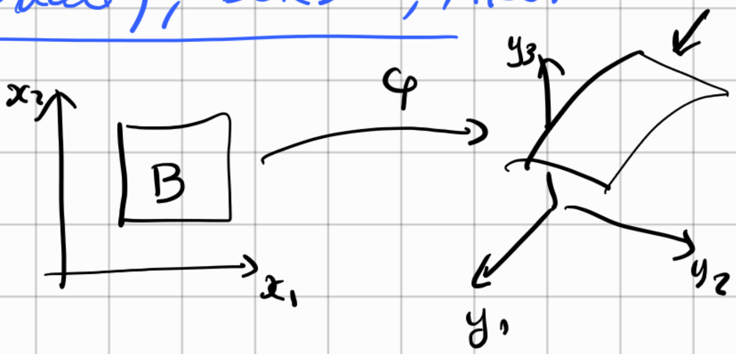
ANALISI MATEMATICA 2
LEZIONE 50 - 15.4.2026

SUPERFICIE (k-dimensionale), BORDO, AREA

$\varphi : B \subseteq \mathbb{R}^k \longrightarrow \mathbb{R}^n$

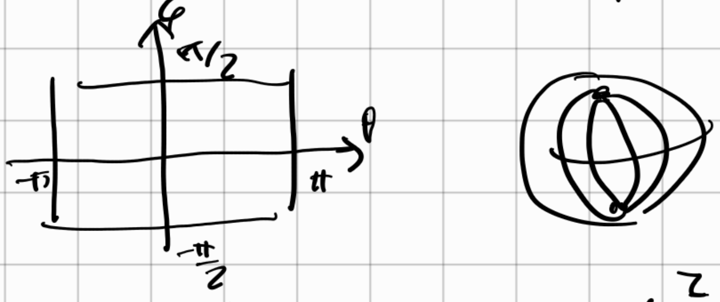


Se $\omega \in \Omega^k(\mathbb{R}^n)$

si definisce $\displaystyle\int_\varphi \omega$

e questo non cambia se cambio parametrizzazione (mantenendo l'orientazione)

Es



$$\begin{cases} x = \cos\theta\cos\varphi \\ y = \sin\theta\cos\varphi \\ z = \sin\varphi \end{cases}$$

$$\begin{cases} x = x \\ y = y \\ z = \sqrt{1-x^2-y^2} \end{cases}$$

calotta boreale.

vorrei poter "sommare" le superfici.

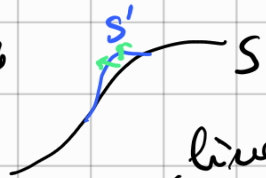
Def Due k-superfici S_1 e S_2 sono uguali se

$$\int_{S_1} \omega = \int_{S_2} \omega \qquad \forall \omega \in \Omega^k(\mathbb{R}^n)$$

ω di classe C^0

Def

$\varphi : B \subseteq \mathbb{R}^k \to \mathbb{R}^n$, B limitato misurabile, $\varphi \in C^1(\bar{B})$



lineare in ω

definisco $[\![\varphi]\!] : \Omega^k(\mathbb{R}^n) \longrightarrow \mathbb{R}$, $[\![\varphi]\!](\omega) = \displaystyle\int_\varphi \omega$

(CORRENTI)

OSS 1 Se ψ è una riparametrizzazione (concorde) di φ

allora $\int_\psi \omega = \int_\varphi \omega \Rightarrow [\![\varphi]\!] = [\![\psi]\!]$

OSS 2 Posso sommare: $\left([\![\varphi_1]\!] + [\![\varphi_2]\!]\right)(\omega) = \int_{\varphi_1} \omega + \int_{\varphi_2} \omega$

OSS 3 Posso avere molteplicità: $[\![\varphi_1]\!] + [\![\varphi_1]\!] = 2[\![\varphi_1]\!]$

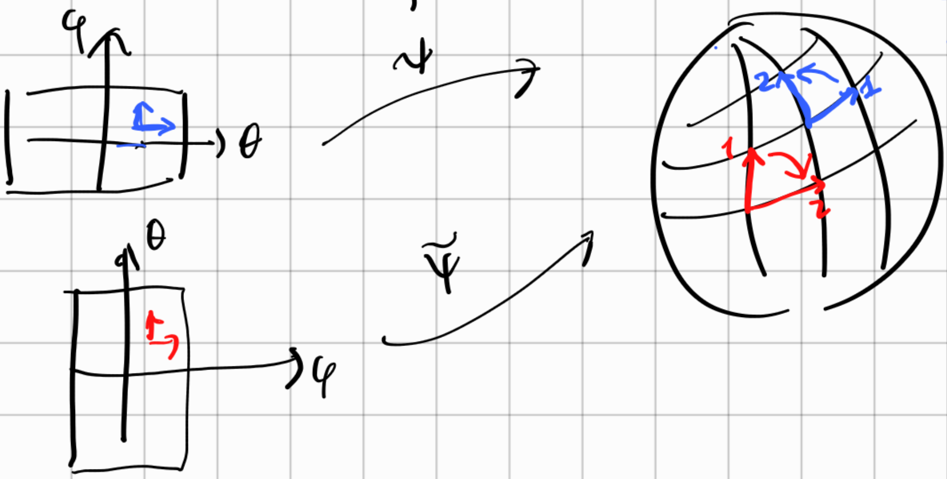
OSS 4 Posso definire $-[\![\varphi]\!] = [\![\tilde{\varphi}]\!]$

