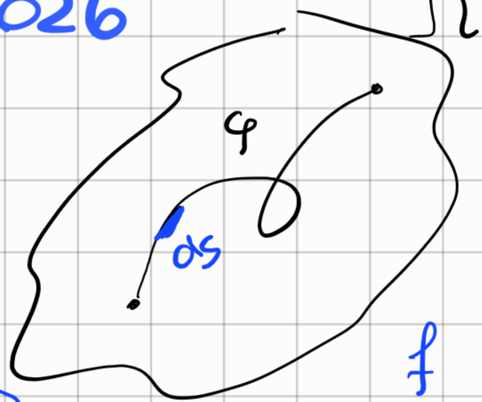
ANALISI MATEMATICA 2
LEZIONE 43 – 18.3.2026



INTEGRALI DI LINEA

f

FUNZIONE SCALARE

$$\varphi : [a,b] \to \Omega \subseteq \mathbb{R}^n$$

$$f : \Omega \to \mathbb{R}$$

lunghezza d'arco infinitesima

$$\int_\varphi f = \int_\varphi f(\varphi(s))\,ds = \int_a^b f(\varphi(t)) \cdot |\varphi'(t)|\,dt$$

giustificazione: $\ell(\varphi) = \int_a^b \underbrace{|\varphi'(t)|\,dt}_{ds}$

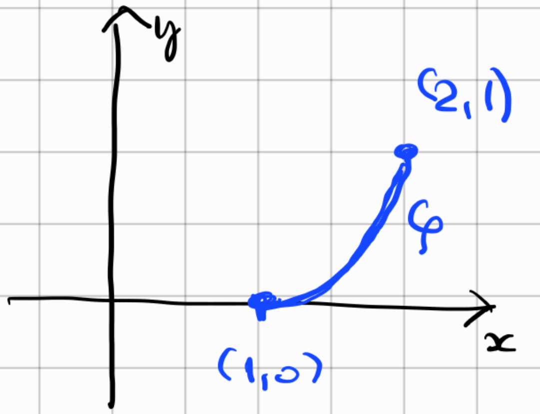
Fisicamente

$\int_\varphi f$ = il peso di un filo con densità f

Esempio

$$\varphi : [0,1] \to \mathbb{R}^2$$

$$\varphi(t) = (1+t, t^3)$$



$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = y$$

$$\varphi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1+t \\ t^3 \end{pmatrix}$$

$$\varphi' = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\int_\varphi f = \int_0^1 f(\varphi(t)) \cdot |\varphi'(t)|\,dt = \dots$$

$x = 1+t$

$y = t^3$

$$ds = |\varphi'(t)|\,dt = \sqrt{(x')^2 + (y')^2}\,dt$$

$$= \sqrt{}$$

$$\left[\begin{array}{c} dx = x'(t)\,dt \\ dy = y'(t)\,dt \\ ds = \sqrt{dx^2 + dy^2} = \sqrt{(x')^2 + (y')^2}\,dt \end{array}\right]$$

$$\begin{cases} dx = 1 \cdot dt \\ dy = 3t^2\,dt \end{cases} \qquad ds = \sqrt{1 + 9t^4}\,dt$$

$$\int_\varphi f = \int_\varphi y\,ds = \int_0^1 t^3 \sqrt{1 + 9t^4}\,dt = (*)$$

$$\begin{cases} u = \sqrt{1 + 9t^4} \\ u^2 = 1 + 9t^4 \\ 2u\,du = 36t^3\,dt, \quad t^3\,dt = \dfrac{u}{18}\,du \end{cases}$$

$$(*) = \int_1^{\sqrt{10}} \frac{u}{18} \cdot u\,du = \frac{1}{18} \int_1^{\sqrt{10}} u^2\,du = \frac{1}{18} \left[\frac{u^3}{3}\right]_1^{\sqrt{10}}$$

$$= \frac{1}{18} \left(\frac{10\sqrt{10} - 1}{3} \right)$$

<u>OSS 1</u> l'integrale non dipende dalla parametrizzazione

Infatti

$$\begin{array}{ccccc} t\in[a,b] & \xrightarrow{\varphi} & \mathbb{R}^n & \xrightarrow{f} & \mathbb{R} \\ \uparrow g & & & & \\ r\in[A,B] & & & & \end{array}$$

$$\int_{\varphi\circ g} f \doteq \int_A^B f((\varphi\circ g)(r))\,|(\varphi\circ g)'(r)|\,dr$$

