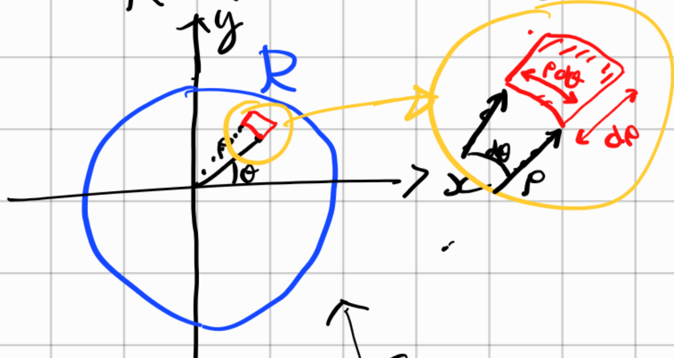
ANALISI MATEMATICA 2
LEZIONE 3~~7~~ - 4.3.2026

CAMBIO di VARIABILE negli INTEGRALI MULTIPLI

Esempio

$B_R = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le R^2\}$

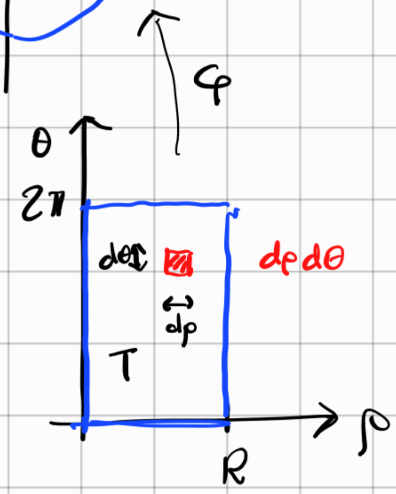


$$\iint_{B_R} |xy| \, dx\, dy$$

vorrei cambiare variabile

coordinate polari

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \qquad dx\,dy \overset{(?)}{=} \rho \, d\rho \, d\theta$$



$$\begin{pmatrix} x \\ y \end{pmatrix} = \varphi(\rho, \theta)$$

$$\iint_{B_R} |xy| \, dx\, dy = \iint_T |\rho\cos\theta \, \rho \sin\theta| \, \rho \, d\rho \, d\theta$$

$B_R \leftarrow$ cerchio T rettangolo

$B_R = \varphi(T)$

$$= \int_0^{2\pi} \int_0^R \rho^2 \, |\sin\theta \cdot \cos\theta| \, \rho \, d\rho \, d\theta$$

$$= \int_0^{2\pi} |\sin\theta \cos\theta| \, d\theta \cdot \int_0^R \rho^3 \, d\rho$$

$$= 4 \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \cdot \int_0^R \rho^3 \, d\rho$$

$$= 4 \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \cdot \left[\frac{\rho^4}{4}\right]_0^R = 4 \, \frac{1}{2} \cdot \frac{R^4}{4} = \frac{1}{2} R^4.$$

In generale

$$\underline{\varphi} : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^n \qquad D\varphi(\underline{x}) \in \mathbb{R}^{n\times n}$$

$$\underline{x} \qquad\qquad \underline{y}$$

$$\begin{cases} \underline{y} = \underline{\varphi}(\underline{x}) \\ d\underline{y} = |\det D\varphi(\underline{x})|\, d\underline{x} \end{cases}$$

$$\boxed{\int_B f(\underline{y})\, d\underline{y} = \int_A f(\underline{\varphi}(\underline{x})) \cdot |\det D\varphi(\underline{x})|\, d\underline{x}}$$

Verifica dell'esempio precedente: $(x,y) = \varphi(\rho,\theta)$

$$\varphi(\rho,\theta) = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad |\det D\varphi(\rho,\theta)| = ?$$

matrice Jacobiana

$$D\varphi(\rho,\theta) = \frac{\partial x,y}{\partial \rho,\theta} = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

