ANALISI MATEMATICA 2
LEZIONE 48 - 1.4.2026

PULL-BACK

$$\begin{tikzcd} \overline{B} \subseteq \mathbb{R}^m \arrow[r, "\varphi"] \arrow[d, "\varphi^* \omega"] & A \subseteq \mathbb{R}^n \arrow[d, "\omega"] \\ \Lambda^k(\mathbb{R}^m) & \Lambda^k(\mathbb{R}^n) \end{tikzcd}$$

B limitato

φ C^1

ω continua

$$\Omega^k(B) \xleftarrow{\varphi^*} \Omega^k(A)$$

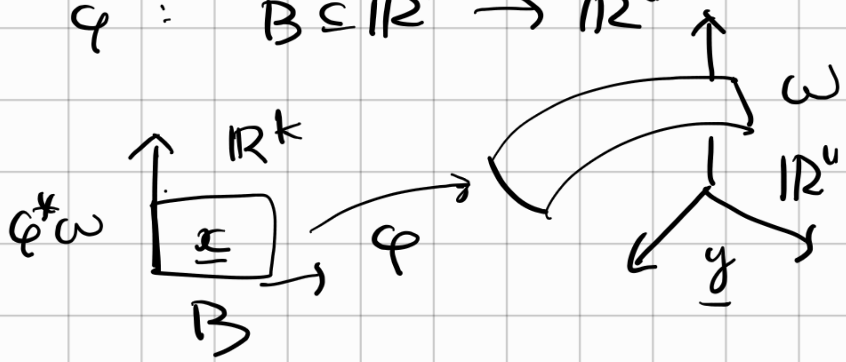
GIÀ VISTO:

$$\int_\varphi \omega \stackrel{\circ}{=} \int_B \varphi^* \omega$$

k forma in $\mathbb{R}^k$

ω k-forma su $\mathbb{R}^n$

$\varphi : B \subseteq \mathbb{R}^k \to \mathbb{R}^n$



Teorema $\varphi^* d\omega = d \varphi^* \omega$

RICAPITOLIAMO il PRODOTTO ALTERNATO:

$dx_{j_1} \wedge \dots \wedge dx_{j_k}$ come è definito?

in generale se $\xi_1, \dots, \xi_k \in (\mathbb{R}^n)^*$

$$\xi_1 \wedge \dots \wedge \xi_k \in \Lambda^k(\mathbb{R}^n)$$

$$(\xi_1 \wedge \dots \wedge \xi_k)[v_1, \dots, v_k] = \sum_{\sigma \in S_k} (-1)^{\sigma} \xi_1(v_{\sigma(1)}) \cdots \xi_k(v_{\sigma(k)})$$

"sgn σ"

ES $(dx \wedge dz)[v_1, v_2] = dx[v_1] \cdot dz(v_2) - dx[v_2] dz[v_1]$

Oss

$v_1 \dots v_k \in \mathbb{R}^m$

$\mathbb{R}^m \xrightarrow{\varphi} \mathbb{R}^n$

$$\varphi^*(dx_{j_1} \wedge \dots \wedge dx_{j_k})[v_1, \dots, v_k]$$

$$\doteq (dx_{j_1} \wedge \dots \wedge dx_{j_k})[d\varphi[v_1], \dots, d\varphi[v_k]]$$

$\mathbb{R}^n$

$$= (dx_{j_1} \wedge \dots \wedge dx_{j_k})\left[\frac{\partial \varphi}{\partial v_1}, \dots, \frac{\partial \varphi}{\partial v_k}\right]$$

$$\frac{\partial(\varphi_{j_1} \dots \varphi_{j_k})}{\partial(v_1 \dots v_k)}$$

$$= \sum_{\sigma \in S_k} (-1)^\sigma \frac{\partial \varphi_{j_1}}{\partial v_{\sigma(1)}} \cdots \frac{\partial \varphi_{j_k}}{\partial v_{\sigma(k)}}$$

$$= (d\varphi_{j_1} \wedge \dots \wedge d\varphi_{j_k})[v_1 \dots v_k]$$

$$\boxed{\varphi^*(dx_{j_1} \wedge \dots \wedge dx_{j_k}) = d\varphi_{j_1} \wedge \dots \wedge d\varphi_{j_k}}$$

