# Teoria dei nodi 6/10/16

$u(D) = \min\{n : \text{con } n \text{ scambi di incrocio } D \text{ diventa diag. di } \bigcirc\}$

$u(K) = \min_{D \text{ diag di } K} u(D)$ numero unknotting

$c(D) = \#$ incroci di diag $D$

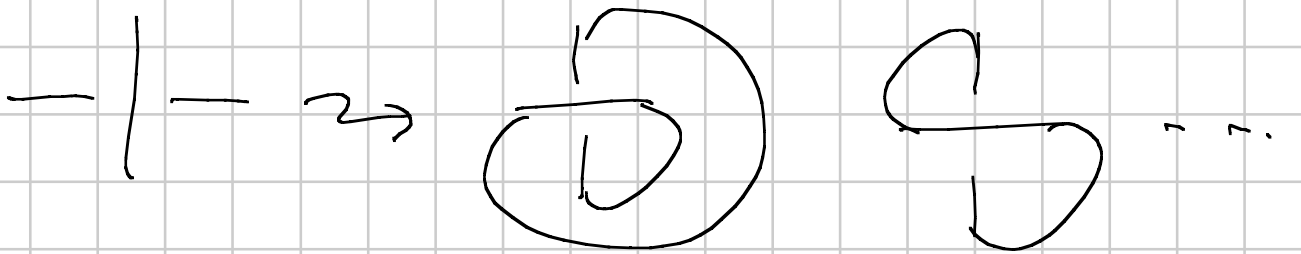
$c(K) = \min_{D \text{ diag di } K} c(D)$

Fatto: ci sono casi in cui il $D$ che dà il minimo per $u(K)$ non dà il minimo per $c(K)$.

Dunque si trova $D$ con $u(D)=1$ e so che $K$ è non banale, concludo che $u(K)=1$; se invece ho $u(D)\geq 2$ con $D$ che realizza $c(K)$ non posso concludere che $u(K)=u(D)$.

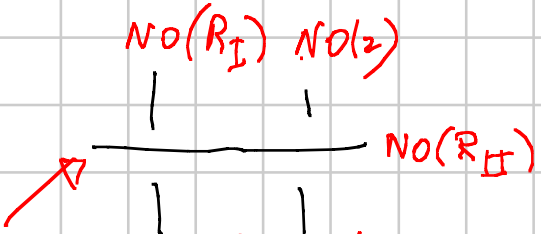
Elenco dei link con $c \leq 4$ escluso componenti banali

$c = 1$

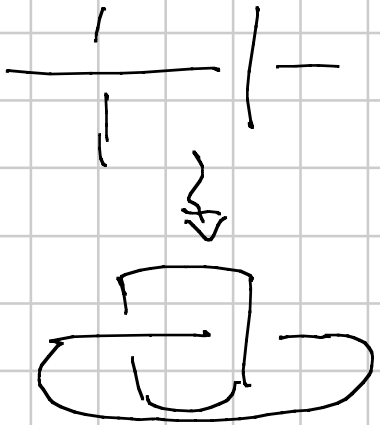


... tutti banali

$c = 2$



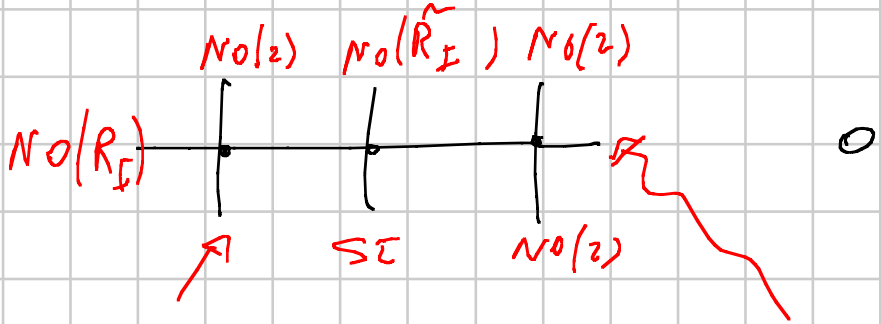

niente



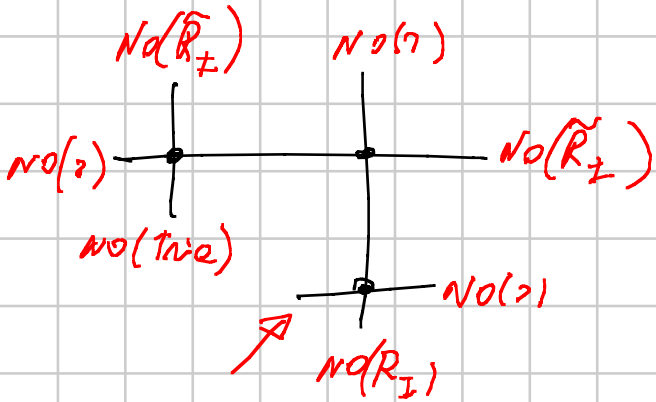
$C = 3$ (i) nessuna regione triangolare

(ii) $\exists$ regione triangolare

(i)

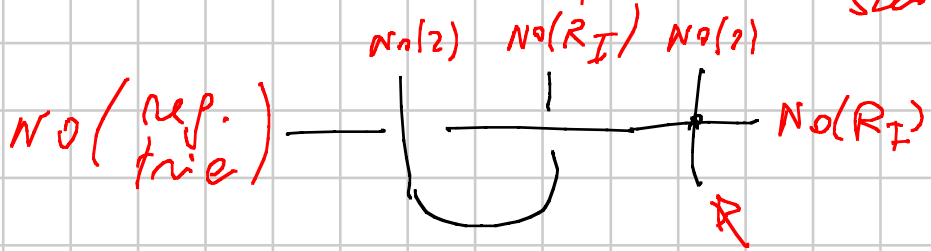



o

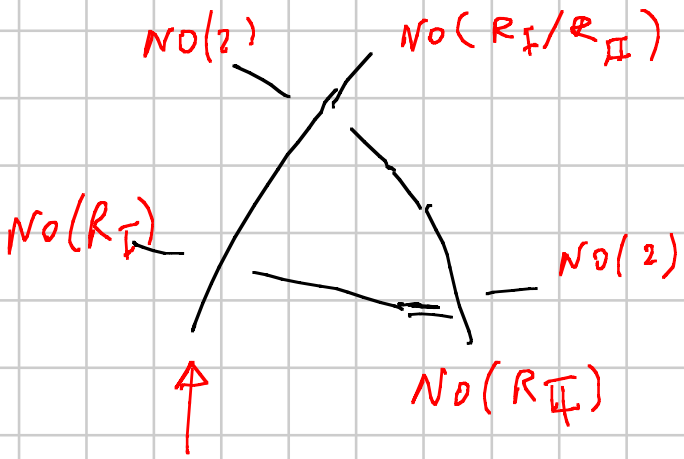


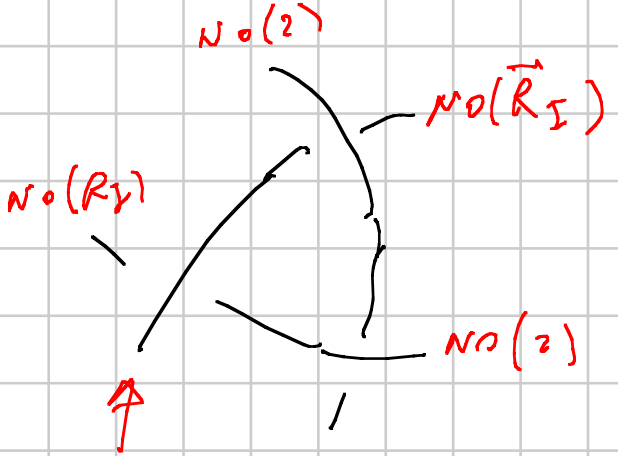

implica

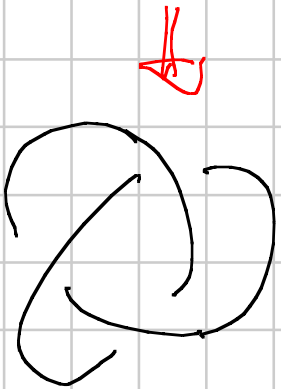
stesso/simile



(ii)

