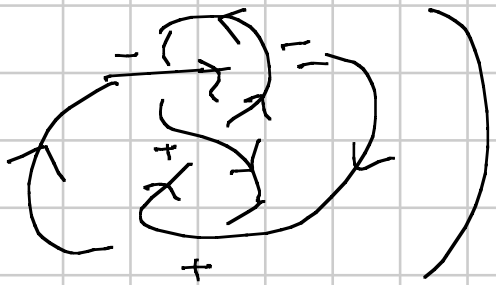
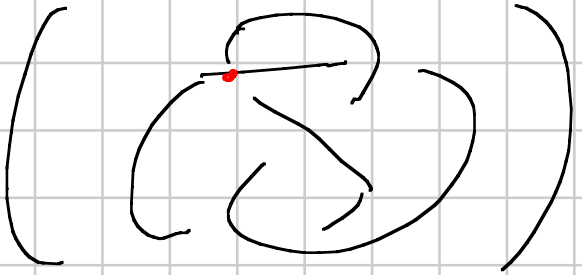
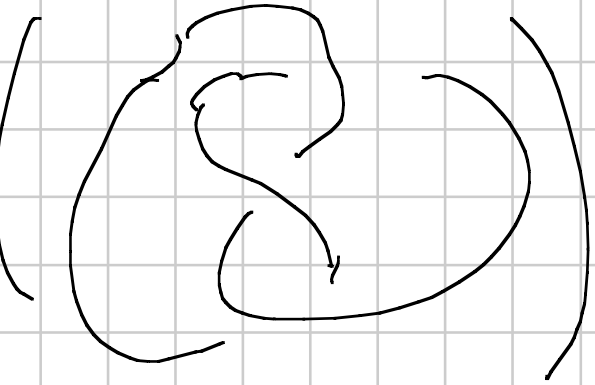
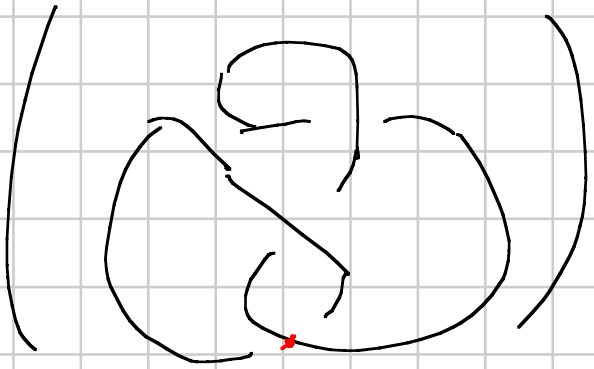
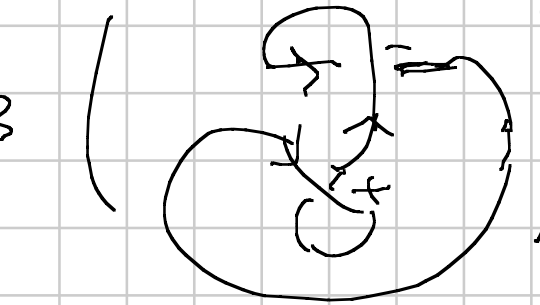
Teoria dei Nodi 24/11/16

P  $= B$ 

$= AB$  $+ A^{-1}B$ 

trifoglio sinistro

$$= A \cdot \left(A^{7} - A^{3} - A^{-5} \right) + A^{-1} \cdot A \cdot B \left(\right) + A^{-1} \cdot A^{-1} \cdot B \left(\right)$$



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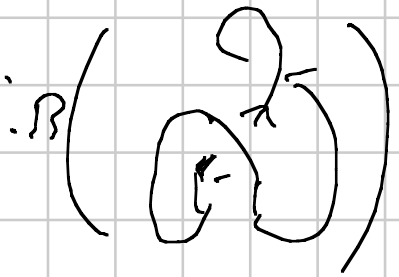
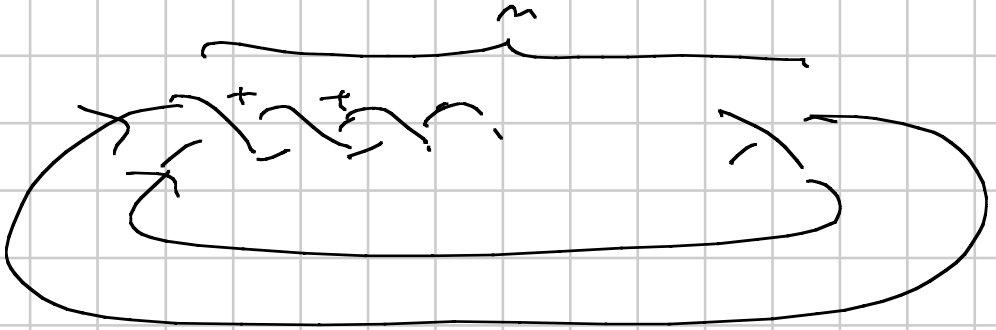


