Teoria dei nodi 3/11/16

$$F \xrightarrow{i} T \xrightarrow{p} B$$

$$\pi_n(F) \xrightarrow{i_n} \pi_n(T) \xrightarrow{p_n} \pi_n(B) \xrightarrow{d_n} \pi_{n-1}(F)$$

$$d_n([\alpha]) = [\tilde{\alpha}|_{S^{n-1}}]$$

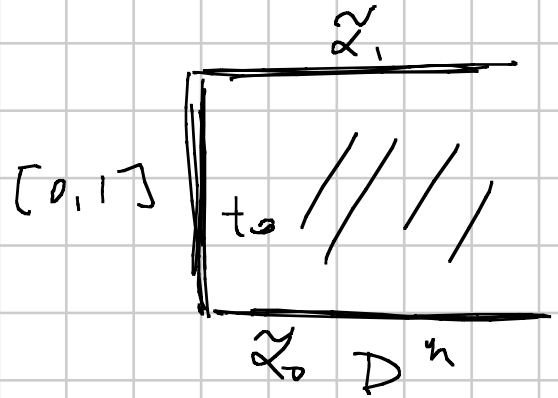
$$\begin{array}{ccc} D^n & \xrightarrow{\tilde{\alpha}} & T \\ \downarrow q & & \downarrow p \\ S^n & \xrightarrow{\alpha} & B \end{array}$$

(0) d_n ben def : $A : S^n \times [0,1] \to B$

$A_0 = \alpha_0 \quad A_1 = \alpha_1$

Considero $\tilde{\alpha}_0, \tilde{\alpha}_1$ sollevamenti a $D^n \to T$

Estendo a $D^n \times [0,1]$



...

$$\pi_n(F) \xrightarrow{i_n} \pi_n(T) \xrightarrow{p_n} \pi_n(B)$$

(1) $\mathrm{Im}(i_n) \subset \ker(p_n)$

$\gamma : S^n \to F$ $\quad i \circ \gamma = \gamma$ $\quad p \circ i \circ \gamma \equiv b_0$

$\Rightarrow p_n(i_n([\gamma])) = 0$

(2) $\ker(p_n) \subset \mathrm{Im}(i_n)$ $\quad \beta : S^n \to T$ $\quad p_n([\beta]) = 0$

$\Rightarrow \exists u \quad S^n \times [0,1] \to T \quad u_0 = p \circ \beta \quad u_1 \equiv b_0$

solleva:

$$\begin{array}{ccc} D^n \times [0,1] & \xrightarrow{\tilde{u}} & T \\ \downarrow q \times \mathrm{id} & & \downarrow \\ S^n \times [0,1] & \xrightarrow{u} & B \end{array}$$

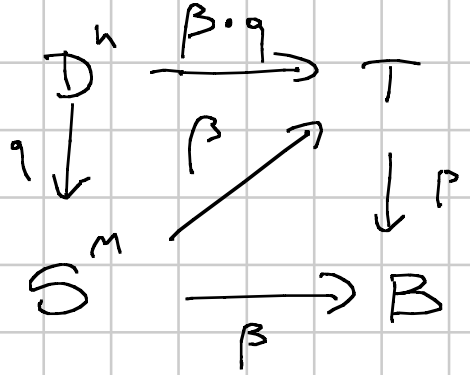
$\tilde{u}_0 = \beta$

$\tilde{u}_1(D^n) \subset p^{-1}(b_0)$

$[\beta] = i_n([\tilde{u}_1])$

$$\pi_n(T) \xrightarrow{p_n} \pi_n(B) \xrightarrow{d_n} \pi_{n-1}(F)$$

(3) $\mathrm{Im}(p_n) \subset \mathrm{Ker}(d_n)$



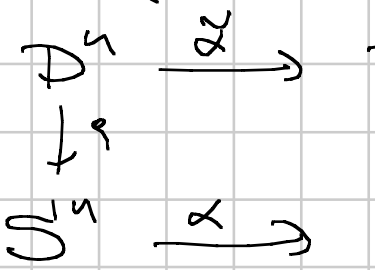
$\beta : S^n \to T$

$\beta \circ q$ solleva $p \circ \beta$

ma $\beta \circ q|_{D^n} \equiv t_0$

$\Rightarrow d_n(p_n([\beta])) = 0$.

