ANALISI MATEMATICA B
LEZIONE 14 - 18.10.2024

Esercizi (NOVAGA 17/10/2024)

Risolvere
1. $z^2 - 4z + 4 - i = 0$
2. $|z|^2 + iz + iz^3 - i\bar{z} = 0$
3. $\mathrm{Re}\,\frac{1}{z} = 1$

$\sqrt{i} = ?$ (**)

ESERCIZIO 2

$$|z|^2 + iz + iz^3 - i\bar{z} = 0 \qquad \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

$$|z|^2 + iz + iz^3 - i\frac{|z|^2}{z} = 0$$

$$|z|^2 + i\left(z + z^3 - \frac{|z|^2}{z}\right) = 0$$

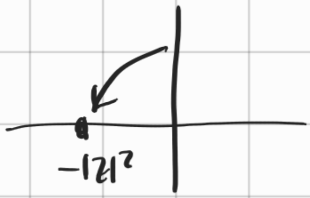
$$\begin{cases} |z|^2 = 0 \\ z + z^3 - \frac{|z|^2}{z} = 0 \end{cases} ?$$

$x + iy = 0 \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

⚠ se $x \in \mathbb{R}$ e $y \in \mathbb{R}$!

$$i\left(z + z^3 - \frac{|z|^2}{z}\right) = -|z|^2$$

$$z + z^3 - \frac{|z|^2}{z} = -\frac{|z|^2}{i}$$



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Altra idea: ⚠ $z^2 \neq |z|^2$

$$|z|^2 + iz + iz^3 - i\bar{z} = 0 \qquad z^2+1 = (z+i)(z-i)$$

$$z \cdot \bar{z} + iz + iz^3 - i\bar{z} = 0$$

$$\bar{z}(z-i) + iz(z^2+1) = 0$$

$$\bar{z}(z-i) + iz(z-i)(z+i) = 0$$

$$(z-i)(\bar{z} + iz(z+i)) = 0 \qquad \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

$$z - i = 0 \qquad z = i$$

$$\bar{z} + iz(z+i) = 0 \Rightarrow iz^2 - z + \bar{z} = 0$$

SOLUZIONE FURBA

$$\bar{z} - z = -2i\,\mathrm{Im}\,z \qquad \begin{cases} \mathrm{Re}\,z = \frac{z+\bar{z}}{2} \\ \mathrm{Im}\,z = \frac{z-\bar{z}}{2i} \end{cases}$$

$$iz^2 - 2i\,\mathrm{Im}\,z = 0 \qquad z^2 = 2\,\mathrm{Im}\,z$$

$z^2 \in \mathbb{R}$

$z^2 \geq 0 \quad z \in \mathbb{R}$

$z = x \in \mathbb{R}$

$z = iy \in i\mathbb{R}$

....

$$z = x + iy$$

$$i(x+iy)^2 - x - iy + x - iy = 0$$

$$i(x^2 + 2ixy - y^2) - 2iy = 0$$

$$ix^2 - 2xy - iy^2 - 2iy = 0$$

$$-2xy + i(x^2 - y^2 - 2y) = 0$$

$$\begin{cases} -2xy = 0 \\ x^2 - y^2 - 2y = 0 \end{cases}$$

$-2xy = 0$: $x = 0$, $y = 0$

$x^2 - y^2 - 2y = 0$: $-y^2 - 2y = 0$, $x^2 = 0 \Rightarrow x = 0$

$$-y(y+2) = 0 \qquad y = 0, \quad y = -2$$

$z = 0 + i\,0 = 0$

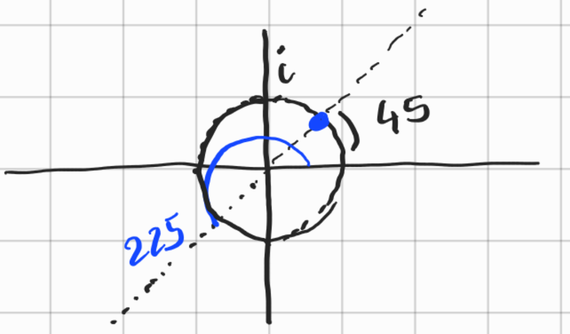
$z = 0 - i2 = -2i$

$z = i$

(**)

