ANALISI MATEMATICA 2
LEZIONE 47 – 27.3.2026

K-FORME in $\mathbb{R}^3$

ω 2-forma in $\mathbb{R}^3$ x,y,z $\xi \in \mathbb{R}^3$

$$\omega = \xi_1 \, dy \wedge dz + \xi_2 \, dz \wedge dx + \xi_3 \, dx \wedge dy$$

$$\omega[v,w] = \xi_1 \cdot (v_2 w_3 - v_3 w_2) + \xi_2 (v_3 w_1 - v_1 w_3) + \xi_3 (v_1 w_2 - v_2 w_1)$$

$$= \det(\xi, v, w) = \det \begin{pmatrix} \xi_1 & v_1 & w_1 \\ \xi_2 & v_2 & w_2 \\ \xi_3 & v_3 & w_3 \end{pmatrix}$$

PRODOTTO VETTORE in $\mathbb{R}^3$

$v, w \in \mathbb{R}^3$ il prodotto vettore $v \times w \in \mathbb{R}^3$ è il vettore che rappresenta l'applicazione lineare:

$$u \longmapsto \det(u, v, w)$$

cioè l'unico vettore tale che:

$$u \cdot (v \times w) = \det(u, v, w) \qquad \forall u \in \mathbb{R}^3.$$

$\uparrow$ prodotto triplo

In coordinate $\quad \xi = v \times w \qquad \xi = (\xi_1, \xi_2, \xi_3)$

$$\xi_1 = \xi \cdot e_1 = \det(e_1, v, w) = \det\begin{pmatrix} 1 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{pmatrix} = v_2 w_3 - v_3 w_2$$

$$\xi_2 = \xi \cdot e_2 = \det(e_2, v, w) = \det\begin{pmatrix} 0 & & \\ 1 & & \\ 0 & & \end{pmatrix} = v_3 w_1 - v_1 w_3$$

$$\xi_3 = \xi \cdot e_3 = \ldots = v_1 w_2 - v_2 w_1$$

Più espressivamente

$$\xi = \det\begin{pmatrix} e_1 & v_1 & w_1 \\ e_2 & v_2 & w_2 \\ e_3 & v_3 & w_3 \end{pmatrix} = e_1 (v_2 w_3 - v_3 w_2) + e_2 (\ldots) + e_3 (\ldots)$$

<u>Nota</u>: $\xi = v \times w$ è ortogonale sia a v che a w

perché $\quad \xi \cdot v = \det(v, v, w) = 0$

$\qquad\qquad \xi \cdot w = \det(w, v, w) = 0$

<u>ROTORE</u>

$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \qquad \partial_x = \frac{\partial}{\partial x} \quad \partial_y = \frac{\partial}{\partial y} \quad \partial_z = \frac{\partial}{\partial z}$$

$\xi : A \subseteq \mathbb{R}^3 \to \mathbb{R}^3$

$$\operatorname{rot} \xi = \nabla \times \xi = \det\begin{pmatrix} e_1 & \partial_x & \xi_1 \\ e_2 & \partial_y & \xi_2 \\ e_3 & \partial_z & \xi_3 \end{pmatrix}$$

$$= \left(\frac{\partial \xi_3}{\partial y} - \frac{\partial \xi_2}{\partial z}\right) e_1 + \left(\frac{\partial \xi_1}{\partial z} - \frac{\partial \xi_3}{\partial x}\right) e_2 + \left(\frac{\partial \xi_2}{\partial x} - \frac{\partial \xi_1}{\partial y}\right) e_3$$

<u>RICORDIAMO</u> $\quad$ grad = gradiente $\qquad$ div = divergenza

$$f : A \subseteq \mathbb{R}^3 \to \mathbb{R} \qquad \operatorname{grad} f = \underline{\nabla} f$$

$$\underline{\xi} : A \subseteq \mathbb{R}^3 \to \mathbb{R}^3 \qquad \operatorname{rot} \xi = \underline{\nabla} \times \underline{\xi}$$

$$\underline{\xi} : A \subseteq \mathbb{R}^3 \to \mathbb{R}^3 \qquad \operatorname{div} \xi = \underline{\nabla} \cdot \underline{\xi}$$

$$= \frac{\partial \xi_1}{\partial x} + \frac{\partial \xi_2}{\partial y} + \frac{\partial \xi_3}{\partial z}$$

grad e div ci sono in $\mathbb{R}^n$:

$$\operatorname{div} \underline{\xi} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial x_k} = t_r D \underline{\xi}$$

$A \subseteq \mathbb{R}^3$

$$0 \xrightarrow{d} \Omega^0(A) \xrightarrow{d} \Omega^1(A) \xrightarrow{d} \Omega^2(A) \xrightarrow{d} \Omega^3(A) \xrightarrow{d} 0$$

$\Omega^0(A) \cong f : A \to \mathbb{R}$, $\omega = f$

$\Omega^1(A) \cong \xi : A \to \mathbb{R}^3$, $\omega = \xi_1 dx + \xi_2 dy + \xi_3 dz$