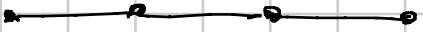


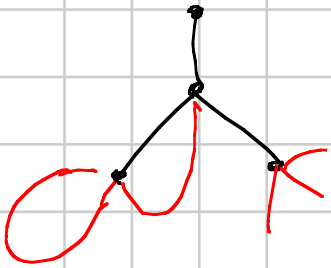



trifoglio

c = 4 . Possibili alberi max





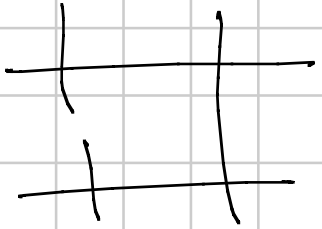
implica che c'è anche l'altro

1

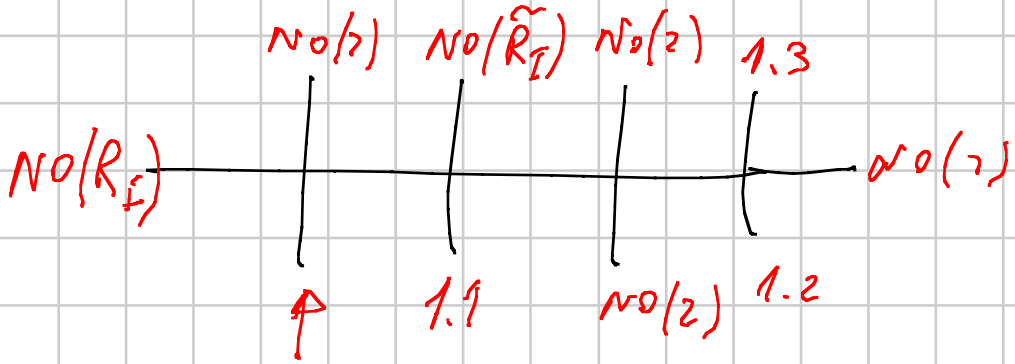
2

3.

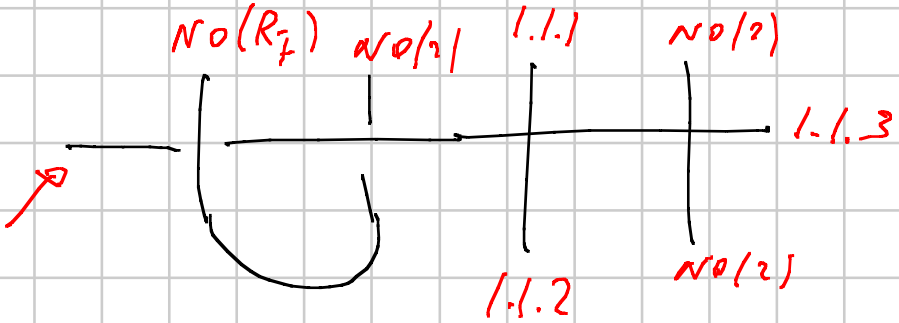
4.



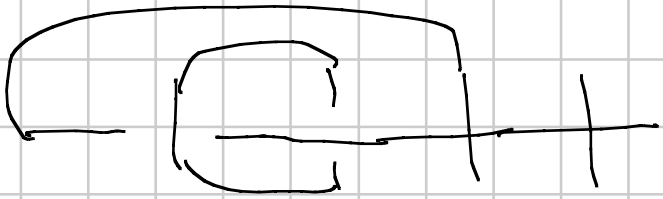
1.




1.1.

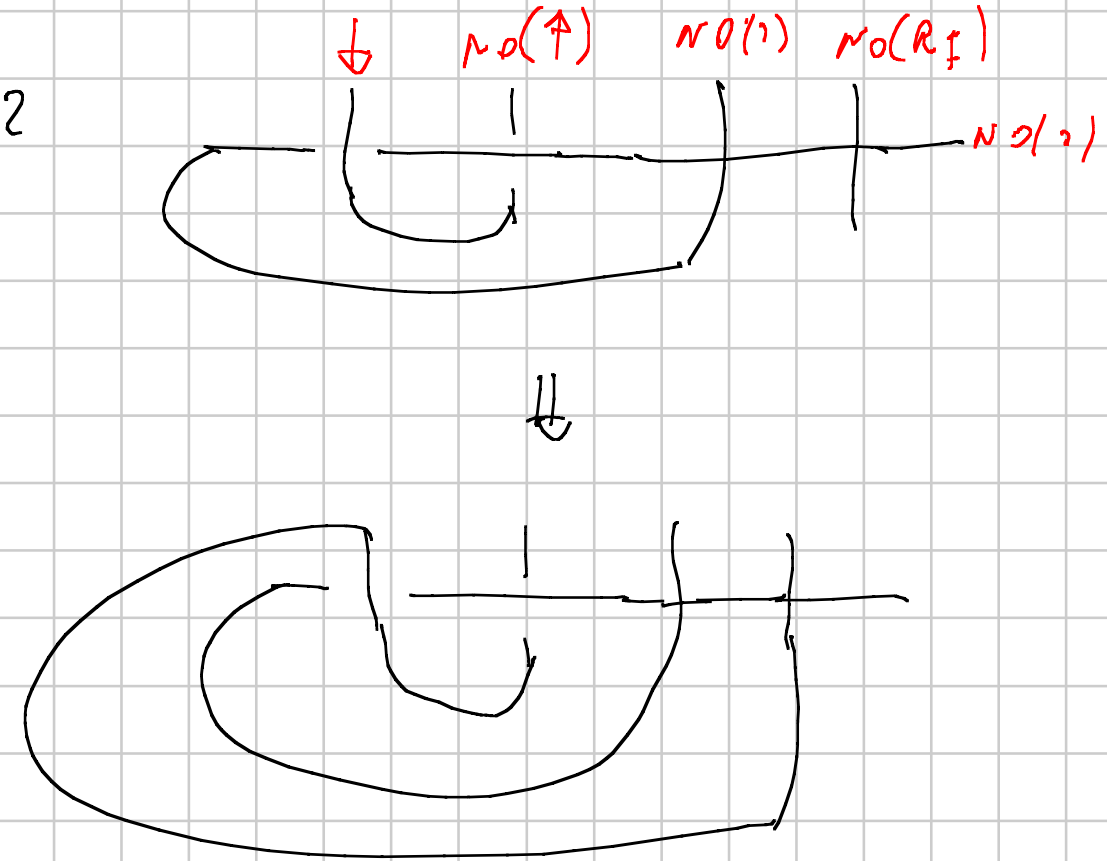


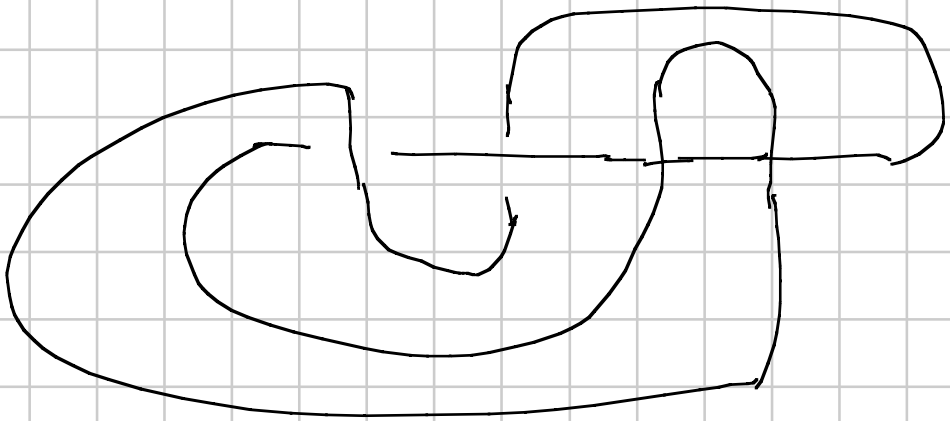

1.1.1

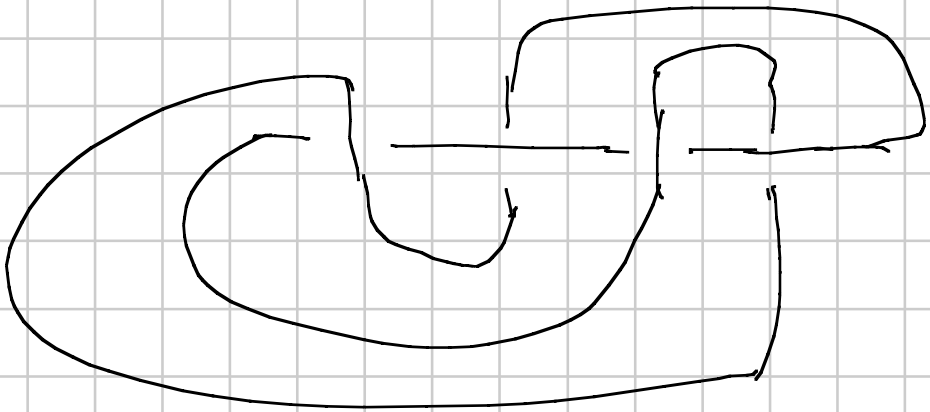


NO (R_I)

