ANALISI MATEMATICA B
LEZIONE 47 - 28.1.2026

Calcolo delle primitive

- teo fondamentale: se f è continua le primitive esistono
- cerchiamo dei "metodi di integrazione" che ci permettono di scrivere esplicitamente le primitive.

Integrali immediati

$x^\alpha \overset{D}{\longmapsto} \alpha x^{\alpha-1} \quad \leadsto \quad \int x^\alpha \, dx \ni \frac{x^{\alpha+1}}{\alpha+1} \quad$ se $\alpha \neq -1$

$\ln|x| \overset{D}{\longmapsto} \frac{1}{x} \quad \leadsto \quad \int \frac{1}{x} \, dx \ni \ln|x|$

$e^x \overset{D}{\longmapsto} e^x \qquad \int e^x \, dx \ni e^x$

$\sin x \overset{D}{\longmapsto} \cos x \qquad \int \cos x \, dx \ni \sin x$

$-\cos x \longmapsto \sin x \qquad \int \sin x \, dx \ni -\cos x$

$\operatorname{arctg} x \longmapsto \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2} \, dx \ni \operatorname{arctg} x$

$\arcsin x \longmapsto \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}} \, dx \ni \arcsin x$

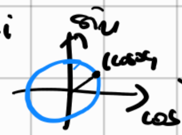
$\int \frac{1}{1-x^2} \, dx \; ? \qquad \int \frac{1}{\sqrt{1+x^2}} = ? \qquad \int \frac{1}{\sqrt{x^2-1}} = ?$

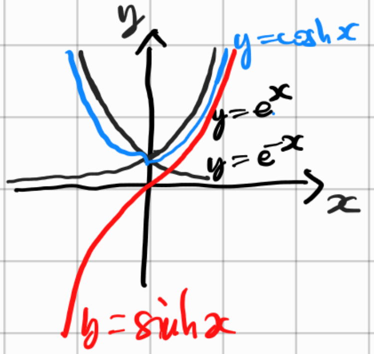
FUNZIONI IPERBOLICHE

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

[Ricordo: $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ $\qquad \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$\cos^2 x + \sin^2 x = 1$]

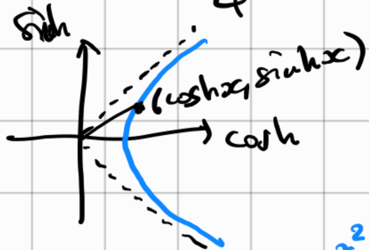


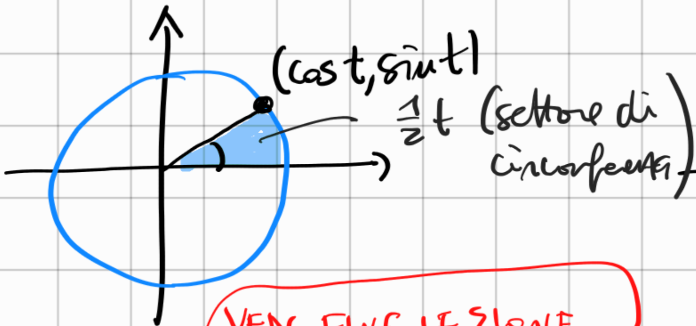


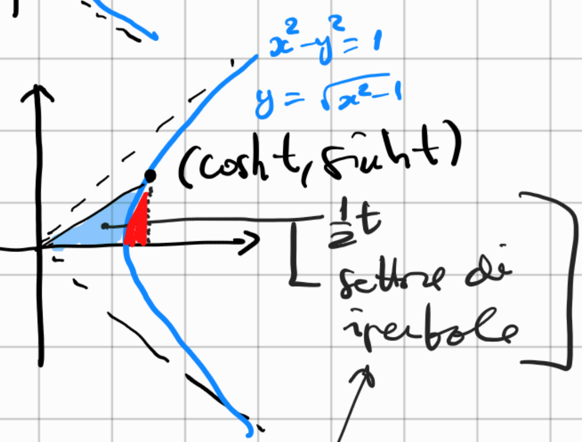
$$\cosh^2 x = \frac{e^{2x} + e^{-2x} + 2}{4}$$

$$\sinh^2 x = \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$\cosh^2 x - \sinh^2 x = 1$$







VEDI FINE LEZIONE

Esercizio ✓ dimostrare che l'area del settore di iperbole con estremo $(\cosh t, \sinh t)$ è proprio $\frac{t}{2}$.