$\Omega^2(A) \cong \xi : A \to \mathbb{R}^3$, $\omega = \xi_1 dy \wedge dz + \xi_2 dz \wedge dx + \xi_3 dx \wedge dy$

$\Omega^3(A) \cong f : A \to \mathbb{R}$, $\omega = f\, dx \wedge dy \wedge dz$

vogliamo mostrare che $\rightarrow$ grad, rot, div

① $\Omega^0 \xrightarrow{d} \Omega^1$

$\omega = f \qquad d\omega \cong \nabla f \qquad$ già visto

③ $\Omega^2 \xrightarrow{d} \Omega^3 \qquad \omega \cong \xi$

$$\omega = \xi_1\, dy \wedge dz + \xi_2\, dz \wedge dx + \xi_3\, dx \wedge dy$$

RICORDIAMO:

$$d(f \cdot dx_1 \wedge \dots \wedge dx_n) = df \wedge dx_1 \wedge \dots \wedge dx_n$$

$$df = \sum_k \frac{\partial f}{\partial x_k} dx_k$$

$$d\omega = \left(\frac{\partial \xi_1}{\partial x} + \frac{\partial \xi_2}{\partial y} + \frac{\partial \xi_3}{\partial z}\right) \cdot dx \wedge dy \wedge dz$$

$$= \operatorname{div} \xi \quad dx \wedge dy \wedge dz \qquad \text{ok !}$$

② $\Omega^1(A) \xrightarrow{d} \Omega^2(A)$ $\qquad \omega \tilde{\simeq} \xi$

$$\omega = \xi_1\, dx + \xi_2\, dy + \xi_3\, dz$$

$$d\omega = \underbrace{\left(\frac{\partial \xi_3}{\partial y} - \frac{\partial \xi_2}{\partial z}\right)}_{\eta_1} dy \wedge dz + \underbrace{\left(\frac{\partial \xi_1}{\partial z} - \frac{\partial \xi_3}{\partial x}\right)}_{\eta_2} dz \wedge dx$$

$$+ \underbrace{\left(\frac{\partial \xi_2}{\partial x} - \frac{\partial \xi_1}{\partial y}\right)}_{\eta_3} dx \wedge dy$$

Se $d\omega \tilde{\simeq} \eta$

quindi $\eta = \operatorname{rot} \xi$ $\qquad \square$

Teorema $d^2 = 0$ cioè $d^2\omega = d(d\omega) = 0$
se $\omega \in C^2$ $\qquad \forall \omega \in \Omega^k(A)$.

dim

Basta mostrarlo per $\omega = f \cdot dx_{j_1} \wedge \dots \wedge dx_{j_k}$

$$d\omega = df \wedge dx_{j_1} \wedge \dots \wedge dx_{j_k}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_{j_1} \wedge \dots \wedge dx_{j_k}$$

$$d^2\omega = d(d\omega) = d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\right) \wedge dx_i \wedge dx_{j_1} \wedge \dots \wedge dx_{j_k}$$

$$= \sum_{i=1}^{n} \sum_{\ell=1}^{n} \frac{\partial^2 f}{\partial x_\ell\, \partial x_i}\, dx_\ell \wedge dx_i \wedge dx_{j_1} \wedge \dots \wedge dx_{j_k}$$

se $\ell = i$ $\quad dx_\ell \wedge dx_i = 0$

Schwarz: $\dfrac{\partial^2 f}{\partial x_\ell \partial x_i} = \dfrac{\partial^2 f}{\partial x_i \partial x_\ell}$

$$= \sum_{i=1}^{n} \sum_{\ell=1}^{i-1} \cancel{\left(\frac{\partial^2 f}{\partial x_\ell \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_\ell} \right)} dx_\ell \wedge dx_i \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_k}$$

$$= 0 \qquad \square$$

Dunque in $\mathbb{R}^3$: $\operatorname{rot} \nabla f = 0$, $\operatorname{div} \operatorname{rot} \xi = 0$

non è zero (ed ha grande rilevanza)

il laplaciano $\Delta f = \operatorname{div} \cdot \nabla f$

$$\begin{tikzcd} \Omega^0(A) \arrow[r] \arrow[r, bend right, "\text{grad}"'] & \Omega^1(A) \arrow[r] \arrow[rrr, bend right, "* \text{ hodge}"'] & \ldots \arrow[r] & & \Omega^{n-1}(A) \arrow[r] \arrow[r, bend right, "\text{div}"'] & \Omega^n(A) \end{tikzcd}$$