$$= \int_A^B f(\varphi(g(r)))\cdot|\varphi'(g(r))\cdot g'(r)|\,dr$$

$$\begin{cases} t = g(r) \\ dt = g'(r)\,dr \\ |g'(r)| = g'(r) \end{cases}$$

$\left[g : [A,B] \to [a,b] \right.$ già crescente

$\left[\begin{array}{l} g(A) = a \\ g(B) = b \end{array} \right.$

$$= \int_a^b f(\varphi(t))\,|\varphi'(t)|\,dt$$

$$=: \int_{\varphi} f$$

<u>OSS</u> se φ è decrescente e $g(A) = b$

$g(B) = a$

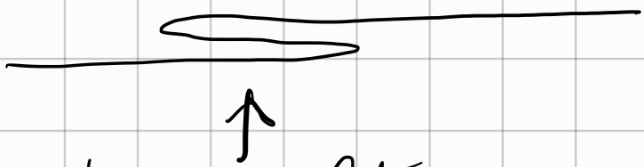
$$|g'(r)| = -g'(r)$$

$$\int_{\varphi\circ g} f = \int_b^a f(\varphi(t))\,|\varphi'(t)|\,(-dt)$$

$$= \int_a^b f(\varphi(t))\,|\varphi'(t)|\,dt$$

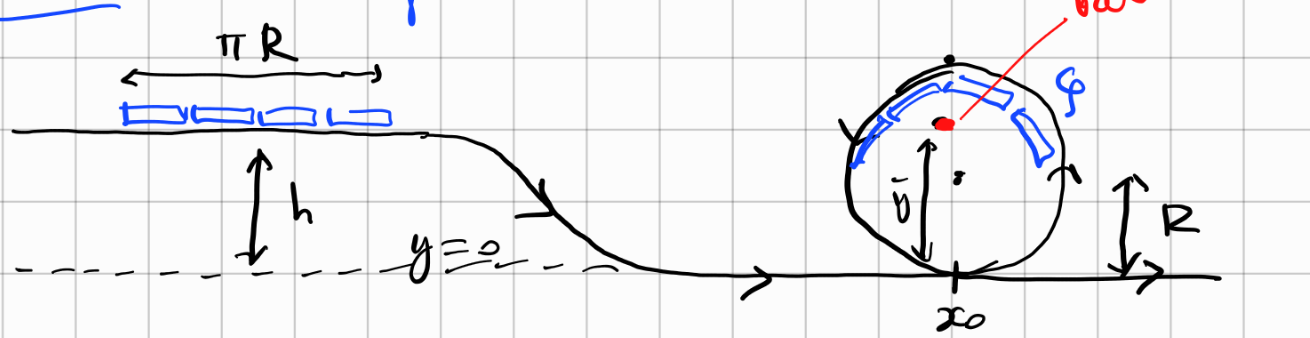
$$= \int_\varphi f.$$

! se g non è iniettiva:



conta 3 volte

Esercizio (montagne russe)



Calcolare h in modo che il baricentro nelle due configurazioni abbia la stessa altezza.

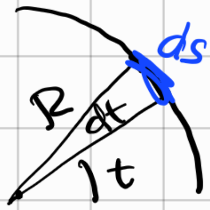
$$\bar{y} = \fint_\varphi y \doteq \frac{\int_\varphi y}{\int_\varphi 1} \qquad \varphi : [0,\pi] \to \mathbb{R}^2$$

$$\varphi(t) = (x_0 + R\cos t,\ R + R\sin t)$$

$$\begin{cases} x = x_0 + R\cos t \\ y = R + R\sin t \end{cases} \qquad \begin{cases} dx = -R\sin t \, dt \\ dy = R\cos t \, dt \end{cases}$$

$$ds = \sqrt{dx^2+dy^2} = \sqrt{R^2\sin^2 t + R^2\cos^2 t}\, dt$$

$$= R\,dt$$



$$\bar{y} = \frac{\int y\, R\, dt}{\ell(\varphi)} = \frac{\int_0^\pi (R+R\sin t)\cdot \cancel{R}\, dt}{\pi \cancel{R}}$$

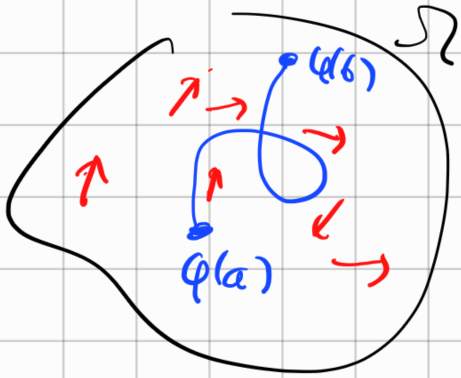
$$= \frac{R}{\pi}\int_0^\pi (1+\sin t)\, dt = \frac{R}{\pi}\Big[t - \cos t\Big]_0^\pi = \frac{R}{\pi}\big[\pi+1+1\big]$$