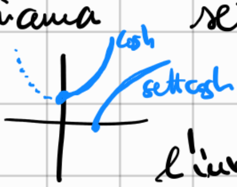
(Serve calcolare $\int \sqrt{x^2-1}\, dx$, lo vedremo più avanti)

$\sinh : \mathbb{R} \to \mathbb{R}$ è bigettiva

l'inversa si chiama $\text{sett}\sinh : \mathbb{R} \to \mathbb{R}$

$\cosh : [0, +\infty) \to [1, +\infty)$

è invertibile



l'inversa si chiama $\text{sett}\cosh : [1, +\infty) \to [0, +\infty)$

Esercizio Risolvere: $y = \sinh x = \dfrac{e^{x} - e^{-x}}{2} = e^{-x}\,\dfrac{(e^{x})^2 - 1}{2}$

$$x = \operatorname{settsinh} y = \ln\left(y + \sqrt{y^2+1}\right)$$

analogamente

$$\operatorname{settcosh} y = \ln\left(y + \sqrt{y^2-1}\right)$$

$$D \cosh x = D\,\frac{e^{x}+e^{-x}}{2} = \frac{e^{x}-e^{-x}}{2} = \sinh x$$

$$D \sinh x = D\,\frac{e^{x}-e^{-x}}{2} = \frac{e^{x}+e^{-x}}{2} = \cosh x$$

$$\begin{tikzcd} e^{x} \arrow[loop right, "D"] \end{tikzcd} \qquad \begin{tikzcd} \cosh x \arrow[d, bend left, "D"] \\ \sinh x \arrow[u, bend left, "D"] \end{tikzcd} \qquad \begin{tikzcd} & \sin x \arrow[dr, "D"] & \\ -\cos x \arrow[ur, "D"] & & \cos x \arrow[dl, "D"] \\ & -\sin x \arrow[ul, "D"] & \end{tikzcd}$$

$$D \operatorname{settsinh} x = \frac{1}{D \sinh(\operatorname{settsinh} x)} = \frac{1}{\cosh(\operatorname{settsinh} x)} =$$

derivata della fn. inversa

$$\left[\cosh^2 x = 1 + \sinh^2 x \qquad \cosh x = \pm\sqrt{1+\sinh^2 x}\right]$$

$$= \frac{1}{\sqrt{1+x^2}}$$

$$\boxed{\int \frac{1}{\sqrt{1+x^2}}\,dx = \operatorname{settsinh} x}$$

∂

$$D \operatorname{settcosh} x = \frac{1}{D\cosh(\operatorname{settcosh} x)} = \frac{1}{\sinh(\operatorname{settcosh} x)} =$$

$$\left[\sinh^2 x = \cosh^2 - 1 \qquad \sinh x = \pm\sqrt{\cosh^2 x - 1}\right]$$

$$= \frac{1}{\sqrt{x^2-1}}$$

$$\boxed{\int \frac{1}{\sqrt{x^2-1}}\,dx = \operatorname{settcosh} x}$$

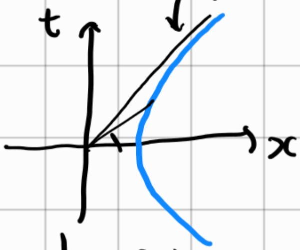
formule di addizione

$$\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta$$

↑ basta fare la verifica.

