-x) = \varphi(x)\cdot\varphi(-x)$

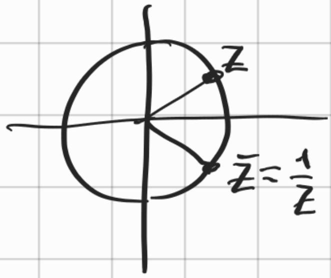
$z \in U \quad |z| = 1$

$\Downarrow$

$$\frac{1}{z} = \frac{\bar z}{z\cdot\bar z} = \frac{\bar z}{|z|^2} = \bar z$$

$$\varphi(-x) = \frac{1}{\varphi(x)} = \overline{\varphi(x)}$$

$$\cos(-x) + i\operatorname{sen}(-x) = \cos(x) - i\operatorname{sen}(x)$$



(5) $\sin : [0, \frac{\tau}{4}] \to \mathbb{R}$ è strett. crescente per costruzione.

$\sin : [-\frac{\tau}{4}, 0] \to \mathbb{R}$ $\qquad \sin(-x) = -\sin x$

$\cos : [0, \frac{\tau}{4}] \to \mathbb{R}$ $\qquad \cos x = +\sqrt{1-\sin^2 x}$ decres.

$$\boxed{\left(\begin{cases}\cos x \geq 0 \\ \sin x \geq 0\end{cases} \quad \text{se } x \in [0, \tfrac{\tau}{4}]\right) \quad (*)}$$

$\cos : [\frac{\tau}{4}, \frac{\tau}{2}] \to \mathbb{R}$

$$\cos\left(\frac{\tau}{2} - x\right) = -\cos x$$

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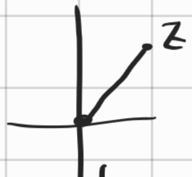
(6) $\varphi(x) \in U$ $\qquad |\varphi(x)| \leq 1$

$\Downarrow$

$|\cos x| \leq 1$

$|\sin x| \leq 1.$



$$\begin{cases}|\operatorname{Re} z| \leq |z| \\ |\operatorname{Im} z| \leq |z|\end{cases}$$

④ $\varphi(0)=1 \Rightarrow \sin(0)=0 \quad \cos(0)=1$

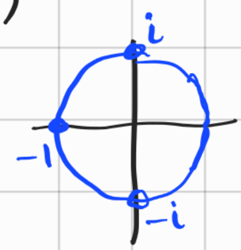
$$\varphi(\tau)=1 \qquad \varphi\left(\frac{\tau}{2}\right) \overset{?}{=} z$$

$$\varphi(\tau) = \varphi\left(\frac{\tau}{2}+\frac{\tau}{2}\right) = \varphi\left(\frac{\tau}{2}\right)\cdot\varphi\left(\frac{\tau}{2}\right) = \left(\varphi\left(\frac{\tau}{2}\right)\right)^2$$

ES

$$\boxed{z^2=1 \;\Rightarrow\; z=1,\; z=-1}$$

$$\underset{180^\circ}{\varphi\left(\frac{\tau}{2}\right)} = \begin{cases} \cancel{1} \leftarrow \text{NO } \left(\frac{\tau}{2}<\tau\right) \\ -1 \end{cases}$$



$$180^\circ \begin{cases} \cos\frac{\tau}{2} = -1 \\ \sin\frac{\tau}{2} = 0 \end{cases}$$

$$\underset{90^\circ}{\varphi\left(\frac{\tau}{4}\right)} = w \qquad \boxed{w^2=-1} \;\; \begin{cases} w=i \\ \cancel{w=-i} \Rightarrow \sin\frac{\tau}{4} = -1 \end{cases}$$

ma sin è crescente.

$$90^\circ \begin{cases} \cos\frac{\tau}{4} = 0 \\ \sin\frac{\tau}{4} = 1. \end{cases}$$

monotonia

$$0 \le \sin t \le 1 \quad \text{per } 0 \le t \le \frac{\tau}{4}.$$

$$-1 \le \sin(-t) \le 0, \qquad \sin(-t) = -\sin(t)$$

Funzioni inverse

$$\sin : \left[-\frac{\tau}{4}, \frac{\tau}{4}\right] \to [-1,1] \text{ è bigettiva}$$

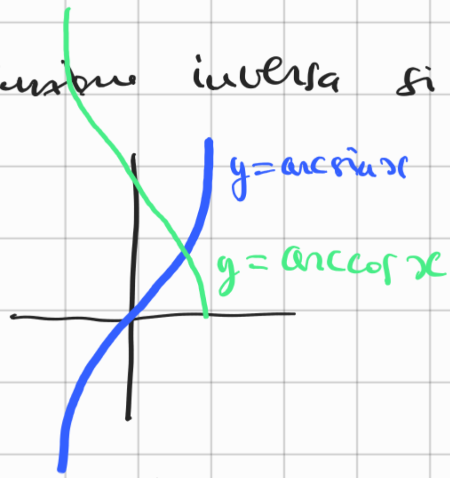
$-90^\circ \quad 90^\circ$ — st. cresc.

la funzione inversa si chiama $\arcsin : [-1,1] \to \left[-\frac{\tau}{4}, \frac{\tau}{4}\right]$

bigettiva str. cresc.