INTEGRALE di una k-forma su una

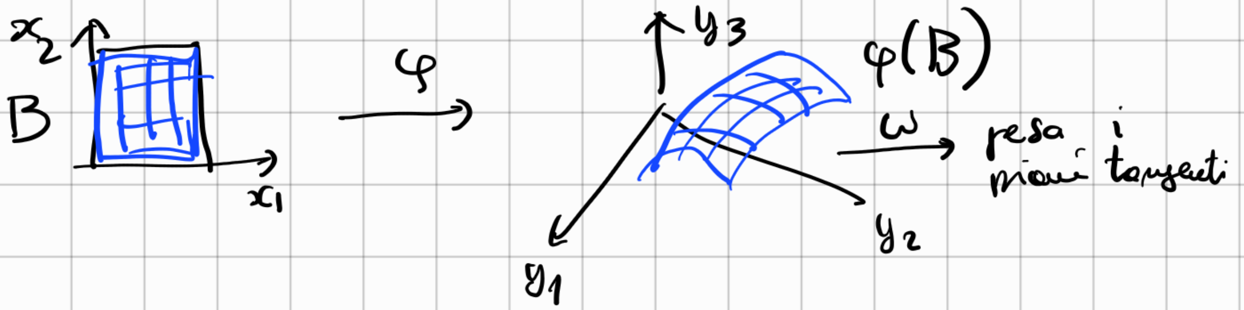
k-superficie.

$$\boxed{\begin{matrix} \varphi \in C^1 \\ \omega \in C^0 \end{matrix}}$$

Dati $\quad B \subseteq \mathbb{R}^k \xrightarrow{\varphi} A \subseteq \mathbb{R}^n$

$$A \xrightarrow{\omega} \Lambda^k(A) \qquad k\text{-forma}$$

$y = \varphi(x)$



$$\int_\varphi \omega \doteq \int_B \omega_{\varphi(x)} [D\varphi(x)]\, dx$$

$$= \int_B \underbrace{\omega_{\varphi(x)} \left[\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_k} \right]} dx_1 \dots dx_k$$

In effetti si definisce il PULL-BACK di ω tramite φ

se $\varphi : B \to A$

$d\varphi(x) : \mathbb{R}^m \to \mathbb{R}^n$

$$\begin{array}{ccc} B \subseteq \mathbb{R}^m & \xrightarrow{\varphi} & A \subseteq \mathbb{R}^n \\ \Big\downarrow \varphi^*\omega & & \Big\downarrow \omega \\ \Lambda^k(\mathbb{R}^m) & & \Lambda^k(\mathbb{R}^n) \end{array}$$

Se ω è una k-forma in $A \subseteq \mathbb{R}^n$
posso definire una k-forma $\varphi^*\omega$ in $B \subseteq \mathbb{R}^m$
(è un cambio di variabile)

$$\varphi^*\omega_x [v_1, \dots, v_k] := \omega_{\varphi(x)} \left[d\varphi_x[v_1], \dots, d\varphi_x[v_k] \right]$$

$y = \varphi(x)$

$$= \omega_{\varphi(x)} [D\varphi(x)]$$

$$\int_\varphi \omega = \int_B \varphi^*\omega$$

siccome $\varphi^*\omega$ è una k-forma in $\mathbb{R}^k$

$$\varphi^*\omega = f\, dx_1 \wedge \dots \wedge dx_k$$

$$\int_B \varphi^*\omega = \int_B f\, dx_1 \dots dx_k = \int_B f(\underline{x})\, d\underline{x}$$

__Esempio__

$$\varphi : B \subseteq \mathbb{R}^2_{u,v} \longrightarrow \mathbb{R}^3_{x,y,z} \qquad (x,y,z) = \varphi(u,v)$$

$$\omega \in \Omega^2(\mathbb{R}^3) \qquad 2\text{-forma}$$

$$\omega = f \; dy \wedge dz \qquad \mathbb{R}^2 = \operatorname{span}\{e_1, e_2\}$$

$$\varphi^* \omega = (\varphi^* \omega)[e_1, e_2] \; du \wedge dv \qquad \mathbb{R}^2 \xrightarrow{d\varphi} \mathbb{R}^3$$

$$= \omega [D\varphi] \; du \wedge dv \qquad D\varphi = \begin{pmatrix} & \\ & \\ & \end{pmatrix}$$

$$= \omega \left[\frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v} \right] du \wedge dv$$

$$= (f \circ \varphi) (dy \wedge dz) \left[\frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v} \right] du \wedge dv$$

$$= f \circ \varphi \cdot \left(\frac{\partial \varphi_2}{\partial u} \cdot \frac{\partial \varphi_3}{\partial v} - \frac{\partial \varphi_3}{\partial u} \frac{\partial \varphi_2}{\partial v} \right) du \wedge dv$$

o meglio $\quad (x,y,z) = \varphi(u,v)$

$$\begin{cases} x = \varphi_1(u,v) = x(u,v) \\ y = \varphi_2(u,v) = y(u,v) \\ z = \varphi_3(u,v) = z(u,v) \end{cases}$$

$$\frac{\partial \varphi_2}{\partial u} = \frac{\partial y}{\partial u} \qquad \frac{\partial \varphi_3}{\partial v} = \frac{\partial z}{\partial v}$$

$$= f\circ\varphi \cdot \left(\frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \frac{\partial y}{\partial v} \right) du \wedge dv$$

$$= f\circ\varphi \cdot \frac{\partial (y,z)}{\partial (u,v)} \; du \wedge dv \qquad \square$$

$$\uparrow$$

determinante $= \det \begin{pmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$