$$= R\left(1+\frac{2}{\pi}\right) \in (R, 2R) \qquad \square$$

INTEGRALE DI LINEA DI UN CAMPO VETTORIALE

$$\varphi : [a,b] \longrightarrow \Omega \subseteq \mathbb{R}^n$$

$$\underline{\xi} : \Omega \rightarrow \mathbb{R}^n$$



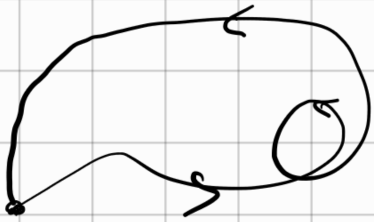
$$\int_\varphi \underline{\xi}\cdot \underline{ds} := \int_a^b \underline{\xi}(\varphi(t))\cdot \underbrace{\underline{\varphi}'(t)\, dt}_{\underline{ds}}$$

<u>Oss</u> anche questo è invariante per riparametrizzazione

<u>ma</u> cambia segno se inverto l'orientazione.

Se $\varphi(b) = \varphi(a)$

φ è un circuito

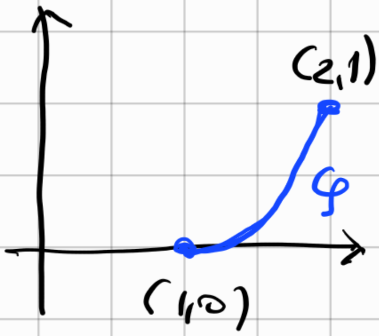


e $\int_\varphi \underline{\xi} \cdot d\underline{s}$ si indica

$\oint_\varphi \underline{\xi}$ e si chiama circuitazione di $\underline{\xi}$ lungo φ.

Esempio $\quad \varphi : [0,1] \to \mathbb{R}^2$

$$\varphi(t) = (1+t,\ t^3)$$



$$\underline{\xi}(x,y) = (y,x)$$

$\varphi'(t) = (1, 3t^2)$

prodotto scalare

$$\int_{\underline{\varphi}} \underline{\xi} \cdot d\underline{s} = \int_0^1 \underline{\xi}(\varphi(t)) \cdot \underline{\varphi}'(t)\, dt$$

$$= \int_0^1 \xi(1+t, t^3) \cdot (1, 3t^2)\, dt$$

$$= \int_0^1 \begin{pmatrix} t^3 \\ 1+t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix} dt = \int_0^1 \left(t^3 + (1+t)3t^2 \right) dt$$

$$= \int_0^1 (4t^3 + 3t^2)\, dt = \left[t^4 + t^3 \right]_0^1 = 2$$

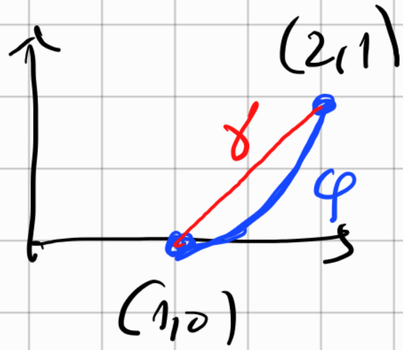
MNEMONICA $\quad \varphi'(t)\, dt = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} dt = \begin{pmatrix} dx \\ dy \end{pmatrix}$

$$\int \begin{pmatrix} y \\ x \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \int \underbrace{(y\, dx + x\, dy)}_{\text{FORMA DIFFERENZIALE}} =$$

$$\begin{cases} x = 1+t \\ y = t^3 \end{cases} \qquad \begin{cases} dx = dt \\ dy = 3t^2\,dt \end{cases}$$

$$= \int_0^1 \left(t^3\,dt + (1+t)\,3t^2\,dt \right) = \ldots = 2.$$

Prendo un'altra curva



$\gamma : [0,1] \longrightarrow \mathbb{R}^2$

$\gamma(t) = (1+t,\ t)$

ξ come prima

$$\int_\gamma \underline{\xi} \cdot \underline{ds} = \int_0^1 \underline{\xi}(\underline{\gamma}(t)) \cdot \underline{\gamma}'(t)\,dt$$

$$= \int_0^1 \begin{pmatrix} t \\ 1+t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} dt$$

$$= \int_0^1 (t + 1+t)\,dt = \int_0^1 (1+2t)\,dt$$

$$= \left[t + t^2 \right]_0^1 = 2$$

IN QUESTO CASO E' VENUTO UGUALE A PRIMA!

dimostreremo che per questo campo ξ l'integrale dipende solo dal punto di partenza e punto di arrivo (non dal percorso).