$$\tilde{\varphi}(x_1, x_2, \dots, x_k) = \varphi(x_2, x_1, x_3 \dots x_k)$$

SCAMBIO

Es Sfera

$$\psi : \begin{cases} x = \cos\theta \cos\varphi \\ y = \sin\theta \cos\varphi \\ z = \sin\varphi \end{cases} \qquad \tilde{\psi}(\theta, \varphi) = \psi(\varphi, \theta)$$



ANTIORARIO o USCENTE

ORARIO o ENTRANTE

Def Se $f : B \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$

$$[\![f]\!] = [\![\varphi]\!]$$

$$\varphi(x_1 \dots x_{n-1}) = (x_1 \dots x_{n-1}, f(x_1 \dots x_{n-1}))$$

Es Sfera: $S = [\![f]\!] - [\![-f]\!]$ con $f(x,y) = \sqrt{1 - x^2 - y^2}$

<u>Esercizio</u> dimostrare che $[\![f]\!] - [\![-f]\!] = [\![\psi]\!]$

$$\psi(\theta,\varphi) = \begin{pmatrix} \cos\theta \cos\varphi \\ \sin\theta \cos\varphi \\ \sin\varphi \end{pmatrix}$$

cioè che l'integrale su ogni forma differenziale è lo stesso.

<u>Def</u> $A \subseteq \mathbb{R}^n$ misurabile limitato

$$[\![A]\!](\omega) = \int_A f \qquad \text{se } \omega = f\, dx_1 \wedge \dots \wedge dx_n$$

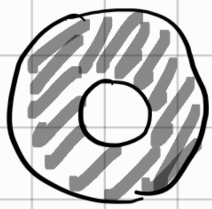
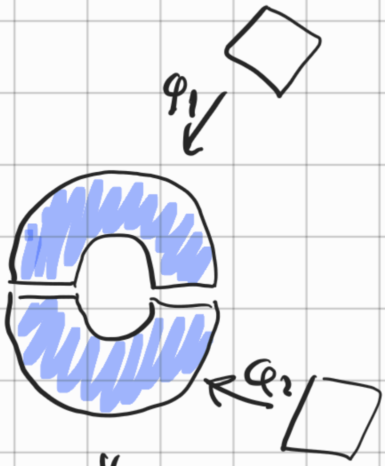
<u>OSS</u> ⚠ la somma di superfici può dare cancellazione.

S_1 

S_2

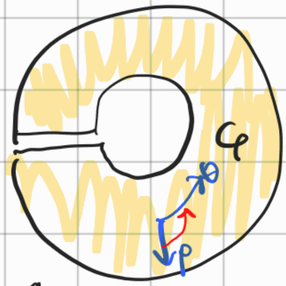
$S_1 + S_2 =$ 

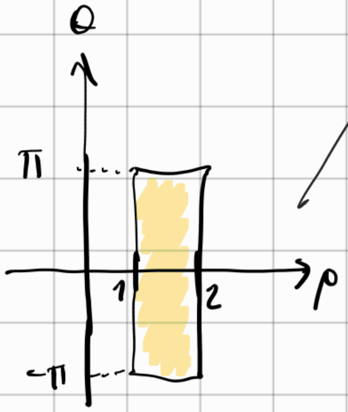
<u>ES</u> $B_\rho = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \le \rho^2 \}$

$[\![B_2]\!] - [\![B_1]\!] =$  $=$ 

$\shortparallel$

$[\![\varphi_1]\!] + [\![\varphi_2]\!]$

$=$  $= [\![\varphi]\!]$



$$\varphi(\rho,\theta) = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \end{pmatrix} \qquad (\rho,\theta) \in [1,2] \times [-\pi,\pi]$$

BORDO

in modo che se S è una k-superficie ∂S deve essere una $(k-1)$-superficie e vogliamo che valga Stokes:

$$\int_S d\omega = \int_{\partial S} \omega \qquad \forall \omega \ (k-1)\text{-forma.}$$