$\sqrt{i} = ?$

$z^2 = i$



$z = x + ix$

$y = x$

$225 \cdot 2 = 450 = 360 + 90$

$z^2 = i$

$|z^2| = |i| = 1$

$\quad \| $

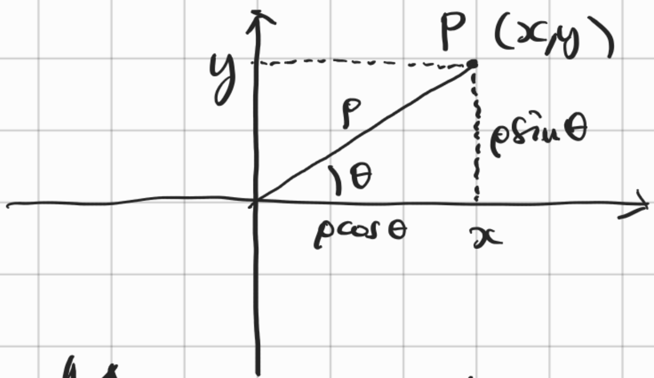
$|z|^2$

$|z| = \sqrt{x^2 + x^2} = \sqrt{2}|x| = 1 \qquad |x| = \frac{1}{\sqrt{2}}$

$$z = \begin{cases} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \end{cases}$$

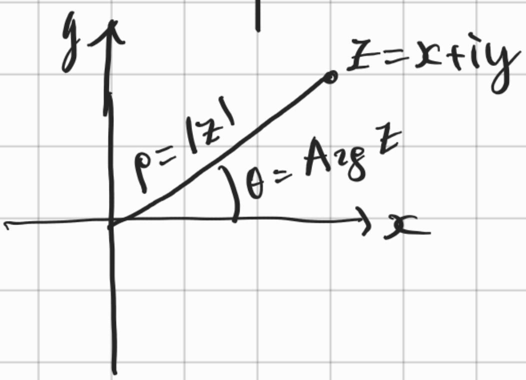
$$\left[\begin{array}{ll} \cos 45° = \frac{\sqrt{2}}{2} & \sin 45° = \frac{\sqrt{2}}{2} \\ \cos 225° = -\frac{\sqrt{2}}{2} & \sin 225° = -\frac{\sqrt{2}}{2} \end{array}\right]$$

RAPPRESENTAZIONE POLARE DEI COMPLESSI



(x, y) = coordinate cartesiane

$(\rho; \theta)$ = coordinate polari



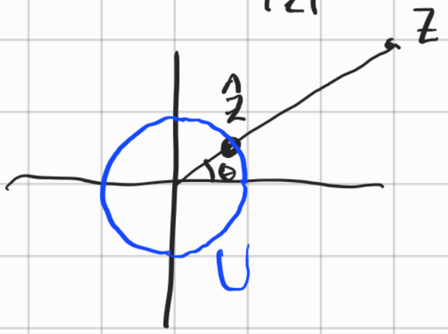
$\rho = \sqrt{x^2 + y^2}$

θ = "angolo identificato da P con l'asse positivo delle x"

$$z = |z| \cdot \frac{z}{|z|} = |z| \cdot \hat{z}$$

$$\hat{z} = \frac{z}{|z|} \qquad |\hat{z}| = 1$$

$\uparrow_e$ in $U = \{w : |w| = 1\}$



$$\hat{z} = \varphi(\theta) \qquad (= \varphi(\theta + \tau))$$

$\tau = 360°$

$\varphi : \mathbb{R} \to U$

$\varphi(x+y) = \varphi(x) \cdot \varphi(y)$

$$z = \rho \cdot \varphi(\theta) \qquad \begin{cases} \rho = |z| \\ \theta \in \varphi^{-1}\left(\frac{z}{|z|}\right) \end{cases} \quad \text{ovvero: } \varphi(\theta) = \frac{z}{|z|}$$