Osservazione se $\xi = \nabla f$ $\qquad f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$

$\varphi : [a,b] \to \Omega \qquad f \in C^1 ,\ \varphi \in C^1$

allora:

$$\int_\varphi \underline{\nabla} f \cdot ds \doteq \int_a^b \underline{\nabla} f(\varphi(t)) \cdot \varphi'(t)\, dt$$

$$= \int_a^b \frac{d}{dt} f(\varphi(t))\, dt = \left[f \circ \varphi \right]_a^b$$

derivata della fcn. composta

$$= f(\varphi(b)) - f(\varphi(a)) \qquad \square$$

In particolare se φ è chiusa cioè $\varphi(b) = \varphi(a)$

allora $\displaystyle\int \nabla f\, ds = 0$

f si dice PRIMITIVA di $\underline{\xi}$ se $\underline{\nabla} f = \underline{\xi}$

POTENZIALE

Esempio $\xi(x,y) = \begin{pmatrix} y \\ x \end{pmatrix}$ ha una primitiva?

Supponiamo $\underline{\nabla} f(x,y) = \underline{\xi}(x,y)$

$$\begin{cases} \dfrac{\partial f}{\partial x} = y \\ \dfrac{\partial f}{\partial y} = x \end{cases} \qquad f = f(x,y)$$

$\int y\,dx$ costante in x

$$\frac{\partial f}{\partial x} = y \implies f = \underbrace{yx}_{\int y\,dx} + \underbrace{c(y)}_{\text{costante in } x}$$

$$\frac{\partial f}{\partial y} = x + c'(y) \overset{!}{=} x$$

se c è costante **OK!**

$$f(x,y) = xy \qquad \nabla f = \begin{pmatrix} y \\ x \end{pmatrix} = \xi \qquad \text{II}$$

OSS 2 Se $\underline{\xi} = \underline{\nabla} f$ $\qquad \underline{\xi} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$

se $\xi \in C^1 \implies f \in C^2$

vale Schwartz $\dfrac{\partial^2 f}{\partial x_k \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_k}$

cioè

$$\boxed{\frac{\partial}{\partial x_k}\xi_j = \frac{\partial}{\partial x_j}\xi_k}$$

ES $\xi(x,y) = \begin{pmatrix} x + 2y^2 \\ y - 2 \end{pmatrix}$ $\qquad \xi_1 = x + 2y^2 \overset{?}{=} \dfrac{\partial f}{\partial x}$

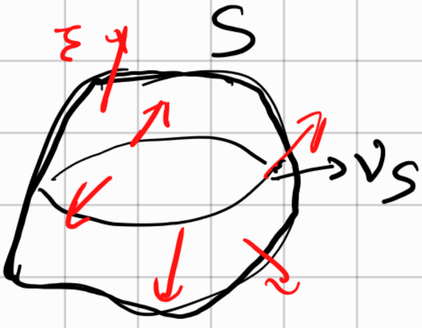
$\qquad \xi_2 = y - 2 \overset{?}{=} \dfrac{\partial f}{\partial y}$

$$\frac{\partial \xi_1}{\partial y} \overset{?}{=} \frac{\partial \xi_2}{\partial x} \qquad \frac{\partial \xi_1}{\partial y} = 4y \qquad \frac{\partial \xi_2}{\partial x} = 0$$

ξ non può avere una primitiva.

FORME DIFFERENZIALI

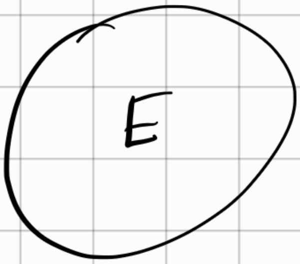
Obiettivo: integrare un campo vettoriale attraverso una superficie



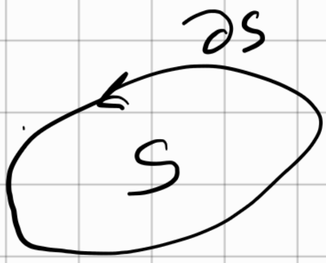
$$\int_S \underline{\xi}\cdot\underline{\nu}_S$$

Voglio definire questo (flusso di ξ attraverso S)

$$\int_E f\,dV$$



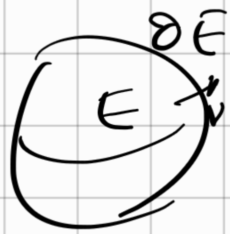
Teoremi di Stokes e Gauss-Green



1-forma, 2-forma

$$\oint_{\partial S} \underline{\xi} = \int_S (\underline{\nabla}\times\underline{\xi})\cdot\nu \qquad \text{Stokes}$$

rotore



$$\int_{\partial E} \underline{\xi}\cdot\nu_E = \int_E \underline{\nabla}\cdot\underline{\xi} \qquad \text{Gauss}$$

2-forma, divergenza, 3-forma

OBIETTIVO definire $\int_S \omega$

ω = k-forma
S = k-superficie in $\mathbb{R}^n$