$\Big(\underline{y} = \varphi(\underline{x}) \qquad \varphi^*(dx_{j_1} \wedge \dots \wedge dx_{j_k}) = dy_{j_1} \wedge \dots \wedge dy_{j_k} \Big)$

dim $(\varphi^* d\omega = d\varphi^*\omega)$

Basta farlo per $\omega = f\, dy_{j_1} \wedge \dots \wedge dy_{j_k}$

$$\varphi^*\omega = (f \circ \varphi)\, d\varphi_{j_1} \wedge \dots \wedge d\varphi_{j_k}$$

$$d\omega = df \wedge dy_{j_1} \wedge \dots \wedge dy_{j_k}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} dy_i \wedge dy_{j_1} \wedge \dots \wedge dy_{j_k}$$

$$\varphi^* d\omega = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} \circ \varphi\right) d\varphi_i \wedge d\varphi_{j_1} \wedge \dots \wedge d\varphi_{j_k}$$

$$d\varphi^* \omega = d(f\circ\varphi) \wedge d\varphi_{j_1} \wedge \dots \wedge d\varphi_{j_k}$$

$$= \sum_{l=1}^{m} \frac{\partial}{\partial x_l}(f\circ\varphi)\, dx_l \wedge d\varphi_{j_1} \wedge \dots \wedge d\varphi_{j_k}$$

$$= \sum_{l=1}^{m} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i}\circ\varphi\right) \cdot \frac{\partial \varphi_i}{\partial x_l}\, dx_l \wedge d\varphi_{j_1} \wedge \dots \wedge d\varphi_{j_k}$$

$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i}\circ\varphi\right) d\varphi_i \wedge d\varphi_{j_1} \wedge \dots \wedge d\varphi_{j_k} \overset{!}{=} \varphi^* d\omega$$

$\square$

Altra osservazione: $\int_\varphi \omega$ non dipende dalla parametrizzazione di φ se mantengo l'orientazione:

Teo (invarianza per riparametrizzazione)

$$\begin{array}{ccc} y \in \overline{B} \subseteq \mathbb{R}^k & \xrightarrow{\varphi} & A \subseteq \mathbb{R}^n \\ \uparrow u & & \downarrow \omega \\ x \in \overline{C} \subseteq \mathbb{R}^k & & \Lambda^k(\mathbb{R}^n) \end{array}$$

B, C, A aperti $\qquad \varphi \in C^1(\overline{B}, A)$

B, C limitati $\qquad u \in C^1(\overline{C}, \overline{B})$

($f \in C^1(K)$ se f si estende C^1 ad un aperto contenente K)

$u : C \to B$ bigettiva

$\det Du > 0$ su C

Allora $\boxed{\int_{\varphi\circ u} \omega = \int_{\varphi} \omega}$ $\left(\int_{\varphi\circ u} \omega = - \int_{\varphi} \omega \text{ se } \det Du < 0 \right)$

<u>dim</u>

$$\int_{\varphi\circ u} \omega = \int_C \omega_{(\varphi\circ u)(x)} \left[D(\varphi\circ u)(x) \right] d\underline{x}$$

$$= \int_C \omega_{(\varphi\circ u)(x)} \left[D\varphi(u(x)) \cdot Du(x) \right] dx$$

$$\overset{(*)}{=} \int_C \omega_{(\varphi\circ u)(x)} \left[D\varphi\left(u(x) \right) \right] \det Du(x) \, dx$$

$$\begin{cases} y = u(x) \\ dy = |\det Du| \, dx \end{cases}$$

$$= \int_B \omega_{\varphi(y)} \left[D\varphi(y) \right] dy = \int_{\varphi} \omega \qquad \square$$

cambio di variabile negli integrali multipli.

$D\varphi$ $\qquad$ Du $\qquad$ ω

$(*)$ Se $L \in \mathbb{R}^{n\times k}$ $\quad M \in \mathbb{R}^{k\times k}$ $\quad \lambda \in \Lambda^k(\mathbb{R}^n)$