$$\left(x<y \Leftrightarrow f(x)<f(y)\right)$$

$$\left(f^{-1}(x)<f^{-1}(y) \Leftrightarrow x<y\right)$$



analogamente $\arccos : [-1,1] \to \left[0, \frac{\tau}{2}\right]$

Esercizio se $\forall x,y$ $\boxed{x<y \overset{(*)}{\Rightarrow} f(x)<f(y)}$

allora $\forall x,y$ $f(x)<f(y) \Rightarrow \underline{x<y}$

dim per assurdo $\exists\, x,y$ $f(x)<f(y)$ e $x \geq y$

se $x=y \Rightarrow f(x)=f(y) \Rightarrow$ NO

$\Rightarrow x>y \overset{*}{\Rightarrow} f(x)>f(y)$ No assurdo

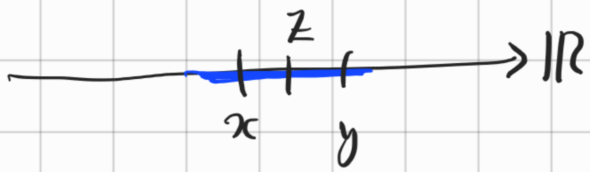
$\square$

[Cose che mi sono dimenticato di dirvi prima]

1. INTERVALLI

Def $I \subseteq \overline{\mathbb{R}}$ si dice essere un intervallo

se presi $x,y \in I$ e $z \in \mathbb{R}$ con



$x<z<y$ allora $z \in I$.

Teorema Se $I \subseteq \overline{\mathbb{R}}$ è un intervallo

$b=\sup I$, $a=\inf I$ $a,b \in \overline{\mathbb{R}}$

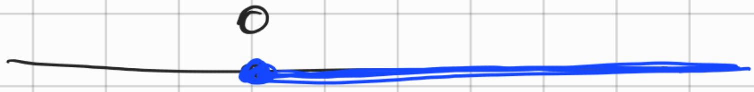
allora I è uno dei seguenti:

- $\{x \in \mathbb{R} : a<x<b\} = (a,b) =]a,b[$ intervallo aperto

- $\{x \in \mathbb{R} : a \leq x \leq b\} = [a, b]$ intervallo chiuso
- $\{x \in \mathbb{R} : a \leq x < b\} = [a, b) = [a,b[$ semichiuso
- $\{x \in \mathbb{R} : a < x \leq b\} = (a, b] =]a, b]$

ES

$[0, +\infty)$



$[-1, 1)$

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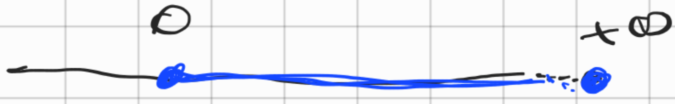
$[0, +\infty] \subseteq \overline{\mathbb{R}}$



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2. Cardinalità

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

$$\#\mathbb{N} \overset{①}{=} \#\mathbb{Z} \overset{②}{=} \#\mathbb{Q} < \#\mathbb{R} = \#\mathbb{C}.$$

I $\quad$ II

① $\mathbb{N} \leftrightarrow \mathbb{Z}$

$$n \mapsto \begin{cases} n/2 & \text{se } n \text{ pari} \\ -\frac{n+1}{2} & \text{se } n \text{ dispari} \end{cases}$$

② $\mathbb{Q} = \left\{\frac{p}{q}\right\}/_{\sim} \simeq \mathbb{Z} \times \mathbb{N}/_{\sim}$

Primo metodo diagonale di Cantor

$$\#(\mathbb{Z} \times \mathbb{N}) \underset{①}{=} \#(\mathbb{N} \times \mathbb{N}) = \#\mathbb{N}$$

	0	1	2	3	4	…
0	(0,0) 0	(0,1) 2	(0,2) 5	9	…	
1	(1,0) 1	(1,1) 4	(1,2) 8			
2	3	7				
3	6					
4						

$$\Rightarrow \quad \# \mathbb{Q} \leq \#(\mathbb{N} \times \mathbb{N}) = \# \mathbb{N}$$

$$\# \mathbb{N} \leq \# \mathbb{Q}$$

Teorema di Cantor-Bernstein $\Downarrow$ $\# \mathbb{Q} = \# \mathbb{N}$

(3) II metodo diagonale di Cantor $\qquad \# \mathbb{Q} \overset{?}{<} \# \mathbb{R}$

Supponiamo per assurdo che $\# \mathbb{R} \leq \# \mathbb{N}$

$$\# [0,1) \leq \# \mathbb{R} \leq \# \mathbb{N}$$

$\mathbb{N}$	$[0,1)$
0	0,12357524....
1	0,500000 ...
2	0,7500000 ...
3	0,000000 00 ...
4	0,7771423...
	0,15349998...

$= \sup\{0, 0.1, 0.12, 0.123, \dots\}$, $\quad 0.1 = \frac{1}{10}$, $\quad 0.12 = \frac{12}{100}$, $\quad 0.123 = \frac{123}{1000}$

0,177773... non sta nell'elenco