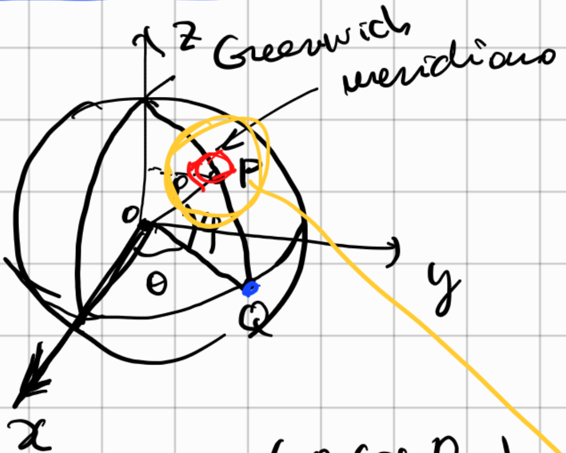
$$= \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix}$$

$$\det(D\varphi(\rho,\theta)) = \rho\cos^2\theta + \rho\sin^2\theta = \rho$$

determinante Jacobiano

$$dx\,dy = |\det D\varphi|\, d\rho\, d\theta = \rho\, d\rho\, d\theta \qquad \text{ok!}$$

ESEMPIO 2: coordinate sferiche



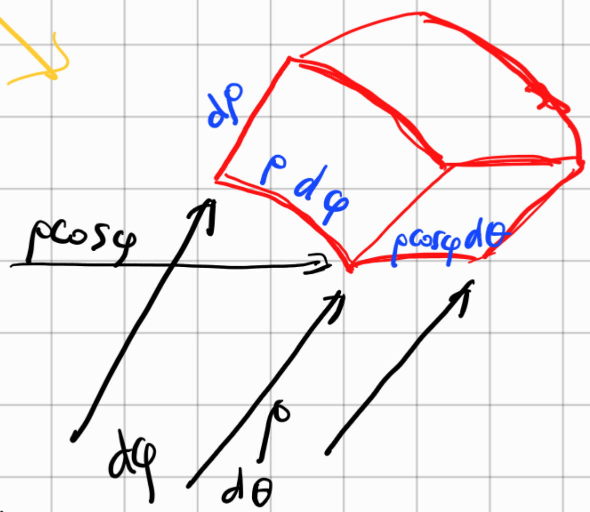
$P = (x, y, z) \in \mathbb{R}^3$

$\rho = |P| = $ distanza da O

$\theta = $ longitudine $\in [-\pi, \pi]$

$\varphi = $ latitudine $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$Q = \begin{pmatrix} \rho \cos\theta \\ \rho \sin\theta \\ 0 \end{pmatrix}$$



$$\Phi\begin{pmatrix} \rho \\ \theta \\ \varphi \end{pmatrix} = \begin{cases} x = \rho \cos\varphi \cdot \cos\theta \\ y = \rho \cos\varphi \cdot \sin\theta \\ z = \rho \sin\varphi \end{cases}$$

$$dx\,dy\,dz = \boxed{\rho^2 \cos\varphi}\, d\rho \cdot d\theta \cdot d\varphi$$

$$D\Phi = \frac{\partial x, y, z}{\partial \rho, \theta, \varphi} = \begin{pmatrix} \cos\varphi \cos\theta & -\rho \cos\varphi \sin\theta & -\rho \sin\varphi \cos\theta \\ \cos\varphi \sin\theta & \rho \cos\varphi \cos\theta & -\rho \sin\varphi \sin\theta \\ \sin\varphi & 0 & \rho \cos\varphi \end{pmatrix}$$

$$\det D\Phi = \sin\varphi \left(\rho^2 \sin\varphi \cos\varphi \sin^2\theta + \rho^2 \sin\varphi \cos\varphi \cos^2\theta\right)$$

$$+ \rho \cos\varphi \left(\rho \cos^2\varphi \cos^2\theta + \rho \cos^2\varphi \sin^2\theta\right)$$

$$= \rho^2 \sin^2\varphi \cos\varphi + \rho^2 \cos^3\varphi = \boxed{\rho^2 \cos\varphi}$$

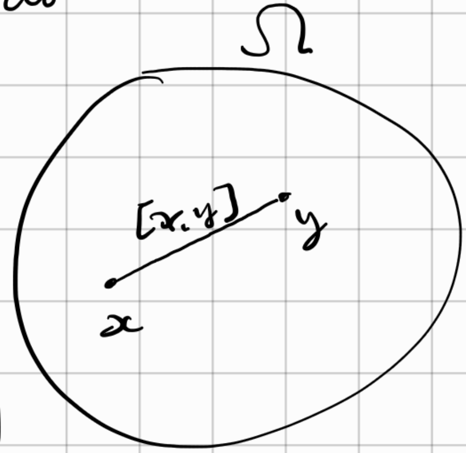
ok!

PROVIAMO a FARE una DIMOSTRAZIONE!