allora $\lambda[LM] = \lambda[L] \det M$

le colonne di LM $\qquad$ le colonne di L

infatti

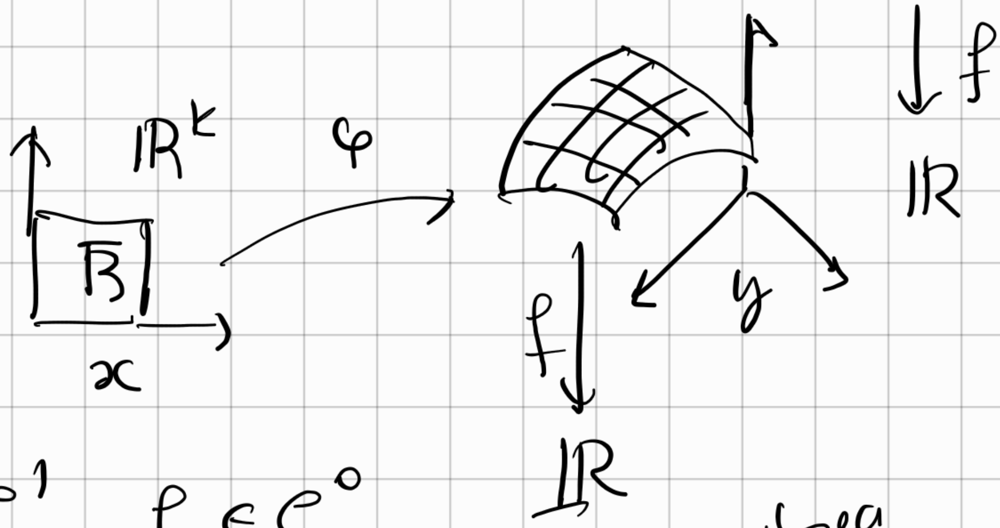
Il lato sinistro è alternante in $M^1, \dots, M^k$

colonne di M.

Ma se $M = Id$ $\qquad \lambda[L\,id] = \lambda[L]$. $\boxed{X}$

INTEGRALI DI SUPERFICIE

Sia $\varphi \in C^1$ $\qquad \bar{B} \subseteq \mathbb{R}^k \xrightarrow{\varphi} A \subseteq \mathbb{R}^n$



$\varphi \in C^1$, $f \in C^0$

B limitato

elemento d'area

Definiamo

$$\int_\varphi f \, d\sigma \doteq \int_B f(\varphi(x)) \underbrace{\sqrt{\det\left((D\varphi(x))^t D\varphi(x)\right)}}_{\text{determinante jacobiano}} \, dx$$

$D\varphi(x) \in \mathbb{R}^{n \times k}$

$D\varphi^t D\varphi \in \mathbb{R}^{k \times k}$

Oss 1 Se $k = n$ $\quad \sqrt{\det(D\varphi^t D\varphi)} = \sqrt{\det D\varphi^t \cdot \det D\varphi}$

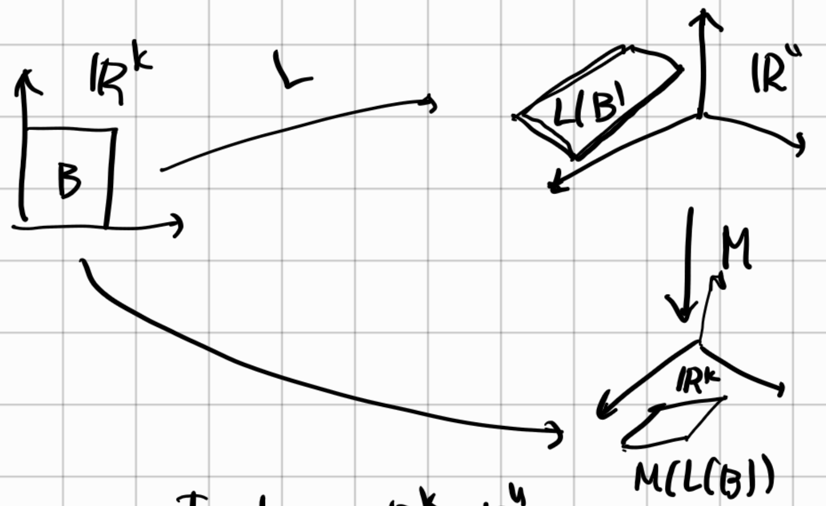
$= \sqrt{(\det D\varphi)^2} = |\det D\varphi|$. (come nel cambio di variabili)

OSS 2 Se $k=1$ $\quad \sqrt{\det D\varphi^t D\varphi} = \sqrt{\det \underbrace{|\varphi'|^2}_{1\times 1}} = |\varphi'|$ (come negli integrali curvilinei)