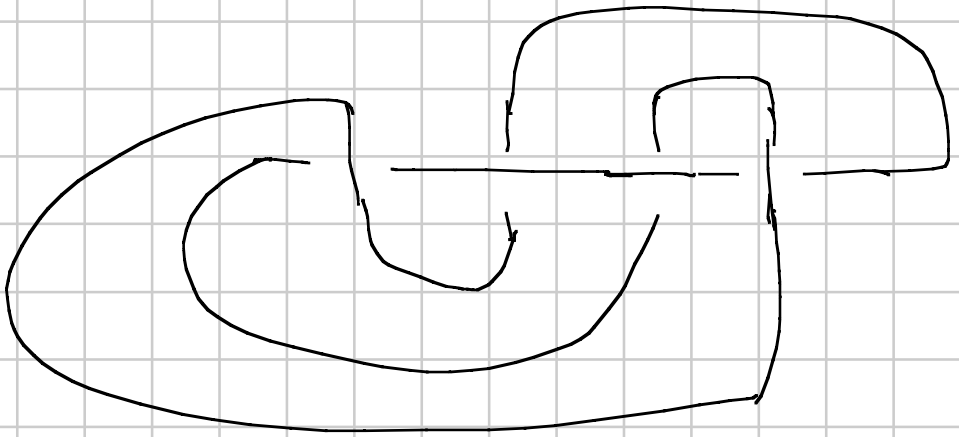
1.1.2





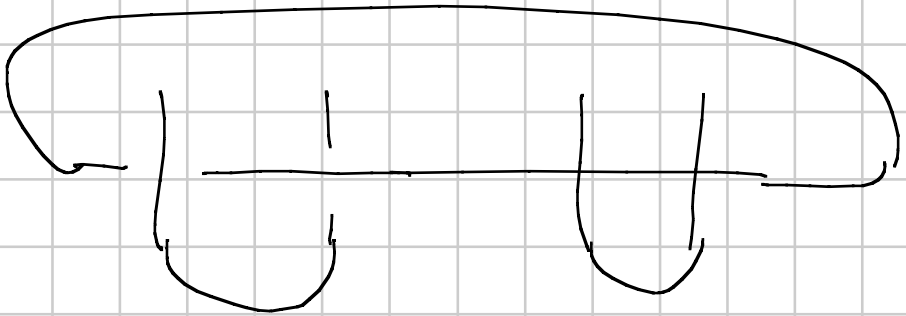


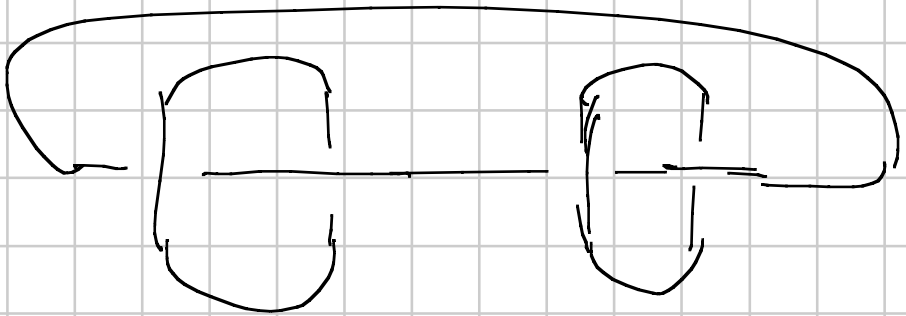
nodo a 8



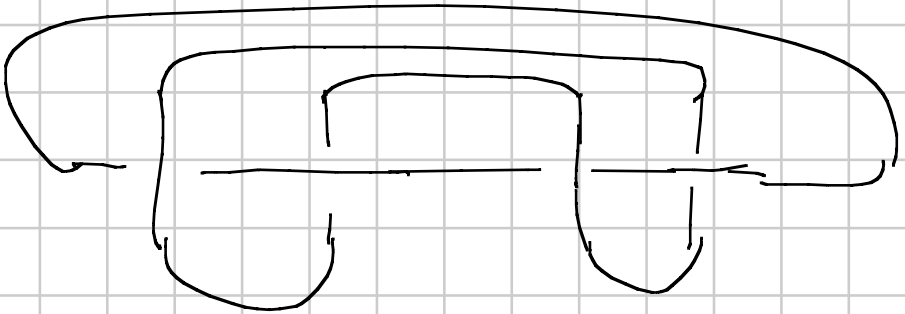
trifoglio

1.1.3

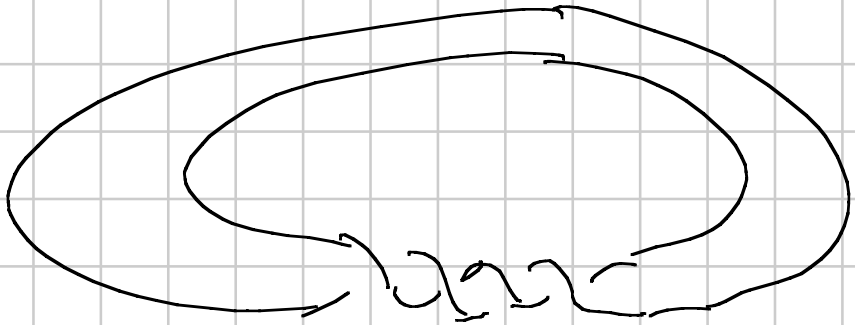




link a 3 comp (nuovo)



T(4,2)

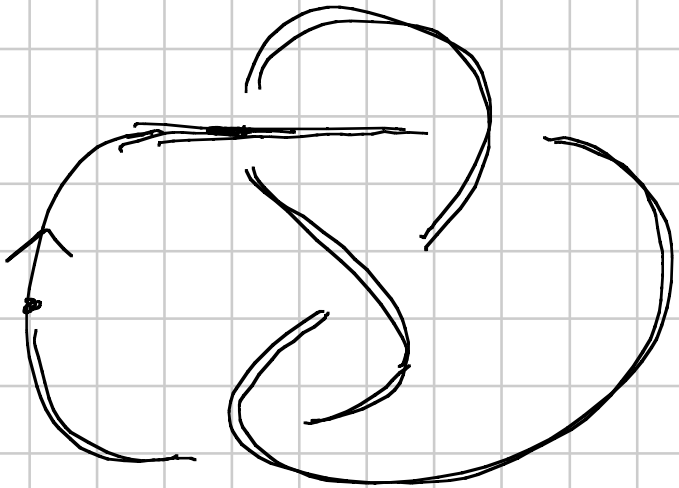


Esercizio: finire provando che non si trova altro

———— o ————

Def: un diagramma si dice alternante se percorrendo ogni componente del link si trovano

alternativamente un incrocio sopra e uno sotto.