(4) $\mathrm{Ker}(d_n) \subset \mathrm{Im}(p_n)$



Supponiamo che $d_n([\alpha]) = 0$

$\tilde{\alpha}|_{S^{n-1}}$ omotopa a cost in F

$\Rightarrow \exists\, u : D^n \to F$

t.c. $u|_{S^{n-1}} = \tilde{\alpha}$

$\beta = \tilde{\alpha} \cup_{\partial} u : S^n \to T$

$p \circ \beta = \alpha \cup_2 (\equiv b_0) \Rightarrow [\alpha] = p_n([\beta])$

$\pi_n(B) \xrightarrow{d_n} \pi_{n-1}(F) \xrightarrow{i_{n-1}} \pi_{n-1}(T)$

(5) $\mathrm{Im}(d_n) \subset \ker(i_{n-1})$

$$\begin{array}{ccc} D^n & \xrightarrow{\tilde{\alpha}} & T \\ \downarrow q & & \downarrow p \\ S^n & \xrightarrow{\alpha} & B \end{array}$$

$d_n([\alpha]) = [\tilde{\alpha}|_{S^{n-1}}]$

a valori in T la classe $\tilde{\alpha} : D^n \to T$

prove che $i_{n-1}([\tilde{\alpha}|_{S^{n-1}}]) = 0 \in \pi_{n-1}(T)$

(6) $\ker(i_{n-1}) \subset \mathrm{Im}(d_n)$

$\gamma : S^{n-1} \to T$ $\quad i_{n-1}([\gamma]) = 0$

$\Rightarrow \exists u : D^n \to T$ t.c. $u|_{S^{n-1}} = \gamma$

$$\begin{array}{ccc} D^n & \xrightarrow{u} & T \\ \downarrow q & & \downarrow p \\ S^n & \xrightarrow{\alpha} & B \end{array}$$

· $p \circ u \,|_{S^{n-1}} = p \circ \gamma \equiv b_0$

è ben def α.

per costruzione $[\gamma] = d_n([\alpha])$ _

$M^{(m)}$, $m \geq 2$

$$\Omega_0^n(M) = \{ (x_1, \dots, x_n) \in M^n : x_i \neq x_j \ \forall i \neq j \}$$

$$\Lambda_0^n(M) = \Omega_0^n(M) / \mathfrak{S}_n$$

$B_n(M) = \pi_1(\Lambda_0^n)$ $\qquad$ $P_n(\mathbb{R}^n) = \pi_1(\Omega_0^n)$

($M = \mathbb{R}^2$: sono i classici) –

$$\Omega_p^n = \Omega_0^n(M \setminus (p \text{ fissati punti}))$$

indipendente dalla scelta dei p'i

<u>Prop</u>: esiste una fibrazione loc. banale se $r < n$

$$\Omega^{n-r}_{p+r} \hookrightarrow \Omega^n_p(M) \xrightarrow{q} \Omega^r_p(M)$$

Dim: $q(x_1, \dots, x_n) = (x_1, \dots, x_n)$

$$\Omega_p^n(M) \simeq \Omega_0^n(M \setminus \{\bar{x}_1, \dots, \bar{x}_p\})$$

$$\Omega_p^n(M) = \Omega_0^n(M \setminus \{\bar{x}_1, \dots, \bar{x}_p\})$$

$$q^{-1}(y_1, \dots, y_n) = \{(y_1, \dots, y_n, x_{n+1}, \dots, x_n) : x_j \neq \bar{x}_1, \dots, \bar{x}_p, y_1, \dots, y_n\}$$

$$= \Omega_{p+n}^{m-n}(M)$$

Esercizio: esibire banalizzazioni locali.