$= \mathrm{Arg}\, z$

$$z = \rho \cdot \varphi(\theta) \qquad z \cdot w = \rho \cdot R \cdot \varphi(\theta) \cdot \varphi(\alpha)$$

$$w = R \cdot \varphi(\alpha) \qquad = \rho \cdot R\, \varphi(\theta + \alpha)$$

$$\boxed{\varphi(\theta) = \cos\theta + i \operatorname{sen}\theta}$$

$$|z \cdot w| = \rho \cdot R = |z| \cdot |w|$$

$$\mathrm{Arg}(z \cdot w) = \theta + \alpha = \mathrm{Arg}\, z + \mathrm{Arg}\, w$$

$$\boxed{z = \rho \cdot (\cos\theta + i \operatorname{sen}\theta)} \longrightarrow z = x + iy \qquad \boxed{\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}}$$

RADICI n-ESIME in $\mathbb{C}$.

Dato $c \in \mathbb{C}$ ($c \neq 0$), $n \in \mathbb{N}$ risolvere $\boxed{z^n = c}$

"$z = \sqrt[n]{c}$"

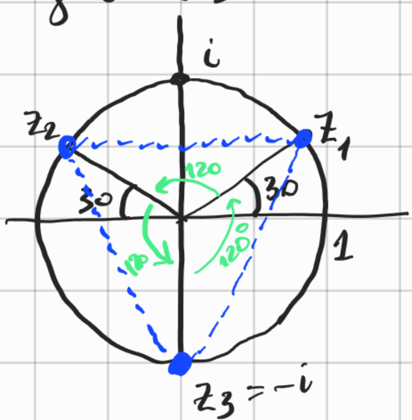
Esempio $z^2 = i$ già visto: $z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ oppure $z = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

Esempio $z^3 = i$

$\Downarrow$

$|z|^3 = 1$

$|z| = 1$



$$\theta_1 = \frac{90°}{3} = 30°$$

$$\theta_2 = \frac{90 + 360}{3} = 150°$$

$$\theta_3 = \frac{90 + 720}{3} = 270°$$

[Verifica: $(-i)^3 = (-1)^3 i^3 = -i\,i^2 = i$]

$\theta_4 = \frac{90+3\cdot360}{3} = 30+360 \simeq \theta_1$

$\varphi(\theta_4) = \varphi(\theta_1)$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

1 3 3 1

1 1
1 2 1
1 3 3 1

Soluzione algebrica

$z = x + iy \qquad z^3 = i$

$(x+iy)^3 = x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$

$= x^3 + 3i x^2 y - 3xy^2 - iy^3$

$= \underbrace{x^3 - 3xy^2} + i\underbrace{(3x^2y - y^3)} \overset{!}{=} i$

$$\begin{cases} x^3 - 3xy^2 = 0 = \mathrm{Re}\, i \\ 3x^2y - y^3 = 1 = \mathrm{Im}\, i \end{cases}$$

$$\begin{cases} x(x^2 - 3y^2) = 0 \\ \text{〃} \end{cases}$$

(stessa cosa che c'era prima)

$\begin{cases} x = 0 \\ -y^3 = 1 \end{cases} \qquad \begin{cases} x = 0 \\ y = -1 \end{cases} \qquad z = -i$

$x^2 = 3y^2$

$|x| = \sqrt{3}|y|$

$\begin{cases} x = \pm\sqrt{3}|y| \\ 3(\sqrt{3}y)^2 y - y^3 = 1 \end{cases} \to 8y^3 = 1$