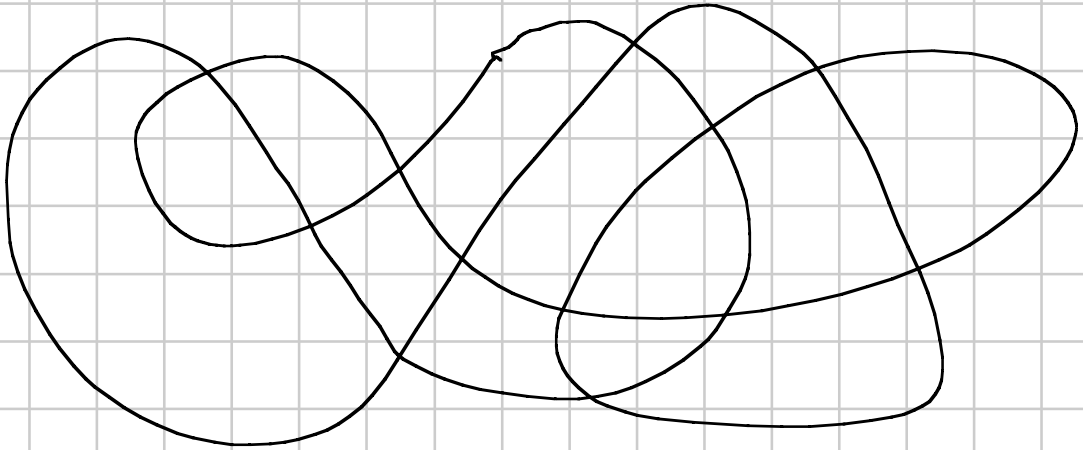


Sì

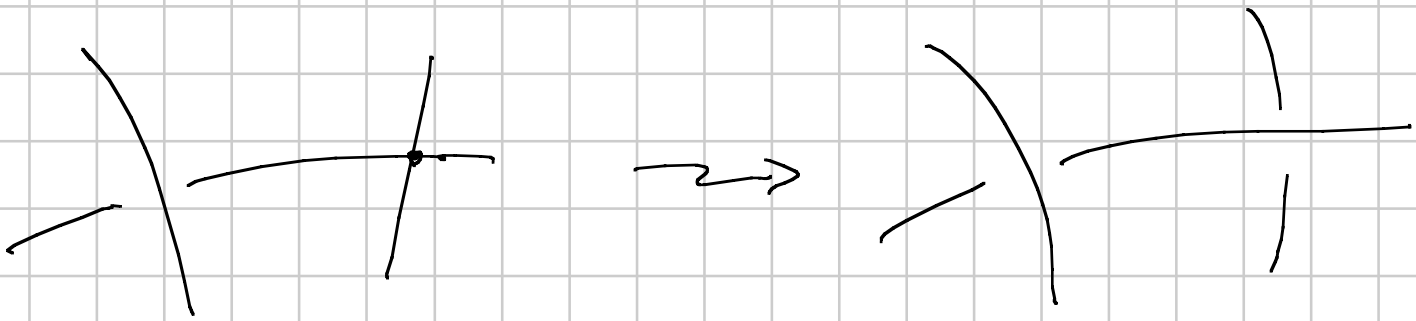
No

Prop: Tramite scambi di incroci ogni diagramma diventa alternante.

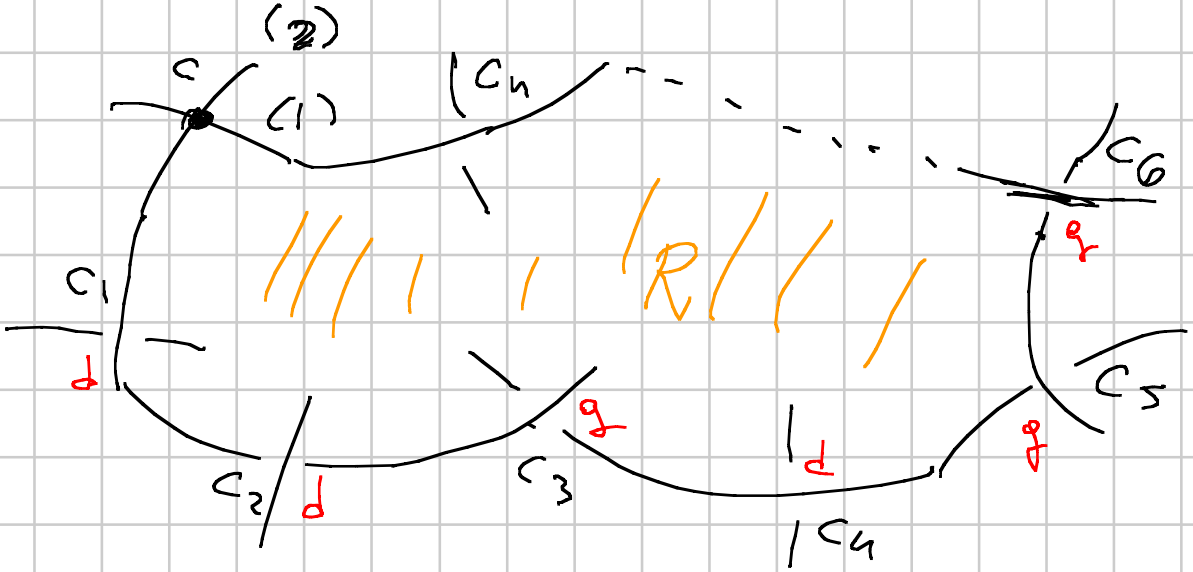
Dim: rendo punti doppi tutti gli incroci.



Comincio a rendere invece un pto doppio e procedo con decisione fornita ogni volta:



Se trovo una contraddizione in un pto doppio c




Prendendo un ciclo controdicente minimale posto supporto semplice.

d → cambio nelle forzature; # dispari di geni di arco entranti in R

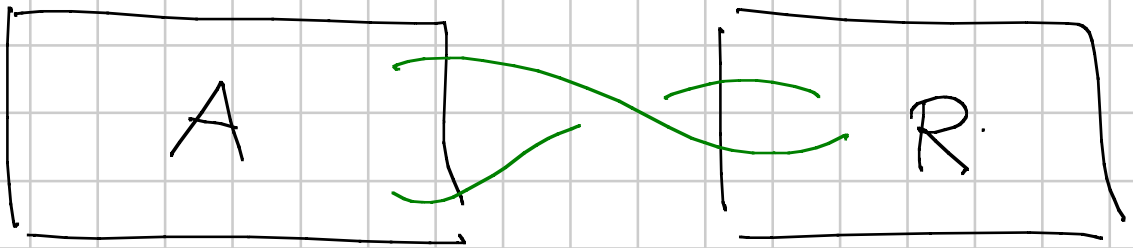
g → mantiene le forzature; # pari di geni

entrants in $R$

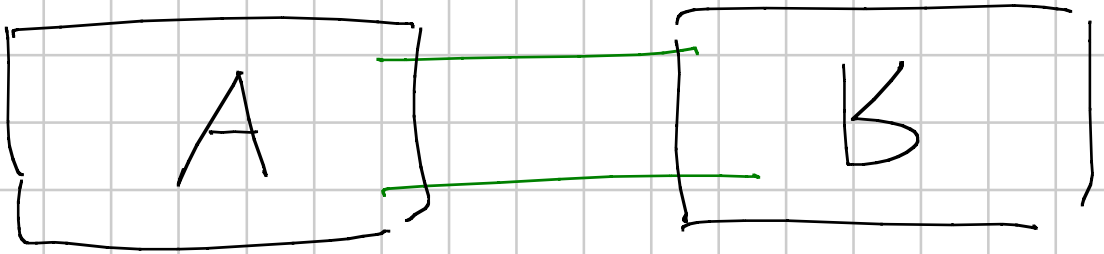
In totale # pari entrants è pari

$\Rightarrow$ # d pari

Def: un incrocio si dice mupatbony se è

$D =$ 


↳ flype

$D' =$ 


$c(D') = c(D) - 1$

Teo: un diagramma alternante senza crossing superfluy realizza il crossing number.