Abbiamo già dimostrato che il teorema di Stokes vale per $\varphi : [0,1]^k \to \mathbb{R}^n$

$$\int_\varphi d\omega = \sum_{i=1}^{k} (-1)^{i-1} \left(\int_{\varphi \circ \psi_i^+} \omega - \int_{\varphi \circ \psi_i^-} \omega \right)$$

dove $\psi_i^\pm : [0,1]^{k-1} \to [0,1]^k \qquad i = 1 \dots k$

$$\psi_i^+(t_1, \dots, t_{k-1}) = (t_1, \dots, t_{i-1}, 1, t_i, \dots, t_{k-1})$$

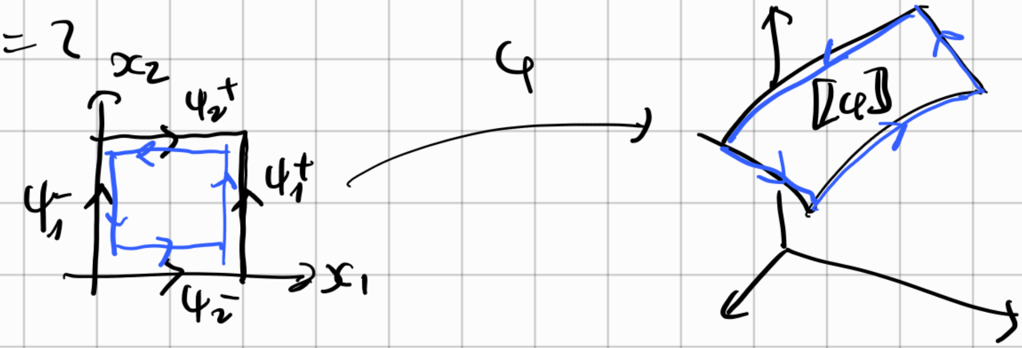
$$\psi_i^-(t_1, \dots, t_{k-1}) = (t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{k-1})$$

$$[\![\varphi]\!](d\omega) \doteq \int_{[\![\varphi]\!]} d\omega \doteq \int_\varphi d\omega = \sum_{i=1}^{k} (-1)^{i-1} \left([\![\varphi \circ \psi_i^+]\!] - [\![\varphi \circ \psi_i^-]\!] \right)$$

Dunque ha senso definire

$$\partial [\![\varphi]\!] =$$

ES $k=2$



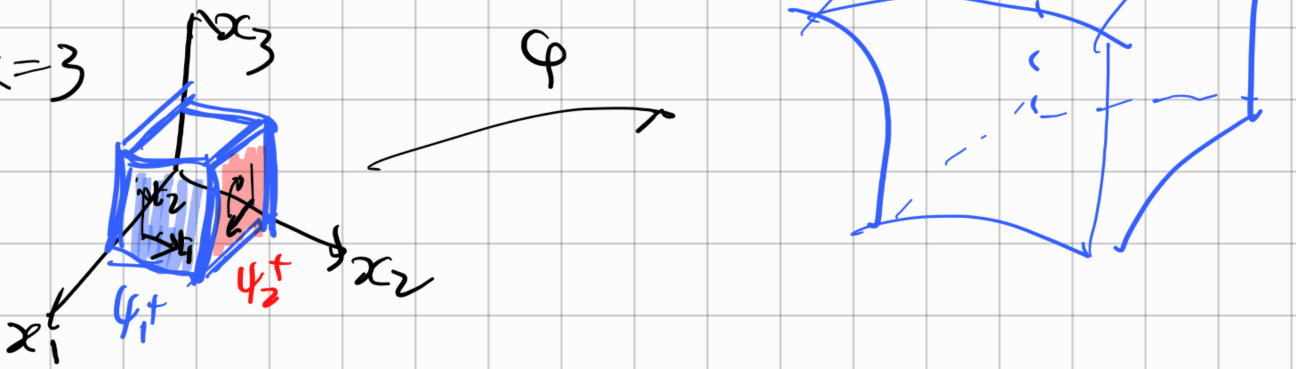
$\psi_1^+(t) = (1,t)$ | $\psi_2^+(t) = (t,1)$

$\psi_1^-(t) = (0,t)$ | $\psi_2^-(t) = (t,0)$

$$\partial[[\varphi]] = [[\varphi\circ\psi_1^+]] - [[\varphi\circ\psi_1^-]] - [[\varphi\circ\psi_2^+]] + [[\varphi\circ\psi_2^-]]$$

$=$ 

ES $k=3$



$$\psi_1^+(t_1,t_2) = (1,t_1,t_2)$$

è la faccie con $x_1 = 1$ orientata in senso antiorario se guardo il cubo dall'esterno.