Lemma $\underline{f} \in C^1(\Omega, \mathbb{R}^n)$ $\quad \Omega \subseteq \mathbb{R}^n$ aperto

$\underline{x}, \underline{y} \in \Omega$, $\quad [\underline{x}, \underline{y}] \subseteq \Omega$

↑ segmento da $\underline{x}$ a $\underline{y}$



$$\left|\underline{f}(\underline{y}) - \underline{f}(\underline{x})\right| \le \max_{[x,y]} \|Df\| \cdot |x-y|$$

esiste perché Df è continua e $[x,y]$ è compatto (Weierstraß)

dim

$$\underline{\varphi}(t) = \underline{f}\left((1-t)\underline{x} + t\underline{y}\right) \qquad t \in [0,1]$$

$$\underline{\varphi}(1) = \underline{f}(\underline{y}) \qquad \underline{\varphi}(0) = \underline{f}(\underline{x})$$

$$\underline{\varphi}'(t) = D\underline{f}\left((1-t)\underline{x} + t\underline{y}\right)[\underline{y} - \underline{x}]$$

$$\left|\underline{f}(\underline{y}) - \underline{f}(\underline{x})\right| = \left|\underline{\varphi}(1) - \underline{\varphi}(0)\right| = \left|\int_0^1 \varphi'(t)\, dt\right|$$

$$\le \int_0^1 \left|Df(\text{—})[\underline{y} - \underline{x}]\right| dt$$

$$\le \int_0^1 \max_{[x,y]} \left|Df(\text{—})[\underline{y} - \underline{x}]\right| dt$$

$$\le \max_{z \in [x,y]} \left|Df(\underline{z})(\underline{y} - \underline{x})\right| \le \max_{[x,y]} \|Df\| \cdot |y-x|$$

$\square$

Def $M \in \mathbb{R}^{m \times n}$, $\quad \|M\| = \max_{v \neq 0} \dfrac{|Mv|}{|v|} = \max_{|v|=1} |Mv|$, $\quad \|\cdot\| = \|\cdot\|_\infty$

NORMA "OPERATORIALE" $\qquad \|M\| \neq \|M\|_2 = \sqrt{\sum m_{ij}^2}$

Proprietà caratterizzante: $|Mv| \le \|M\| \cdot |v|$

<u>Esercizio</u> se $M(t)$ è continua allora anche $\|M(t)\|$ è continua.

<u>Corollario</u> se $f \in C^1(\Omega, \mathbb{R}^n)$, K compatto, convesso $\subseteq \Omega$

allora $\exists L$ f è L-lipschitz su K:

$$\forall \underline{x}, \underline{y} \in K \quad |\underline{f}(\underline{y}) - \underline{f}(\underline{x})| \le L\,|\underline{y} - \underline{x}|$$

<u>dim</u> basta prendere $L = \max\limits_{K} \|Df\|$. □

<u>Esercizio</u> (curiosità)

Come si può estendere Lagrange in + variabili?

<u>Teorema</u> (Lagrange in + variabili)

$f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ f differenziabile su Ω aperto.

$\underline{x}, \underline{y} \in \Omega$, $[\underline{x}, \underline{y}] \subseteq \Omega$ allora $\exists\, \underline{z} \in [\underline{x}, \underline{y}]$ t.c.

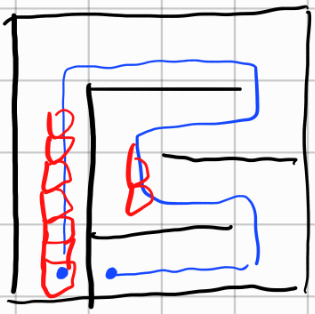
$$|\underline{f}(\underline{y}) - \underline{f}(\underline{x})|^2 = \underbrace{Df(z)}_{m\times n} \underset{\substack{\uparrow\\ \text{prodotto matrice vettore}}}{\,} \underbrace{[y - x]}_{n\times 1} \underset{\substack{\uparrow\\ \text{prodotto scalare}}}{\cdot} \underbrace{(f(y) - f(x))}_{m\times 1}$$

<u>dim</u> per esercizio!

Obiettivo $C^1 \to$ Lipschitz

Esempio cattivo

$f \in C^1(\Omega)$



$\Omega =$ interno del labirinto

Teorema "$C^1 \Rightarrow$ lipschitz"

Sia $\Omega \subseteq \mathbb{R}^n$ aperto, $\varphi \in C^1(\Omega, \mathbb{R}^n)$.