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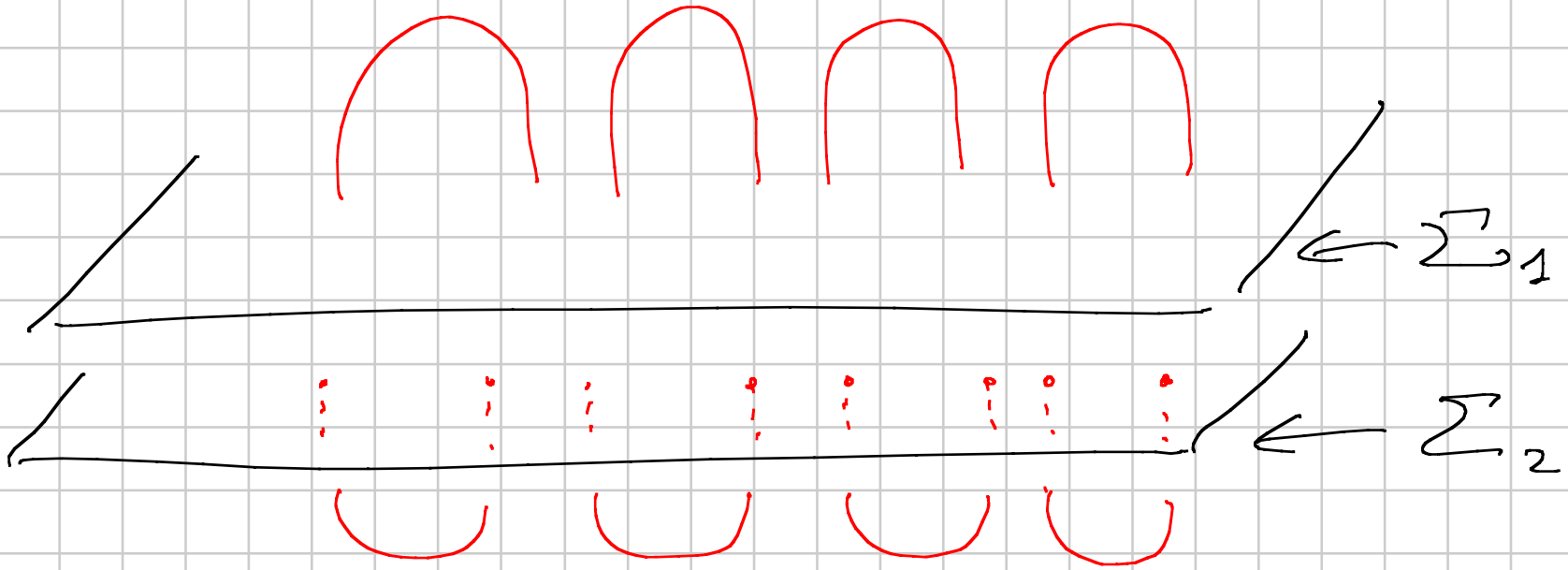
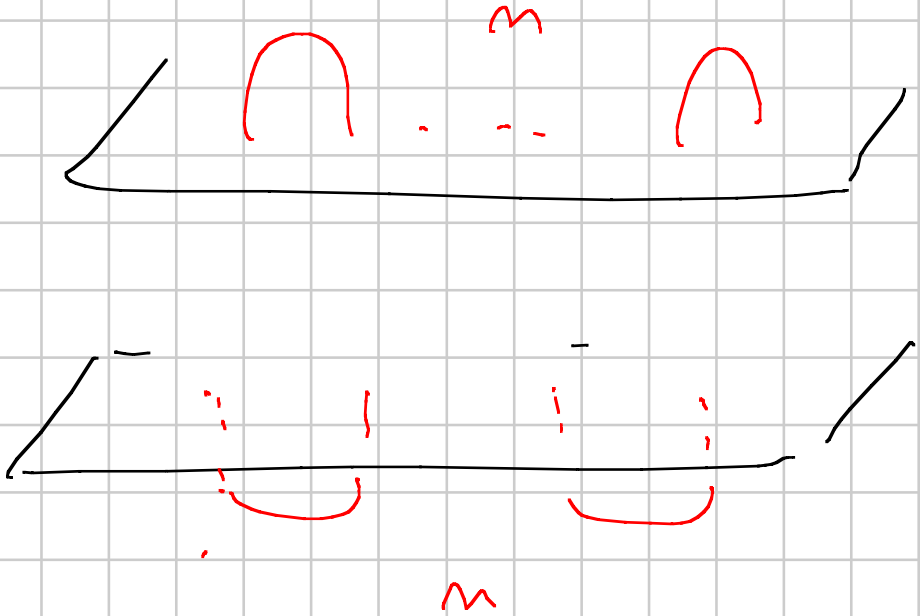
Bridge number:

$$b(L) = \min_n \left\{ \exists\, \Sigma \subset S^3 : \Sigma \cong S^2,\ \Sigma = \partial D_1 = \partial D_2,\ D_1 \cong D_2 \cong D^3,\ L \cap D_i \cong_+ \right\}$$