$y^3 = \frac{1}{8}$

$y = \frac{1}{2}$

$\begin{cases} y = \frac{1}{2} \\ x = \pm\frac{\sqrt{3}}{2} \end{cases}$

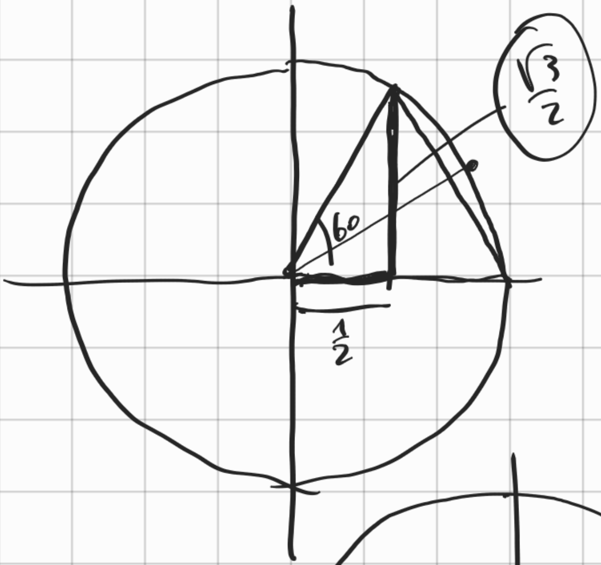
$$\begin{cases} z_1 = \frac{\sqrt{3}}{2} + \frac{i}{2} \\ z_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2} \\ z_3 = -i \end{cases}$$

$$\left[\begin{array}{ll} \cos 30° = \frac{\sqrt{3}}{2} & \sin 30° = \frac{1}{2} \\ \cos 150° = -\frac{\sqrt{3}}{2} & \sin 150° = \frac{1}{2} \\ \cos 270 = 0 & \sin 270 = -1 \end{array}\right]$$

DA SAPERE:

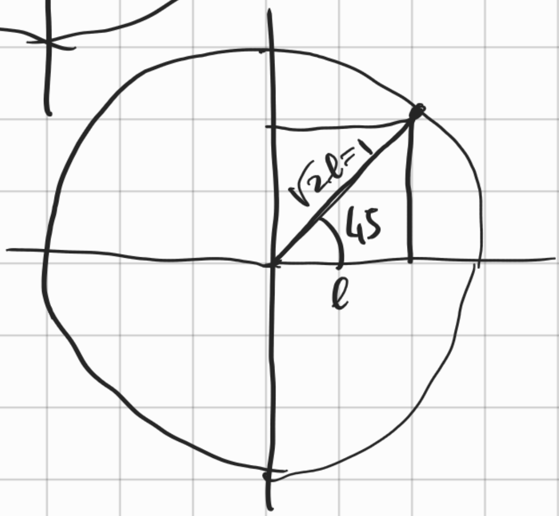
$\sin 30° = \frac{1}{2}$ $\quad$ $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60° = \frac{1}{2}$



$$\begin{cases} \cos(90-\alpha) = \sin\alpha \\ \sin(90-\alpha) = \cos\alpha \end{cases}$$

$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad$ $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$



Torniamo a $z^n = c$ $\quad$ $c \neq 0$, $R > 0$

$z = \rho\,\varphi(\theta) = \rho(\cos\theta + i\,\text{sen}\,\theta)$

$c = R\,\varphi(\alpha) = R(\cos\alpha + i\,\text{sen}\,\alpha)$

$\rho = |z| \quad \theta = \text{Arg}\, z$

$R = |c| \quad \alpha = \text{Arg}\, c$

$$(\rho\cdot\varphi(\theta))^n = R\cdot\varphi(\alpha)$$

$$\rho^n\varphi(n\theta) = R\cdot\varphi(\alpha)$$

$$\begin{cases} \rho^n = R \\ n\theta = \alpha + k\tau \end{cases} \quad k\in\mathbb{Z}, \quad \tau = 360°$$