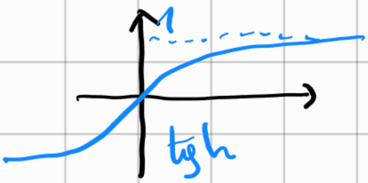
$$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta$$

$$\operatorname{tgh} x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



$$\begin{pmatrix} \cosh & -\sinh \\ \sinh & \cosh \end{pmatrix}$$

$$D \operatorname{tgh} x = \frac{\cosh x \cdot \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$



$$= 1 - \operatorname{tgh}^2 x$$

$$D \operatorname{setttgh} x = \frac{1}{1-\operatorname{tgh}^2(\operatorname{setttgh} x)} = \frac{1}{1-x^2}$$

$$\boxed{\int \frac{1}{1-x^2}\,dx = \operatorname{setttgh} x}$$

↑

(non servirà saperla a memoria)

Esercizio

$$\operatorname{setttgh} x = \ln\sqrt{\frac{1+x}{1-x}}$$

REGOLE DI INTEGRAZIONE

LINEARITA'

$c \in \mathbb{R}$

$$D(f+g) = Df + Dg, \qquad Dc \cdot f = c\, Df$$

$$\int f+g = \int f + \int g \qquad \int cf = c \int f$$

Esempio

$$\int (2x-1)^2\, dx = \int (4x^2 - 4x + 1)\, dx$$

$$= 4 \int x^2\, dx - 4 \int x\, dx + \int 1\, dx$$

$$= 4 \cdot \frac{x^3}{3} - 4 \frac{x^2}{2} + x \qquad (+c)$$

2

CAMBIO DI VARIABILE

$$\frac{d}{dx} F(g(x)) = F'(g(x))\, g'(x)$$

$$\int F'(g(x))\, g'(x)\, dx = F(g(x))$$

MNEMONICA
$$\boxed{\begin{cases} y = g(x) \\ dy = g'(x)\, dx \end{cases}}$$

$$\int F'(g(x))\, g'(x)\, dx = \int F'(y)\, dy = \Big[F(y)\Big]_{y=g(x)}$$

$$= F(g(x))$$

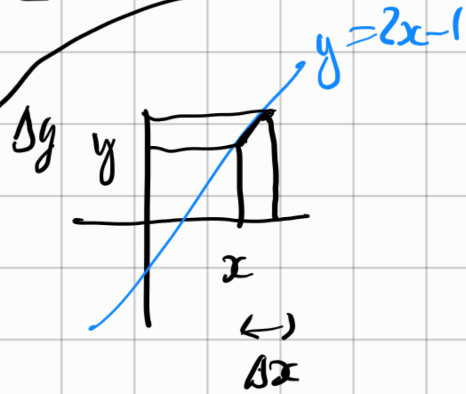
$f = F'$ $\qquad F = \int f$

sostituzione diretta

$$\boxed{\int f(g(x))\, g'(x)\, dx = \int f(y)\, dy} = \left[\int f\right]_{y=g(x)}$$

$$\int (2x-1)^2\, dx = \frac{1}{2} \int \underbrace{(2x-1)^2}_{y^2}\, \underbrace{2\, dx}_{dy} =$$

$$\begin{cases} y = 2x-1 \\ dy = 2 \cdot dx \end{cases}$$



$$= \frac{1}{2}\left[\int y^2\, dy\right]_{y=2x-1} = \frac{1}{2}\left[\frac{y^3}{3}\right]_{y=2x-1}$$

$$\frac{\Delta y}{\Delta x} = y'$$

$$\Delta y = y' \cdot \Delta x$$

$$= \frac{1}{2}\,\frac{(2x-1)^3}{3} = \frac{1}{6}(2x-1)^3$$

(confronta esercizio precedente)

NOTA che differiscono per una costante!