$b(L) \leq n$ se riesco a costruire $L$ da

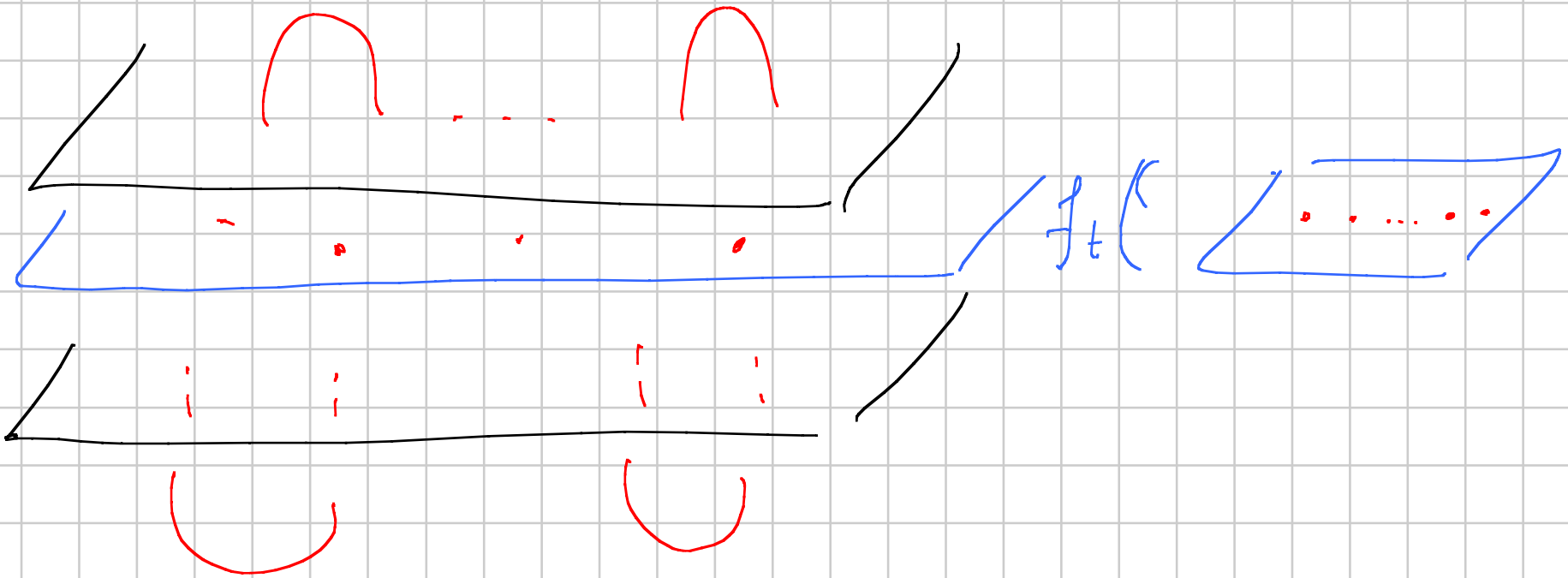


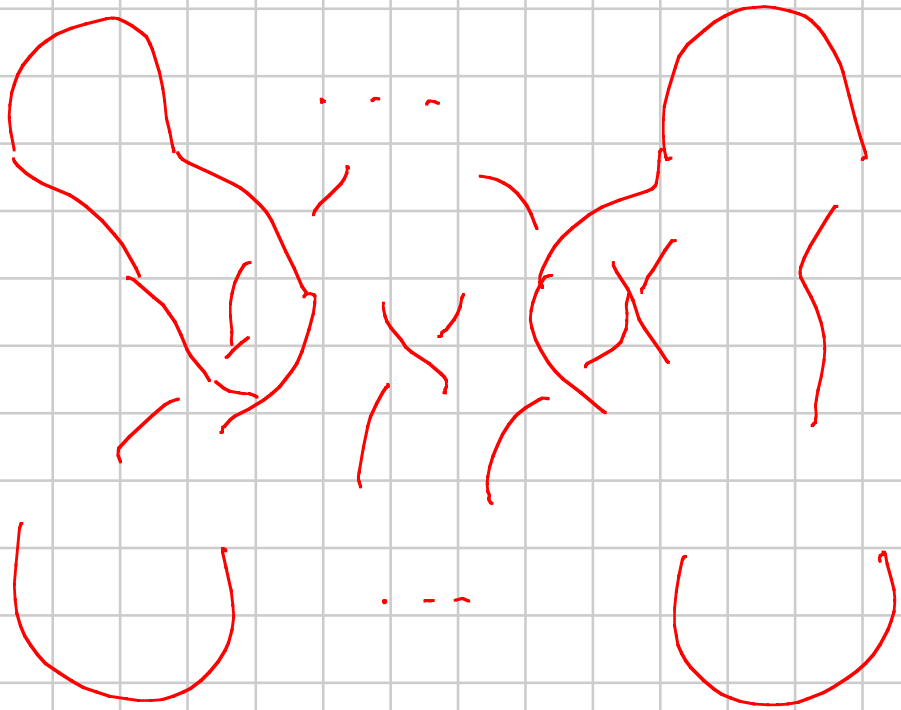

← completando qui con fili che scendono in modo monotono

$f : \partial D_2 \to \partial D_1$ omeo con cui reincollo

Posso vedere $f: \Sigma \to \Sigma$ preserve orientazione

$\Rightarrow \exists (f_t)_{t \in [0,1]}$ $\quad f_t = id \quad f_1 = f$



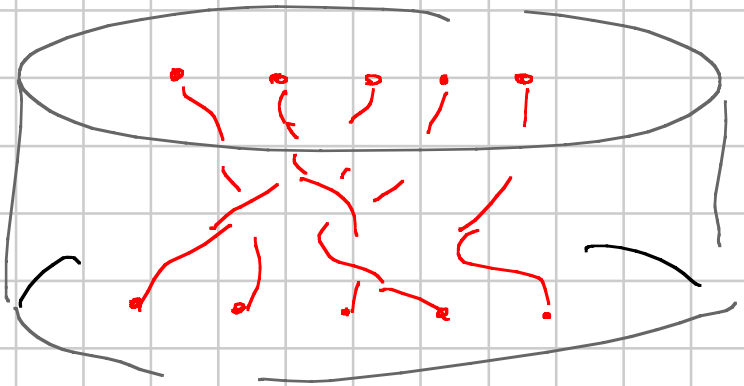


**Def:** chiamo treccia con $k$ fili un embedding (proprio) di

$\underbrace{[0,1] \sqcup \dots \sqcup [0,1]}_{n \text{ copie}}$ in $D^2 \times [0,1]$, con estremi prefissati su $D^2 \times \{0,1\}$

con ogni $[0,1]$ monotono crescente in $[0,1]$

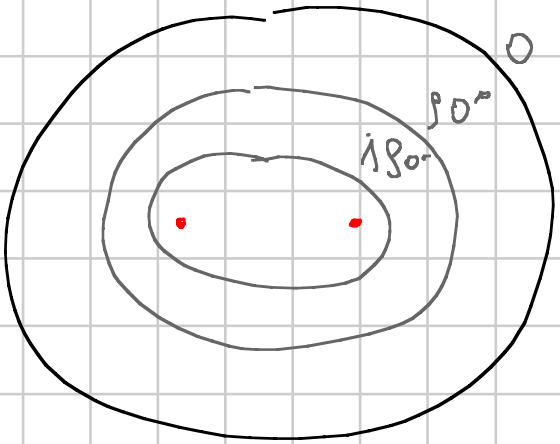
a meno di isotopie tra oggetti dello stesso tipo



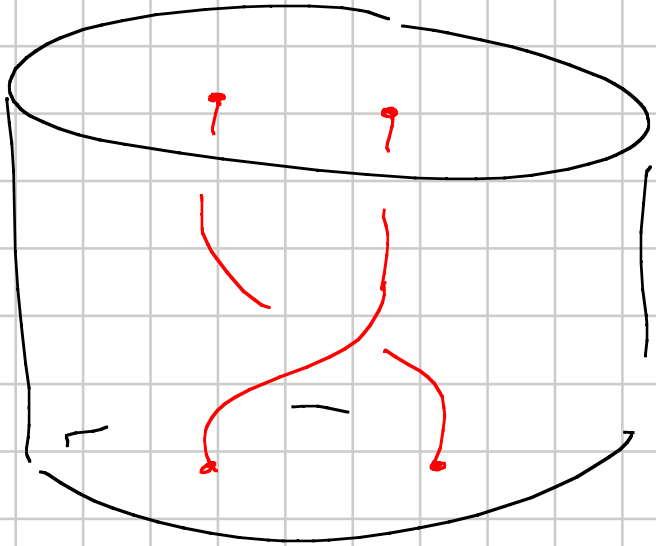
Oppure : $\{ f : D^2 \to D^2$ automorfismo $+$

$f(\cdot\ \cdot\ \cdot\ \cdot\ \cdot) = \cdot\ \cdot\ \cdot\ \cdot\ \cdot \}$ / isotopie che fisse $\cdot\ \cdot\ \cdot\ \cdot$

Es:

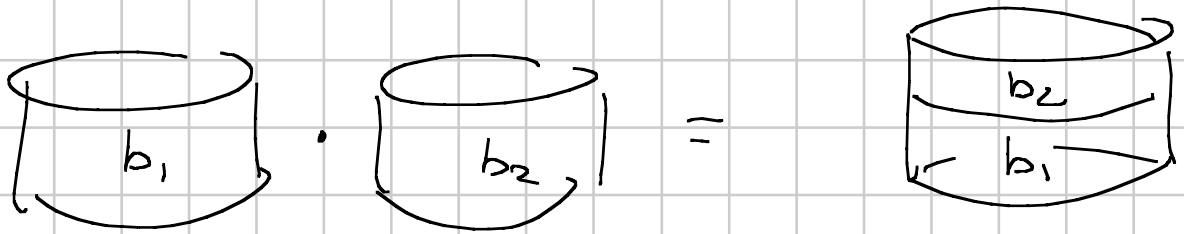



$\longleftrightarrow$

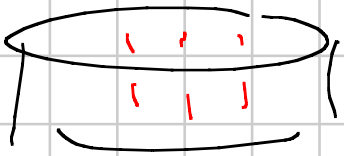


$B_k$ = {trecce con $k$ fili}

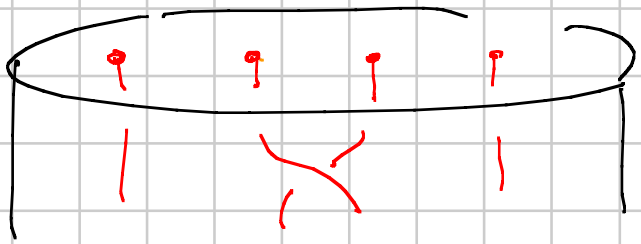
Fatto: $B_k$ è un gruppo tramite l'operazione di




$e$ =



$b^{-1}$ = la stessa trecca rispetto a $D^2 \times \{1/2\}$ :

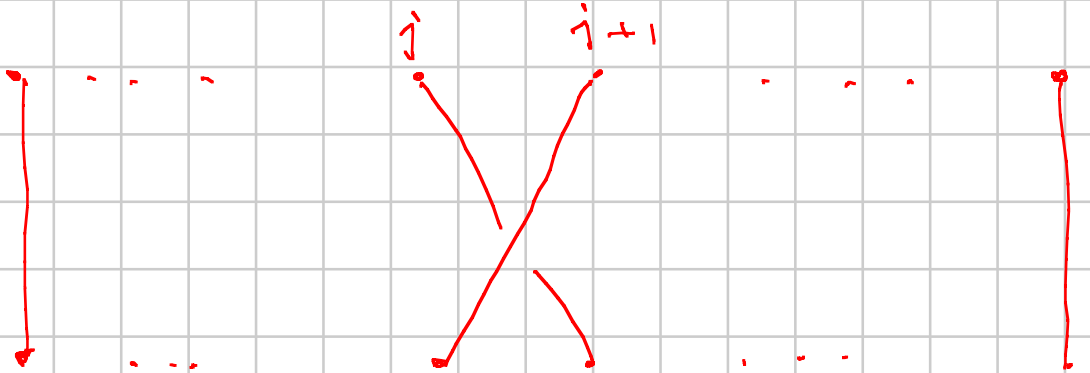


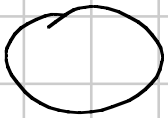
Dunque: una realizzazione a $n$ punti di $L$ è