Cor: Se $\pi_2(M \setminus (p\ p+1)) = \pi_3(M \setminus (p\ p+1)) = 0 \quad \forall p \geq 1$

$\Rightarrow \pi_2(\Omega_0^n(M)) = 0 \quad \forall n \geq 1$

Dim: Uso $\Omega_{p+1}^{n-1}(M) \hookrightarrow \Omega_p^n(M) \to \Omega_p^1(M) = M \setminus (p\ p+1)$

$\Rightarrow \pi_3(M \setminus (p\ p+1)) \to \pi_2(\Omega_{p+1}^{n-1}) \to \pi_2(\Omega_p^n) \to \pi_2(M \setminus p\ p+1)$

$\pi_3(M \setminus (p\ p+1)) = 0$ e $\pi_2(M \setminus p\ p+1) = 0$

$$\pi_2(\Omega_p^n) = \pi_2(\Omega_{p+1}^{n-1}) = \cdots = \pi_2(\underbrace{\Omega_{p+n-1}^1}_{M \setminus (p+n-1\ p+1)}) = 0$$

□

Prop: nelle stesse ipotesi ho successione esatta

$$1 \to \pi_1(M \setminus (n-1 \text{ pti})) \to \Gamma_n(M) \to \Gamma_{n-1}(M) \to 1$$

Dim: uso $\Omega^1_{n-1}(M) \hookrightarrow \Omega^n_0(M) \to \Omega^{n-1}_0(M)$

$\Omega^1_{n-1}(M) = M \setminus (n-1 \text{ pti})$

$$\pi_2(\Omega^{n-1}_0) \to \pi_1(M \setminus (n-1 \text{ pti})) \to \pi_1(\Omega^n_0) \to \pi_1(\Omega^{n-1}_0) \to \pi_0(M \setminus (n-1 \text{ pti}))$$

$$\pi_2(\Omega^{n-1}_0) = 1 \qquad \pi_1(\Omega^n_0) = \Gamma_n \qquad \pi_1(\Omega^{n-1}_0) = \Gamma_{n-1} \qquad \pi_0(M \setminus (n-1 \text{ pti})) = 1$$

□

Prop: se $\tilde{X} \xrightarrow{p} X$ rivestimento allora $p_n : \pi_n(\tilde{X}) \to \pi_n(X)$ è isomorfismo $\forall n \geq 2$.

Dim: $[\alpha] \in \pi_n(X)$ $\alpha : S^n \to X$ $n \geq 2 \Rightarrow \pi_1(S^n) = 1$

$\Rightarrow \exists \tilde{\alpha} : S^n \to \tilde{X}$; $j_n([\alpha]) = [\tilde{\alpha}]$

ben definito perché anche le omotopie si sollevano.

$j_n = p_n^{-1}$.

□

Oss: non è vero che se K è un complesso simpliciale / cellulare di $\dim p$ allora $\pi_n(K) = 0$ per $n > p$.

Ad es. $\pi_3(S^2) = \mathbb{Z}$.

Oss: se M è superficie opt allora

$M \setminus (n \text{ pti})$ ($n \geq 1$ se chiusa, $n \geq 0$ se con ∂)

si retrae su un grafo Γ; ora $\tilde{\Gamma}$ è contraibile

$\Rightarrow \pi_n(M \setminus (n \text{ pti})) = 0 \quad \forall n \geq 2$

Con: per $M = \mathbb{R}^2$ troviamo

$$1 \longrightarrow \pi_1(\mathbb{R}^2 \setminus (n-1 \text{ pti})) \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow 1$$

$$\| \wr$$

$$\mathbb{Z}^{*(n-1)}$$

Fatto: $\exists\, i_n : P_{n-1} \to P_n$ che solleva $P_n \to P_{n-1}$

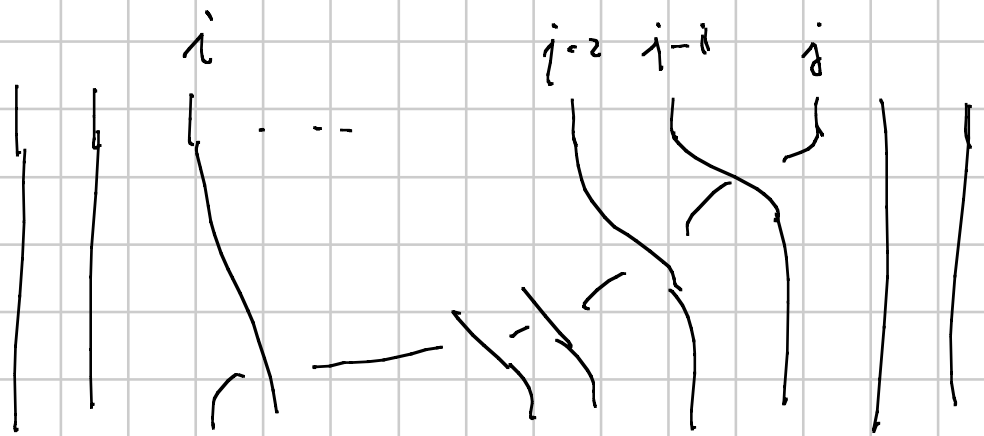
$$\Rightarrow P_n = \mathbb{Z}^{*(n-1)} \rtimes P_{n-1}$$