SOSTITUZIONE INVERSA

$$\boxed{\int f(x)\,dx = \int f(g(y))\,g'(y)\,dy}$$

$$\begin{cases} x = g(y) \\ dx = g'(y)\,dy \end{cases}$$

Esercizio

$$\int (2x-1)^2\,dx = \int y^2\,\frac{1}{2}\,dy = \frac{1}{2}\,\frac{y^3}{3}$$

$$y = 2x-1$$

$$\begin{cases} x = \dfrac{1+y}{2} \\ dx = \dfrac{1}{2}\,dy \end{cases}$$

Esercizio

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx =$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \int \frac{1}{2}\,dx + \frac{1}{2}\int \cos 2x\,dx =$$

$$\left[\int \cos 2x\,dx = \int \cos y\,\frac{1}{2}\,dy = \frac{1}{2}\sin y = \frac{1}{2}\sin 2x\right]$$

$$y = 2x \qquad x = \frac{y}{2} \qquad dx = \frac{1}{2}\,dy$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x = \frac{x}{2} + \frac{1}{2}\sin x \cos x \qquad \square$$

Esercizio

$$\int \sqrt{1-x^2}\, dx = ?$$

$x = \sin t$

$dx = \cos t \, dt$

$t = \arcsin x$

$$\int \sqrt{1-x^2}\, dx = \int \sqrt{1-\sin^2 t} \cdot \cos t \, dt$$

$$= \int \cos t \cdot \cos t \, dt$$

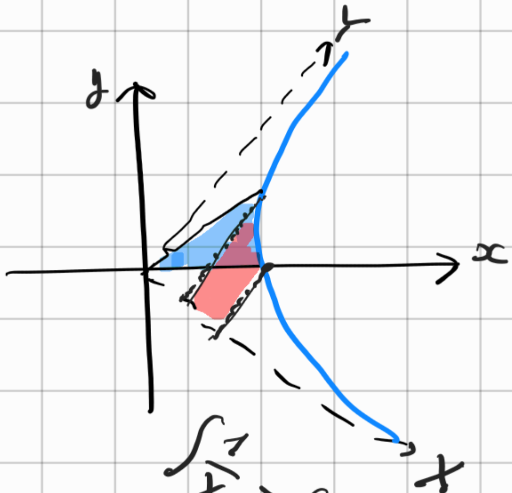
$$= \int \cos^2 t \, dt \underset{\text{ESERCIZIO PRECEDENTE}}{=}$$

$$= \frac{t}{2} + \frac{1}{2}\sin t \cos t = \frac{1}{2}\arcsin x + \frac{1}{2} x \sqrt{1-x^2}$$

$$\begin{cases} t = \arcsin x \\ x = \sin t \\ \cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2} \end{cases}$$

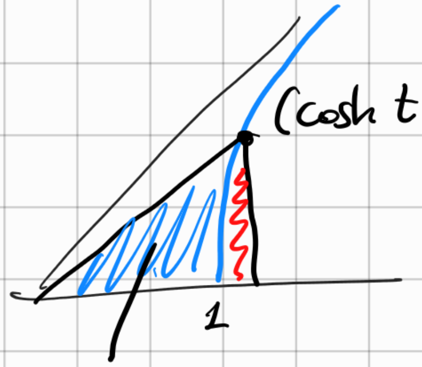
Esercizio $\square$

$$\int \sqrt{1+x^2}\, dx = ?$$

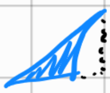


$$\int \frac{1}{x} = \ln x$$

$x^2 - y^2 = 1$



$\sinh 2t = 2 \sinh t \cosh t$

 $=$  $-$  $= \dfrac{\cosh t \cdot \sinh t}{2} - \displaystyle\int_{\cosh 0}^{\cosh t} \sqrt{x^2-1}\; dx$

$$= \frac{1}{4} \sinh 2t - \left[\frac{1}{4} \sinh 2t - \frac{1}{2} t \right]_0^t$$

$$= \cancel{\frac{1}{4} \sinh(2t)} - \cancel{\frac{1}{4} \sinh 2t} + \frac{1}{2} t = \frac{1}{2} t.$$