$b \in B_{2n}$

Fatto $B_k$ è generato da $\sigma_1, \dots, \sigma_{k-1}$ dove

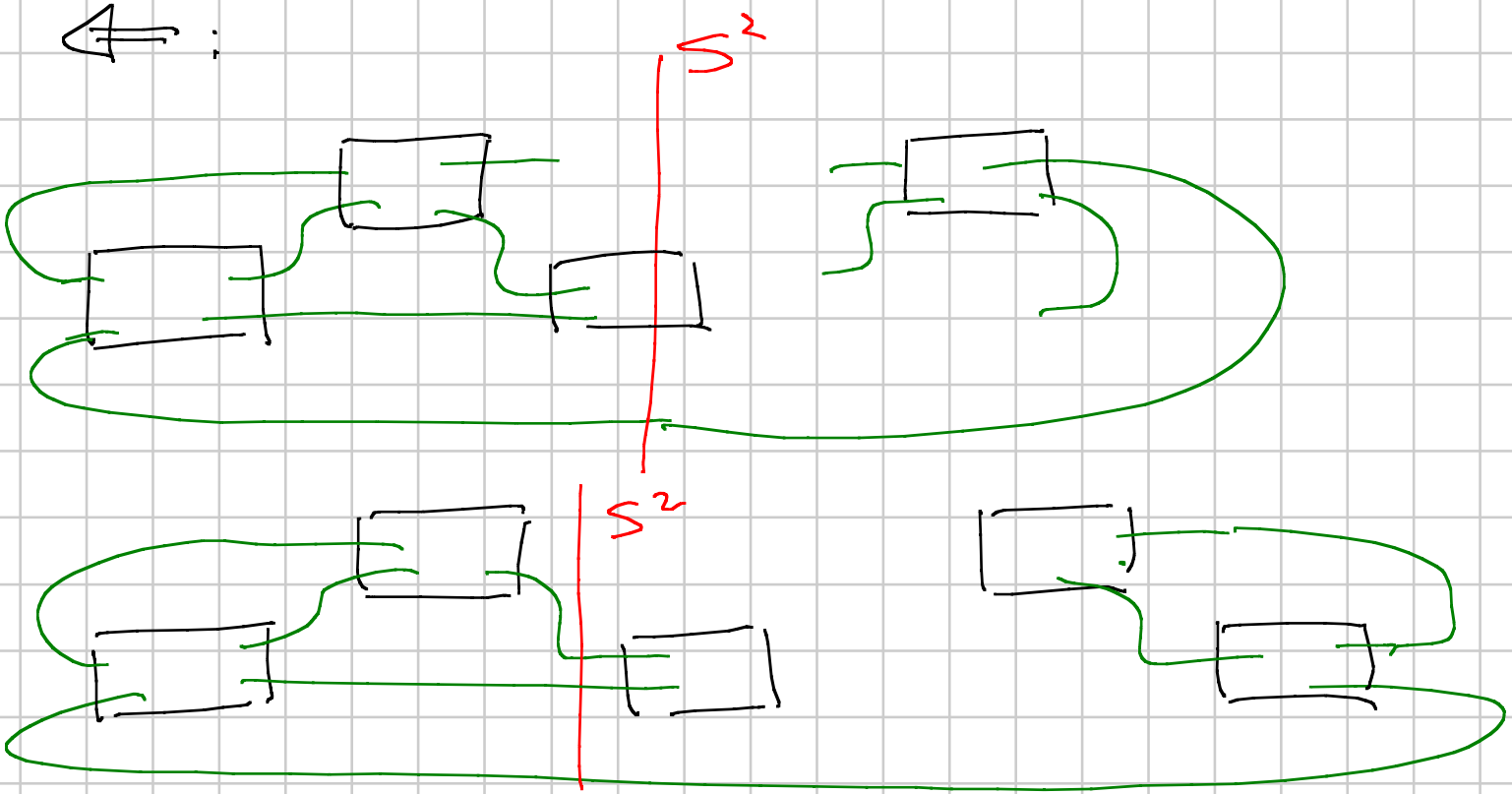
$\sigma_j =$




Oss: $b(L) = 1 \Rightarrow L =$  . Infatti

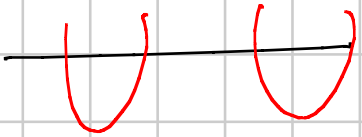
**Prop:** $b(L) = 2 \iff L$ link razsovzemlje.

**Dim:** $\Leftarrow$:

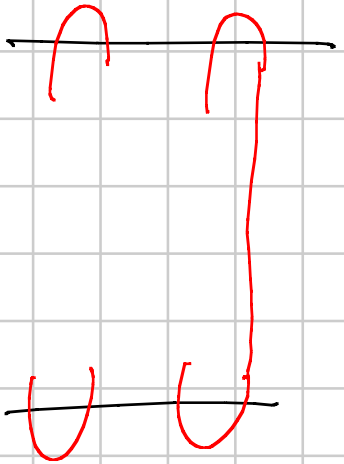


$\Rightarrow$ devo provare che $d_4$

$b_4$

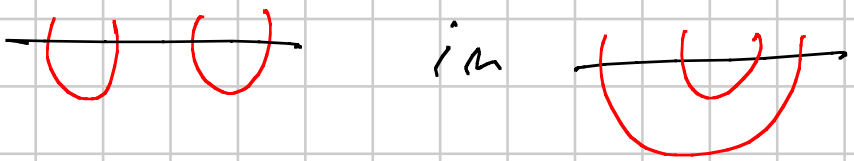


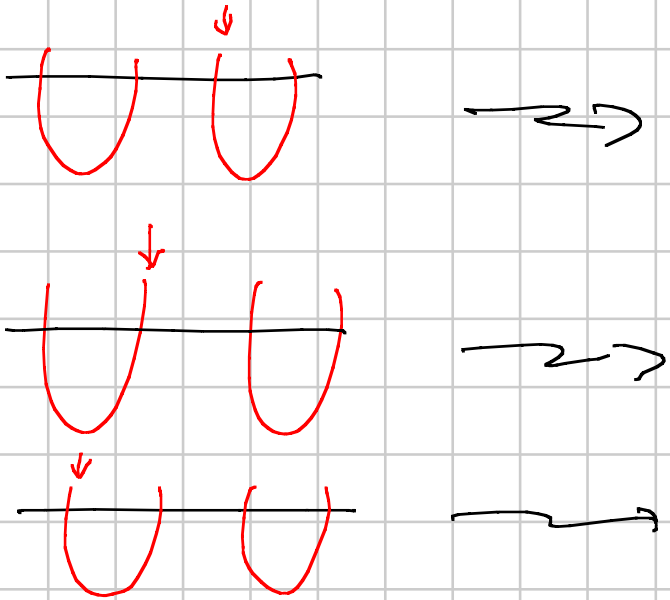
posso ricordarmi

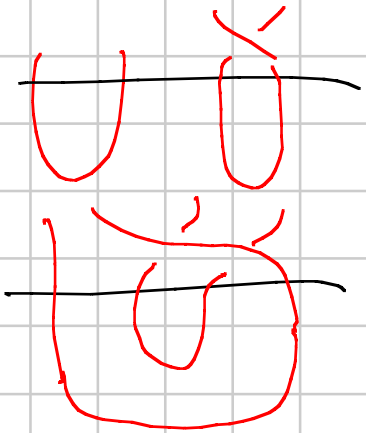


Passi:

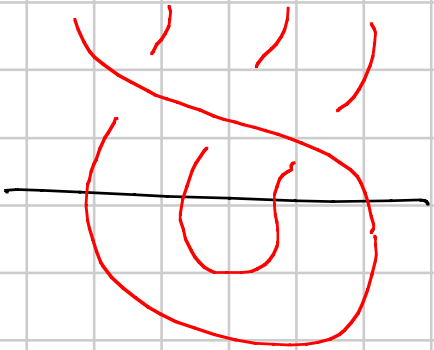
1. Posso supporre che il filo più a dx scende su quello più a dx a meno di cambiare sotto