Interpretazione geometrica del determinante jacobiano

Supponiamo φ lineare! (in "piccolo" lo è). $L = D\varphi$

Teo Sia $L : \mathbb{R}^k \to \mathbb{R}^n$ lineare $\quad M : \mathbb{R}^n \to \mathbb{R}^n$ ortogonale



M manda $\operatorname{Im} L$ in $\mathbb{R}^k \subseteq \mathbb{R}^n$. $\quad \Pi : \mathbb{R}^n \to \mathbb{R}^k$ proiezione ortogonale

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "L"] \arrow[rrr, bend right, "\tilde{L}"'] & \mathbb{R}^n \arrow[r, "M"] & \mathbb{R}^n \arrow[r, "\Pi"] & \mathbb{R}^k \end{tikzcd}$$

Allora $\quad |\det \tilde{L}| = \sqrt{\det(L^t L)}$.

overo $\quad B \subseteq \mathbb{R}^k \quad$ "k-area"$(L(B)) = \sqrt{\det(L^t L)} \cdot |B|$

misura del ruotato di $L(B)$ in $\mathbb{R}^k$

misura di P-J.

<u>dim</u>

$$|\det \tilde{L}| = \sqrt{\det (\tilde{L})^t \tilde{L}}$$

$$= \sqrt{\det ((\Pi M L)^t \Pi M L}$$

$$= \sqrt{\det (L^t M^t \Pi^t \Pi M L)}$$

$$\Pi = \left(\begin{array}{c} Id \\ \hline 0 \end{array}\right) \begin{array}{l} \updownarrow k \\ \updownarrow n-k \end{array} \qquad \text{su } Im\, ML \quad \text{perché } Im\, ML \subseteq \mathbb{R}^k \qquad \Pi^t \Pi = Id$$
$$\underbrace{\qquad}_{k}$$

$$|\det \tilde{L}| = \dots = \sqrt{\det(L^t \cancel{M^t M} L)} = \sqrt{\det L^t L} \qquad \square$$

M ortogonale $\quad M^t M = Id$

Esempio (area della sfera)

$$\begin{cases} x = R \cdot \cos\varphi \cdot \cos\theta \\ y = R \cdot \cos\varphi \cdot \sin\theta \\ z = R \cdot \sin\varphi \end{cases} \qquad \begin{array}{ll} -\pi \leq \theta \leq \pi & \text{longitudine} \\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} & \text{latitudine} \end{array}$$

$$\psi(\theta,\varphi) = R(\cos\varphi \cos\theta, \cos\varphi \sin\theta, \sin\varphi)$$

$$\text{Area(Sfera)} = \int_\psi 1 \, d\sigma \doteq \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{\det D\psi^t D\psi} \, d\theta \, d\varphi$$

$$D\psi = \left(\begin{array}{c|c} -R\cos\varphi \sin\theta & -R\sin\varphi\cos\theta \\ R\cos\varphi\cos\theta & -R\sin\varphi\sin\theta \\ 0 & R\cos\varphi \end{array}\right) = \left(\frac{\partial\psi}{\partial\theta} \,\middle|\, \frac{\partial\psi}{\partial\varphi}\right)$$

$$D\psi^t D\psi = R^4 \begin{pmatrix} \left|\frac{\partial\psi}{\partial\theta}\right|^2 & \cancel{\frac{\partial\psi}{\partial\theta}\cdot\frac{\partial\psi}{\partial\theta}} \\ \cancel{\frac{\partial\psi}{\partial\theta}\cdot\frac{\partial\varphi}{\partial\theta}} & \left|\frac{\partial\psi}{\partial\varphi}\right|^2 \end{pmatrix} = \dots = R^4 \left(\begin{array}{c|c} \cos^2\varphi & 0 \\ \hline 0 & 1 \end{array}\right)$$

$$\sqrt{\det (D\varphi)^t D\varphi} = R^2 \cos\varphi$$

$$d\sigma = R^2 \cos\varphi \; d\theta \, d\varphi$$

$$\int_\varphi d\sigma = R^2 \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\pi/2} \cos\varphi \; d\theta \, d\varphi = 2\pi R^2 \overbrace{\int_{-\frac{\pi}{2}}^{\pi/2} \cos\varphi \, d\varphi}^{\overset{2}{\shortparallel}}$$

$$= 4\pi R^2$$

$\square$