Se $E \subset\subset \Omega$ (significa che $\bar{E}$ è compatto e $\bar{E} \subset \Omega$)

allora $\exists L \in \mathbb{R}$ t.c. $\varphi_{|E}$ è L-lipschitz

dim Fissato $E \subset\subset \Omega \Rightarrow \delta = \operatorname{dist}(E, \partial\Omega) > 0$

$$\delta = \inf_{\substack{x \in E \\ y \in \partial\Omega}} |x-y|$$

cubo di lato 2^{-N}

$\exists N \in \mathbb{N}$ t.c. $Q_j^N \cap E \neq \emptyset \Rightarrow Q_j^N \subset\subset \Omega$

(basta che sia $\sqrt{n} \cdot 2^{-N} < \delta$)

E si ricopre con un numero finito di cubetti

$$Q_j^N = \frac{j + [0,1]^n}{2^N} \qquad j \in \mathbb{Z}^n$$

Voglio trovare L per cui φ è L-lipschitz su E:

$\forall x_1, x_2 \in E \quad |\varphi(x_1) - \varphi(x_2)| \overset{?}{\leq} L\, |x_1 - x_2|$

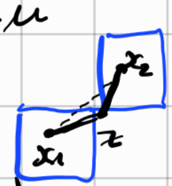
⚠ AGGIUSTATO DOPO LA LEZIONE

Caso 1 se x_1, x_2 stanno in due cubetti adiacenti:

Prendo $E^N = \bigcup \{ Q_j^N : Q_j^N \cap E \neq \emptyset \}$

E^N è un compatto $\subseteq \Omega$, $\lambda := \max_{E^N} \|D\varphi\|$.

Sia $[x_1, z] \cup [z, x_2]$ la spezzata più corta che congiunge x_1 a x_2.



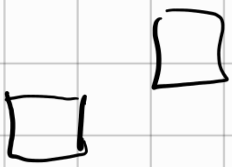
$$|x_1 - z| \leq |x_1 - x_2|, \quad |z - x_2| \leq |x_1 - x_2|$$

$$|\varphi(x_1) - \varphi(x_2)| \leq |\varphi(x_1) - \varphi(z)| + |\varphi(z) - \varphi(x_2)|$$

$$\leq \lambda \cdot |x_1 - z| + \lambda \cdot |z - x_2| \leq 2\lambda \cdot |x_1 - x_2|$$

Caso 2 se x_1, x_2 stanno in cubetti non adiacenti

$\Rightarrow |x_1 - x_2| > 2^{-N}$



$$|\varphi(x_1) - \varphi(x_2)| \leq R \leq \frac{R}{|x_1 - x_2|} \cdot |x_1 - x_2|$$

$B_R \supseteq \varphi(E)$

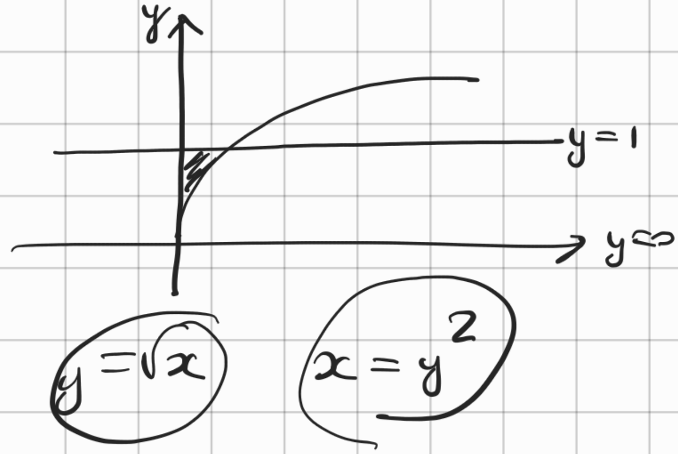
$\uparrow$ compatto $\quad \uparrow$ compatto

$$\leq \frac{R}{2^{-N}} |x_1 - x_2|$$

$$= 2^N \cdot R\, |x_1 - x_2|$$

$L = 2^N R \vee 2\lambda$

$\square$



$$\iint \sin(y^3)\,dx\,dy$$

$0 \le x \le y^2$

$0 \le y \le 1$

$$\int_0^1 \int_0^{y^2} \underline{\sin(y^3)}\,dx\,dy$$

$$= \int_0^1 (\sin y^3)\, y^2\, dy = \left[-\frac{1}{3}\cos(y^3)\right]_0^1 = 1-\frac{\cos 1}{3}$$