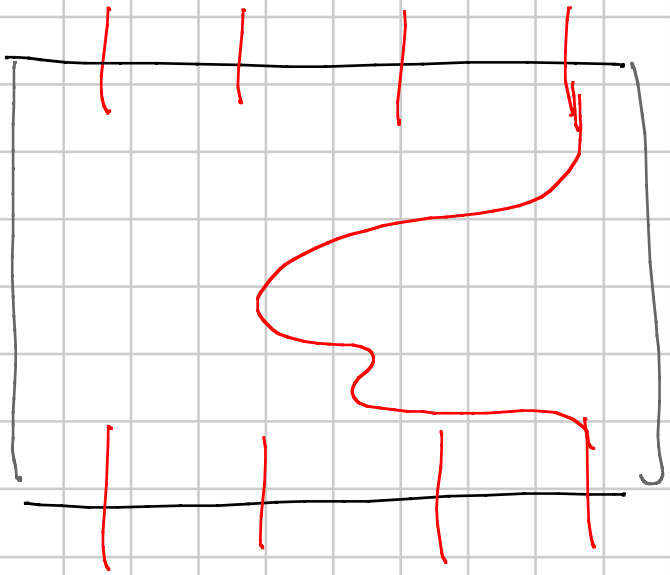



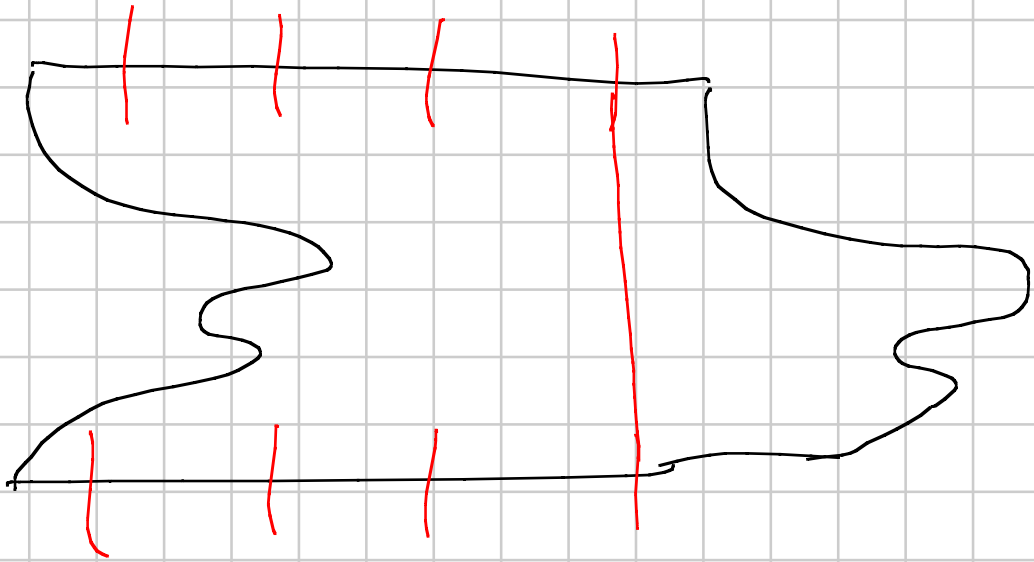






2. Raddrizzo senza cambiare nulla

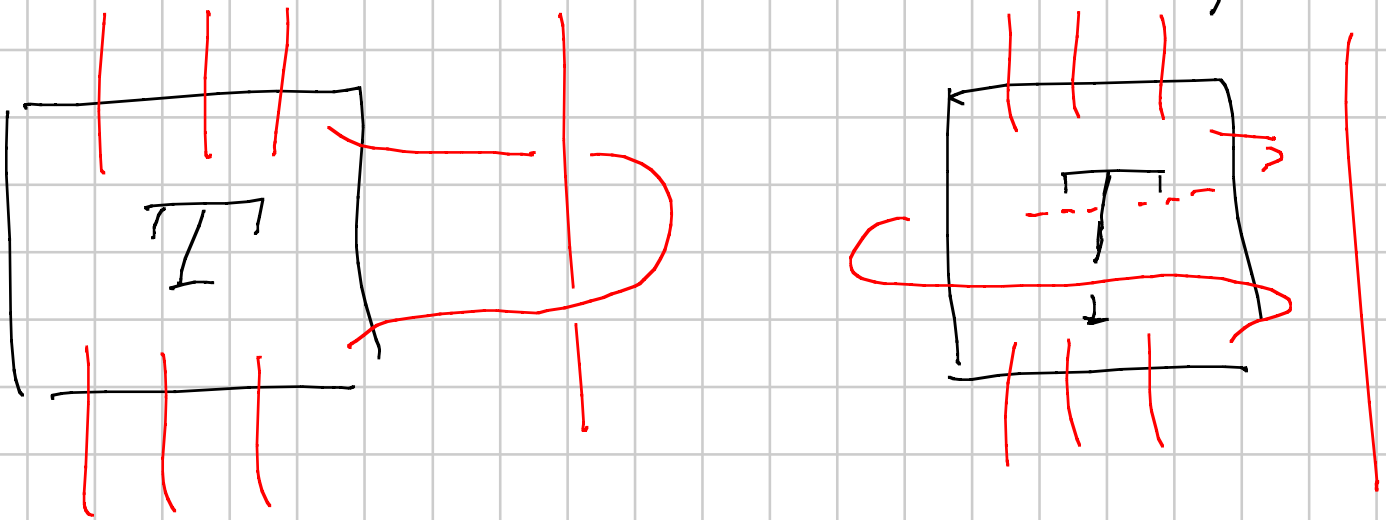




3. Elimino eventuali incroci a destra di pulsante tasto e dx:

( ... )

4. Sposto eventuali archi che passano a dx di quello dato



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Presentazione di $\pi_1(E(L))$ e invariants da essa ottenuti.

**Def:** gruppo libero con generatori $x_1, \dots, x_n$ è l'insieme delle parole (finite) nei simboli $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ a meno di cancellazione

$$w \cdot x_j \cdot x_j^{-1} \cdot u \sim w \cdot u$$

$$w \cdot x_j^{-1} \cdot x_j \cdot u \sim w \cdot u$$

con operazione di concatenazione e elemento neutro parola vuota 1. Indicato con $F(x_1, \dots, x_n)$.

**Oss:** se $G$ è gruppo, dati $g_1, \dots, g_n \in G$

$\exists!$ $\varphi : F(x_1, \dots, x_n) \to G$ omomorfismo

t.c. $\varphi(x_j) = g_j$ .

**Cor:** $\mathrm{Hom}(F(x_1, \dots, x_n), G) \cong G^n$

**Cor:** $F(x_1, \dots, x_n) \not\cong F(y_1, \dots, y_m)$ per $n \neq m$

poiché $\#\,\mathrm{Hom}(F(x_1, \dots, x_n)/\mathbb{Z}/2) = 2^n$.

Gruppo finitamente presentato:

$r_1, \dots, r_k \in F(x_1, \dots, x_n)$

$\langle x_1, \dots, x_n \mid r_1, \dots, r_k \rangle = F(x_1, \dots, x_n) / N(r_1, \dots, r_k)$

$N(r_1, \dots, r_k)$ = il più piccolo sottogruppo normale che contiene $r_1, \dots, r_k$

= tutti i prodotti di tutti i coniugati di $r_1^{\pm}, \dots, r_k^{\pm 1}$.

Q: Come differiscono due presentazioni dello stesso $G$?

Mosse di Tietze:

I: $\langle x_1, \dots, x_m \mid r_1, \dots, r_k \rangle \rightsquigarrow \langle x_1, \dots, x_m \mid r_1, \dots, r_k, r \rangle$

se $r \in N(r_1, \dots, r_k)$

(se $r$ è conseguenza di $r_1, \dots, r_k$)

II: $\langle x_1, \dots, x_m \mid r_1, \dots, r_k \rangle \rightsquigarrow \langle x_1, \dots, x_m, x \mid r_1, \dots, r_k, x^{-1} \cdot g \rangle$

$g \in F(x_1, \dots, x_m)$

Teo: due presentazioni dello stesso G differiscono per una serie di Tietze $I^{\pm 1}, II^{\pm 1}$.

Oss (mosse inverse):

$I^{-1}$: $\langle x_1, \dots, x_n \mid r_1, \dots, r_k \rangle \rightsquigarrow \langle x_1, \dots, x_n \mid r_1, \dots, r_k, r_{k+1} \rangle$

se $r_{k+1} \in N(r_1, \dots, r_k)$ ovvero se

$N(r_1, \dots, r_k) = N(r_1, \dots, r_{k+1})$

$\text{II}^{-1}$ $\quad \langle x_1, \dots, x_m, x \mid \pi_1, \dots, \pi_k, x^{-1} \cdot \rho \rangle$

$\rightsquigarrow \langle x_1, \dots x_n \mid \pi_1 \dots \pi_k \rangle$

se $x$ non compare in $\pi_1, \dots, \pi_k$

Tramite $k$ applicazioni di $I$ e $k$ di $I^{-1}$ equivale a

$\langle x_1, \dots, x_n, x \mid \pi_1 \dots \pi_k, x^{-1} \cdot \rho \rangle \rightsquigarrow \langle x_1 \dots x_m \mid \tilde{\pi}_1, \dots, \tilde{\pi}_k \rangle$

$\tilde{\pi}_j$ = ottenuto da $\pi_j$ sostituendo $x$ con $\rho$.