Ricordo: $P_n \lhd B_n$ $\qquad P_n = \ker(B_n \twoheadrightarrow S_n)$

$\Rightarrow B_n / P_n \cong S_n$

Esibisco un insieme di rappresentanti:

$$\gamma_{ji} = \sigma_{j-1}\,\sigma_{j-2} \cdots \sigma_i \qquad 1 \le i < j \le n \quad (\sigma_{ii} = id)$$



Fatto: $\mathcal{R} = \left\{ \prod_{j=1}^{n} \tau_{j,i_j} : 1 \le i_j \le j \qquad j = 1 \dots n \right\}$

è un insieme di rappresentanti per B_n / I_n :

ce ne sono $n! = 1 \cdot 2 \cdots n$. Per $n = 3$:

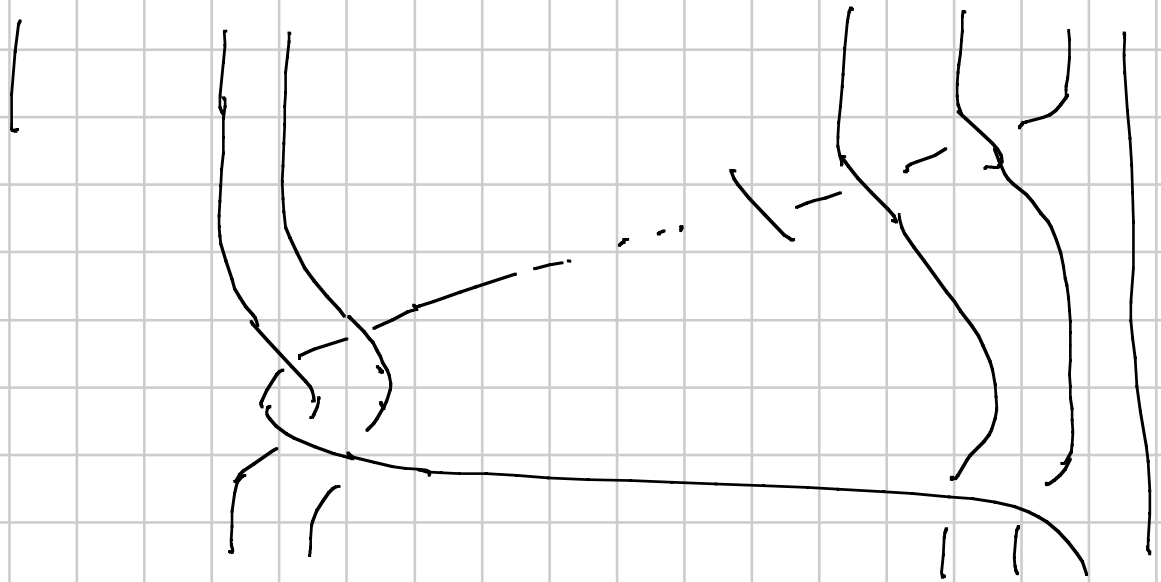
$\rho_n(\tau_{11}\tau_{22}\tau_{33}) = id$
$\rho_n(\tau_{11}\tau_{22}\tau_{32}) = (23)$
$\rho_n(\underbrace{\tau_{11}\tau_{22}\tau_{31}}_{\sigma_2\sigma_1}) = (23)(12) = (132)$

$\rho_n(\tau_{11}\tau_{21}\tau_{33}) = \rho_n(\sigma_1) = (12)$
$\rho_n(\tau_{11}\tau_{21}\tau_{32}) = (12)(23) = (123)$
$\rho_n(\tau_{11}\tau_{21}\tau_{31}) = (12)(23)(12) = (13)$.