φ è τ periodica

$$\begin{cases} \rho = \sqrt[n]{R} & \text{in } \mathbb{R} \\ \theta = \frac{\alpha}{n} + k\frac{\tau}{n} & k\in\mathbb{Z} \end{cases}$$

$$\begin{cases} \theta_0 = \frac{\alpha}{n} & k=0 \\ \theta_1 = \frac{\alpha}{n} + \frac{\tau}{n} & k=1 \\ \vdots \end{cases}$$

$$z_k = \sqrt[n]{R}\left(\cos\theta_k + i \operatorname{sen}\theta_k\right)$$

$$\left| \begin{array}{l} \theta_k = \frac{\alpha}{n} + k \cdot \frac{\tau}{n} \\ \vdots \\ \theta_{n-1} = \frac{\alpha}{n} + (n-1)\frac{\tau}{n} \end{array} \right.$$

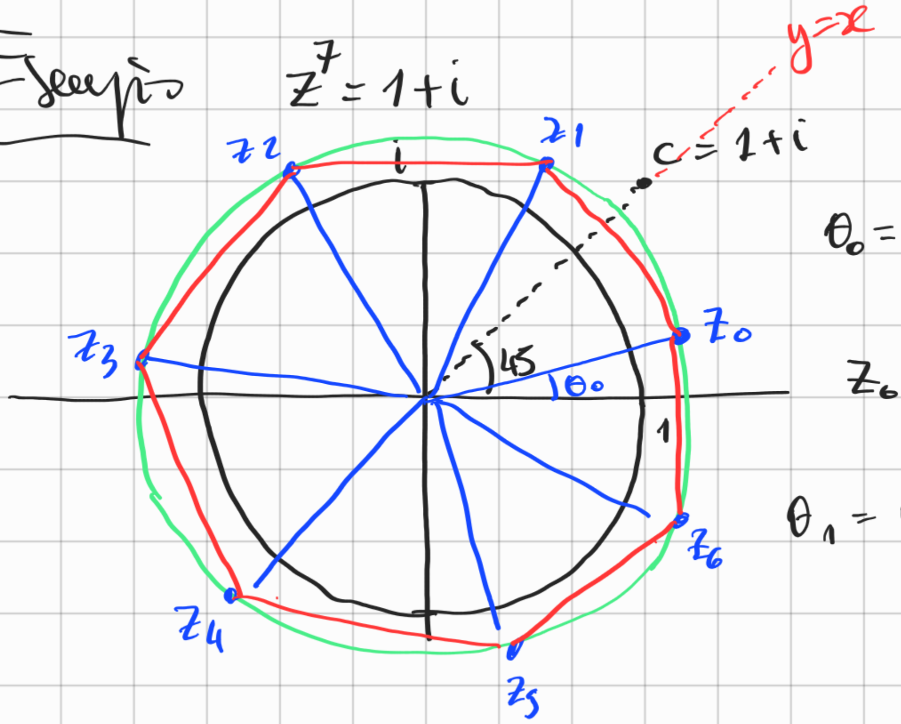
$k = 0, \dots, n-1$

sono distinti

$(z_{k+n} = z_k)$

$$\left[\begin{array}{l} \theta_n = \frac{\alpha}{n} + \tau \Rightarrow \varphi(\theta_n) = \varphi(\theta_0) \\ \theta_{n+1} = \frac{\alpha}{n} + \tau + \frac{\tau}{n} \Rightarrow \varphi(\theta_{n+1}) = \varphi(\theta_1) \\ \vdots \end{array} \right]$$

Esempio $\quad z^7 = 1+i$



$|z| = \sqrt[7]{|c|} = \sqrt[7]{\sqrt{2}} = \sqrt[14]{2}$

$\alpha = 45°$

$\theta_0 = \frac{45°}{7}$

$z_0 = \sqrt[14]{2} \cdot \left(\cos\frac{45°}{7} + i \operatorname{sen}\frac{45°}{7}\right)$

$\theta_1 = \theta_0 + \frac{360°}{7}$