$$\psi_2^+(t_1,t_2) = (t_1,1,t_2)$$

Def Una k-superficie orientata in $\mathbb{R}^n$ è (per noi)

una $S : \Omega^k(\mathbb{R}^n) \to \mathbb{R}$

tale che esistono $\varphi_1, \dots, \varphi_M : [0,1]^k \to \mathbb{R}^n$

$$S = [\![\varphi_1]\!] + \dots + [\![\varphi_M]\!]$$

Si definisce ∂S come:

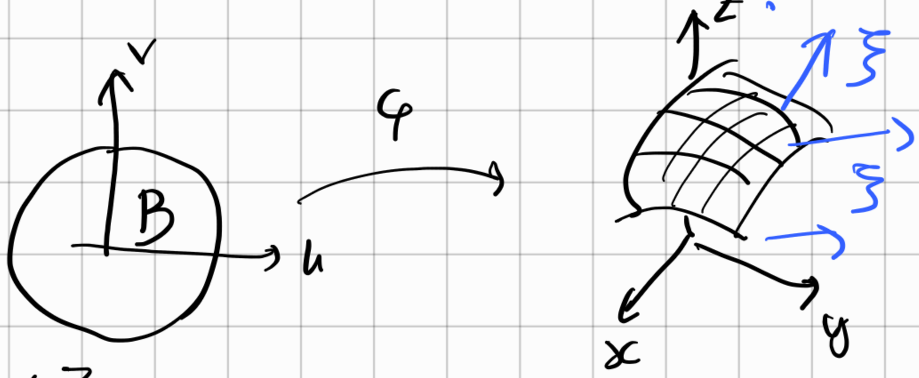
$$\partial S = \partial [\![\varphi_1]\!] + \dots + \partial [\![\varphi_M]\!].$$

Oss

$$\partial(\partial S) = 0 \qquad \text{perché} \qquad d^2\omega = 0$$

$$\int_{\partial(\partial S)} \omega = \int_{\partial S} d\omega = \int_S d(d\omega) = 0$$

Esercizio $\quad S : \begin{cases} x = u+v \\ y = v^2 \\ z = 1-u \end{cases} \qquad u^2+v^2 \leq 1$



$\xi = \begin{pmatrix} z \\ x \\ x^2 \end{pmatrix} \qquad \xi : \mathbb{R}^3 \to \mathbb{R}^3$

Verificare

$$\boxed{\begin{array}{ccc} \oint_{\partial S} \xi \cdot \tau_{\partial S}\, ds & = & \int_S \operatorname{rot} \xi \cdot \nu_S \, d\sigma \\ \| & & \| \\ \int_{\partial S} \omega & = & \int_S d\omega \end{array}}$$

∂S ← circuitazione

$\uparrow$ flusso

$$\xi \overset{?}{\longleftrightarrow} \omega$$

ω è una 1-forma in $\mathbb{R}^3$
$d\omega$ è una 2-forma in $\mathbb{R}^3$

$$\omega = \xi_1\, dx + \xi_2\, dy + \xi_3\, dz$$

$$d\omega \longleftrightarrow \operatorname{rot} \xi$$

$$d\omega = (\operatorname{rot}\xi)_1\, dy \wedge dz + (\operatorname{rot}\xi)_2\, dz \wedge dx + (\operatorname{rot}\xi)_3\, dx \wedge dy.$$

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1) $\boxed{\int_S d\omega}$ $\quad \xi = \begin{pmatrix} z \\ x \\ x^2 \end{pmatrix} \rightsquigarrow \omega = z\, dx + x\, dy + x^2 dz$

$$d\omega = dz \wedge dx + dx \wedge dy + 2x\, dx \wedge dz$$
$$= dx \wedge dy + (2x-1)\, dx \wedge dz$$

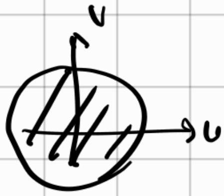
$$S: \begin{cases} x = u+v \\ y = v^2 \\ z = 1-u \end{cases} \qquad u^2+v^2 \le 1 \qquad (u,v)$$

$$\int_S d\omega = \int_S \left(dx \wedge dy + (2x-1)\, dx \wedge dz \right)$$

$$= \int_S \Big[(du + \cancel{dv}) \wedge (2v\, dv) + \big(2(u+v)-1\big)(\cancel{du} + dv) \wedge (-du) \Big]$$

$$= \int_S \Big[2v \; du \wedge dv + (2u+2v-1)\, du \wedge dv \Big]$$

$$= \int_S (2v + 2u + 2v - 1)\, du \wedge dv$$



$$= \int_{u^2+v^2 \le 1} (\cancel{4v} + \cancel{2u} - 1)\, du\, dv = - \text{Area del cerchio}$$

simmetria

$$= -\pi$$