——— ∘ ———

Def : $\tau_{ij} = \sigma_{j-1}\,\sigma_{j-2}\,\cdots\,\sigma_{i+1}\,\sigma_i^{2}\,\sigma_{i+1}^{-1}\,\cdots\,\sigma_{j-1}^{-1}$

Obs : $\tau_{ij} \in P_n$, $\rho_n(\tau_{ij}) = (\;)(\;)(\;)(ij)^2(\;)^{-1}(\;)^{-1} = id$



Fatto: T_{ij} $1 \le i < j \le n$ generano P_n

$i_n : P_{n-1} \longrightarrow P_n$ inclusione

$k_n : P_n \longrightarrow P_{n-1}$ della mcc. esatta $\tilde{t}$

$$\alpha_n(T_{ij}) = T_{ij} \quad \text{se } 1 \le i < j \le n-1$$

$$\alpha_n(T_{in}) = 1 \quad \forall i < n$$

induttivamente ciò dà presentazione di P_n —

Prop: $Z(B_n) = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n$

$= \tau_{12} \cdot \tau_{12}\tau_{23} \cdot \tau_{14}\tau_{24}\tau_{34} \cdots$

(No dimo).

Oss: (1) $Z(B_n) < P_n$ poiché $z \in Z(B_n)$

$\rho_n(z) \in Z(\mathfrak{S}_n) = \{1\} \Rightarrow z \in P_n \quad (n \geq 3)$

(2) $(\sigma_1 \cdots \sigma_{n-1})^n \in P_n$: $\rho_n(\ ) = (1\,2 \cdots n) = id$

(3) verificare che vale l'$=$

(4) $(\sigma_1 \cdots \sigma_{n-1})^n$ è centrale ; per $n=3$

$$(\sigma_1\sigma_2)^3\sigma_1 = \sigma_1 \underbrace{\sigma_2 \sigma_1 \sigma_2}\underbrace{\sigma_1 \sigma_2}\sigma_1$$

$$= \sigma_1\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 = \sigma_1(\sigma_1\sigma_2)^3$$

stesso con σ_2 –

Prop: è ben def una rappresentazione

$$\phi : B_n \longrightarrow Aut(F_n) \qquad F_n = \mathbb{Z}^{*n} = \langle x_1 \cdots x_n | - \rangle$$

$$\phi(\sigma_i) : x_j \longmapsto \begin{cases} x_i x_{i+1} x_i^{-1} & j = i \\ x_i & j = i+1 \\ x_j & j \neq i, i+1 \end{cases}$$

Dim: provo che sono rispettate le relazioni

$\phi(\sigma_i) \circ \phi(\sigma_j) = \phi(\sigma_i) \circ \phi(\sigma_j)$ $\quad |i-j| \geq 2$ facile

$\phi(\sigma_i) \circ \phi(\sigma_{i+1}) \circ \phi(\sigma_i) = \phi(\sigma_{i+1}) \circ \phi(\sigma_i) \circ \phi(\sigma_{i+1})$:

f $\qquad\qquad$ g

Facile: $f(x_j) = g(x_j)$ per $j \neq i, i+1, i+2$

$f: x_i \longmapsto x_i x_{i+1} x_i^{-1} \longmapsto x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} \longmapsto x_i x_{i+1} x_i^{-1} x_i x_{i+2}$
$x_i^{-1} x_i x_{i+1}^{-1} x_i^{-1}$
$= x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1}$

$g: x_i \longmapsto x_i \longmapsto x_i x_{i+1} x_i^{-1} \mapsto x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1}$ OK

Analogamente per x_{i+1} e x_{i+2}. ☒

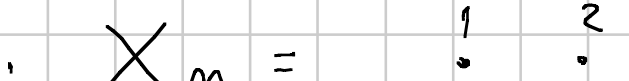
Fatto: tale Φ è iniettiva (fedele)

Non lo provo; verifico che $\ker(\Phi) < P_n$: infatti è ben def

$$\bar{\phi} : B_n \longrightarrow Aut\left(\mathbb{Z}^n = F_n/[F_n, F_n]\right)$$

$$\bar{\phi}(\sigma_i): \quad \begin{aligned} x_i &\longmapsto x_{i+1} \\ x_{i+1} &\longrightarrow x_i \\ x_j &\longmapsto x_j \end{aligned} \qquad \Rightarrow \bar{\phi} = \rho_n .$$

Interpretazione geometrica di ϕ

Ricordo : $X_n =$  $\subset D^2$

$$B_n = \left\{ f : D^2 \to D^2 \text{ omeo} : f(X_n) = X_n \right\} \Big/ \substack{\text{omotopia tra} \\ \text{tali } f}$$
$$f|_{\partial D^2} = id$$

Corrispondenza con treccia geom: data f

considero $(f_t)_{t \in [0,1]}$ $\quad f_t : D^2 \to D^2 \quad f_t|_{\partial D^2} = id$

con $f_0 = id$ e $f_1 = f$; la treccia che corrisponde a f è

$$\mathrm{Im}\left(t \mapsto (f_t(X_n), t) \right)$$

$\phi : B_n \to Aut(F_n)$ si interpreta così:

$f : (D^2, X_n) \to (D^2, X_n)$

$\Rightarrow f| : D^2 \setminus X_n \to D^2 \setminus X_n$

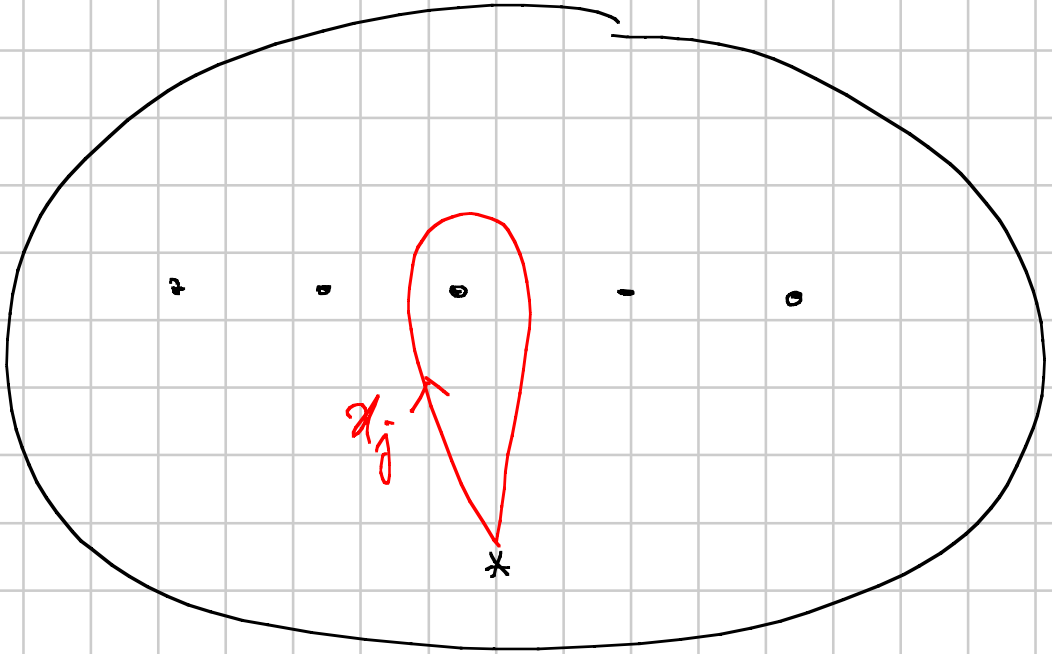
$\Rightarrow (f|)_* : \pi_1(D^2 \setminus X_n) \circlearrowleft$

$\parallel$

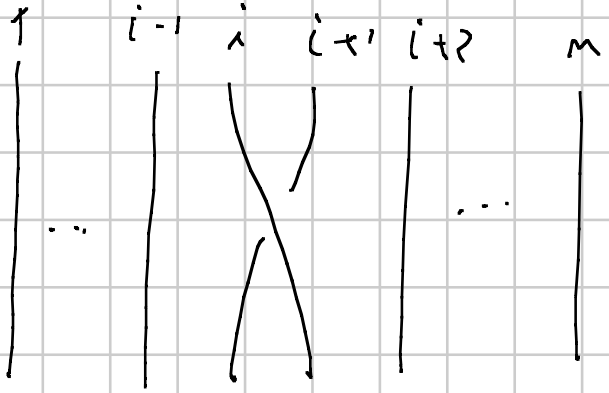
F_n

Provo che $\phi([f]) = (f|_\cdot)_* \in Aut(F_n)$:

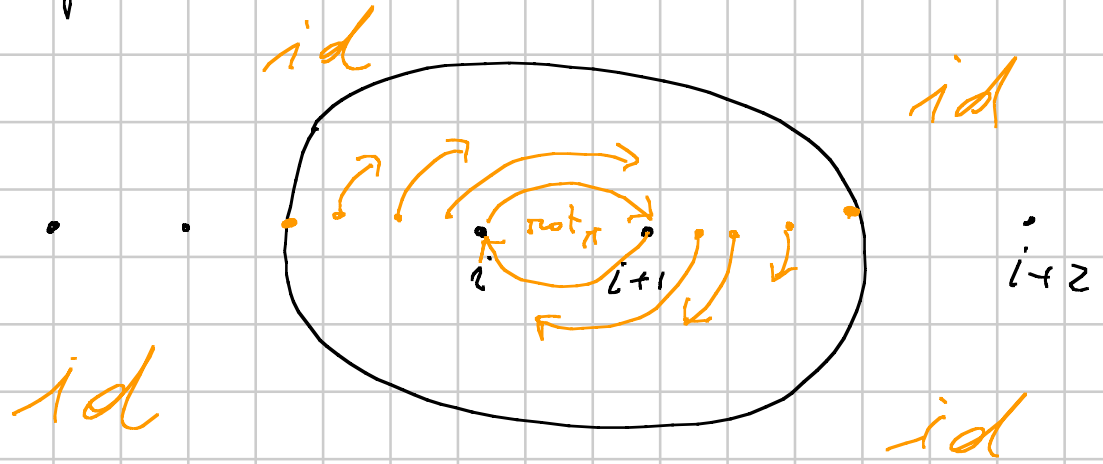
$$\pi_1(D^2 \setminus X_n) = \langle x_1, \dots, x_n \mid \rightarrow \rangle$$



La $f \in Aut(D^2)$ che corrisponde a σ_i è:

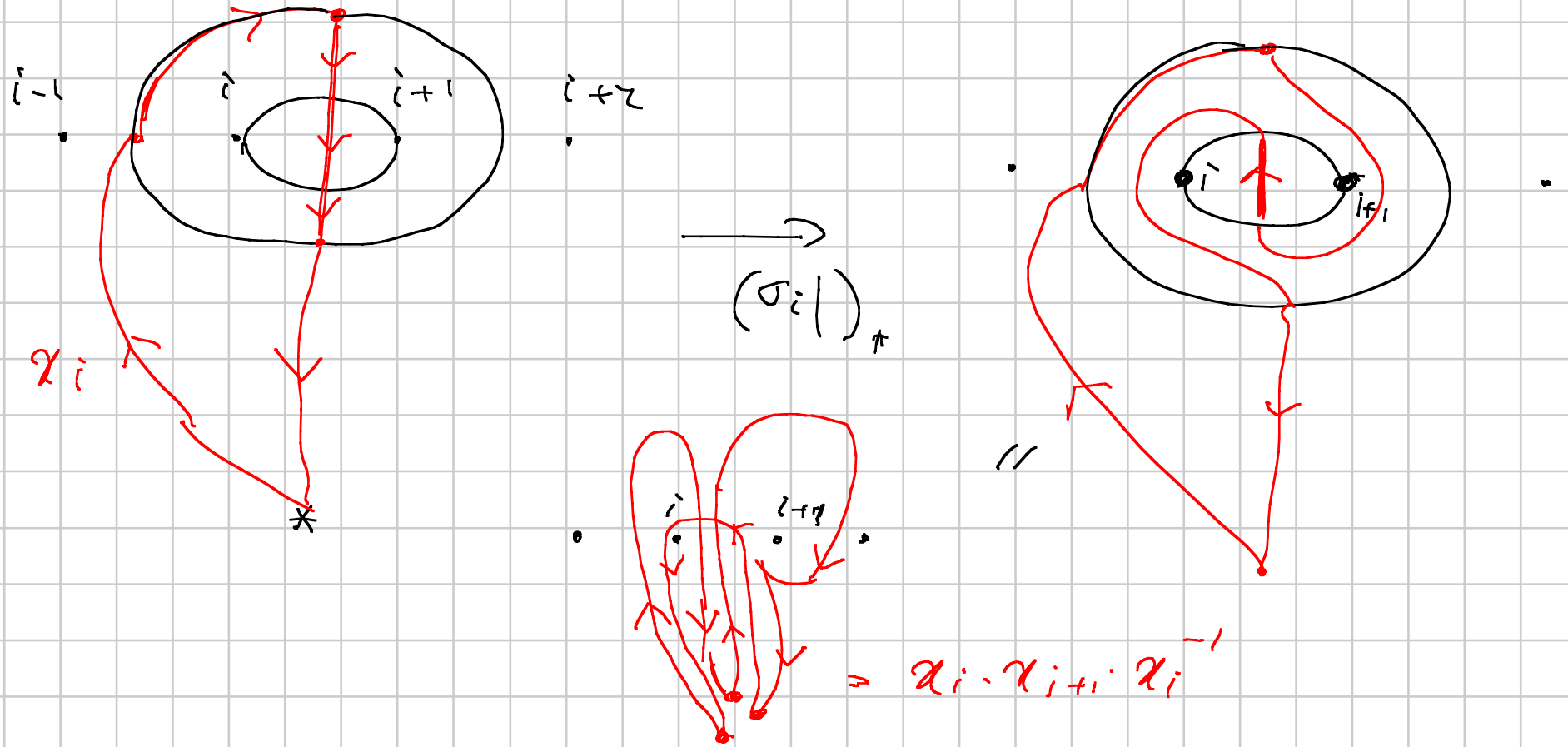


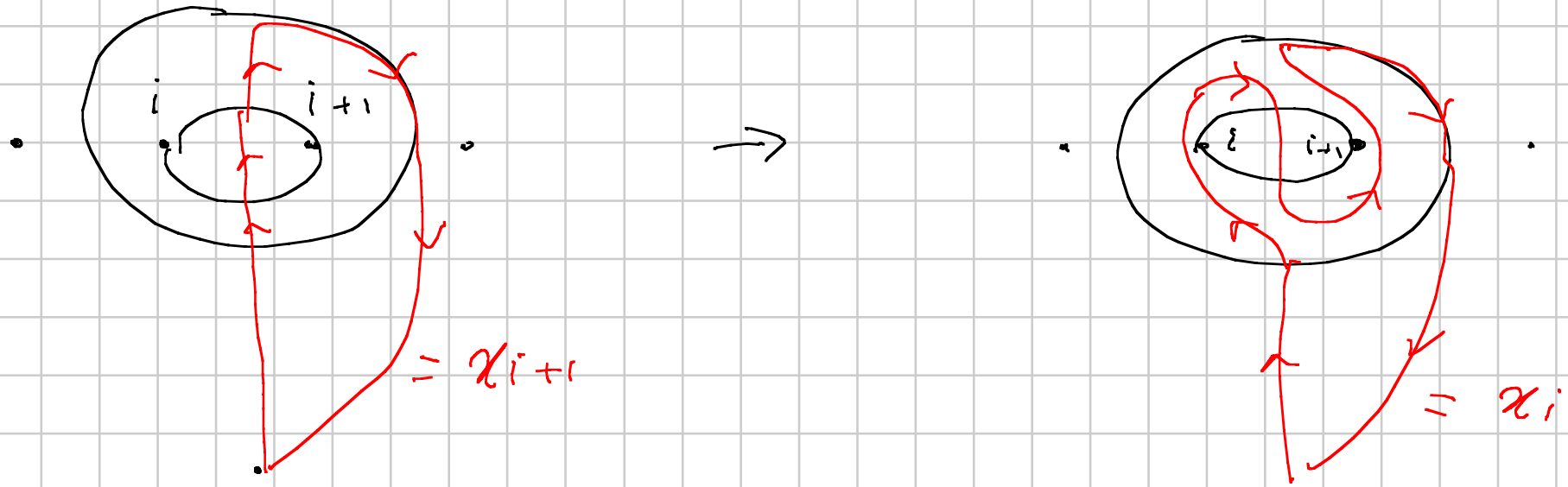
$=$



Azione sul cammino x_j è x_j se $j \neq i, i+1$

Provo che su i e $i+1$ l'azione è quella di $\phi(\sigma_i)$ definita algebricamente.


i-1
i
i+1
i+2
γ_i
$(\sigma_i|)_*$
i
$i+1$
=
$\gamma_i \cdot \gamma_{i+1} \cdot \gamma_i^{-1}$


i
i+1
= x_{i+1}
i
i-1
= x_i