$(-A^3)^{-2}$

$$= A^{8} - A^{4} + 1 - A^{-4} + A^{-8}$$

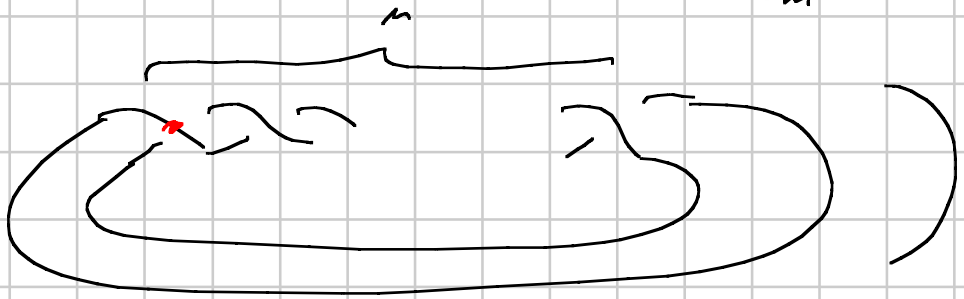
non cambia con $A \longleftrightarrow A^{-1}$

$K(m,2) =$

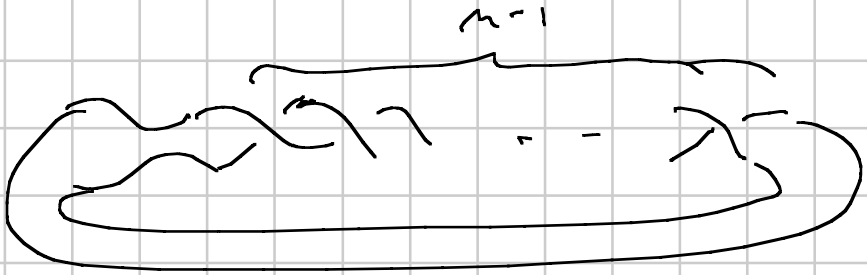
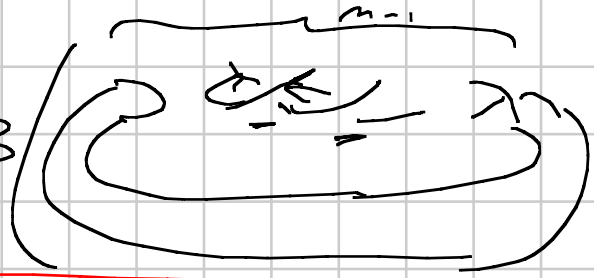


$$p_m = P(K(m,2)) = (-A^3)^{-m} \cdot \underbrace{B(K(m,2))}_{b_m}$$

$b_m = B($



$)$

$= A \cdot B$  $+ A^{-1} \cdot B$ 

$= A \cdot b_{n-1} + A^{-1} \cdot (-A^{-3})^{n-1} = A \cdot b_{n-1} + (-1)^{n-1} \cdot A^{2-3n} = b_n$

$b_3 =$ è quello del trifoglio

$b_1 = B$ $= -A^3$

Prop: $b_n = -A^{n+2} + (-1)^n \cdot A^{-3n-2} \cdot \left(\frac{1-(-A^4)^n}{1+A^4} - 1 \right)$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\sum_{i=1}^{n-1}(-A^4)^i}$$

Pf: Use C_n RHS.

$C_1 = -A^3 + 0$ ✓

$$A \cdot C_{n-1} + (-1)^{n-1} \cdot A^{2-3n}$$

$$= A \cdot \left(-A^{n+1} + (-1)^{n-1} A^{-3n+3-2} \left(\frac{1-(-A^4)^{n-1}}{1+A^4} - 1 \right) \right) + (-1)^{n-1} \cdot A^{2-3n}$$

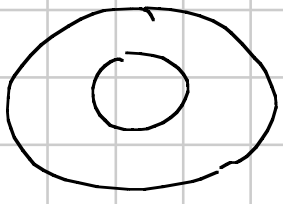
$$= -A^{n+2} + (-1)^{n-1} \cdot A^{-3n+2} \left(\frac{1-(-A^4)^{n-1}}{1+A^4} - 1 + 1 \right)$$

$$= -A^{n+2} + (-1)^n \cdot A^{-3n-2} \cdot (-A^4) \sum_{i=0}^{n-2} (-A^4)^i$$

$$= -A^{n+2} + (-1)^n \cdot A^{-3n-2} \cdot \sum_{i=1}^{n-1} (-A^4)^i \quad \checkmark$$

Oss: vero anche per $n=0$

$$\beta_0 = b_0 = B\left(\quad\right) = -A^2 - A^{-2}$$



$$\beta_n = (-A)^{-3n} \cdot b_n = (-A^{-3n})\left(-A^{n+2} + (-1)^n A^{-3n-2}\left(\frac{1-(-A^4)^n}{1+A^4} - 1\right)\right)$$

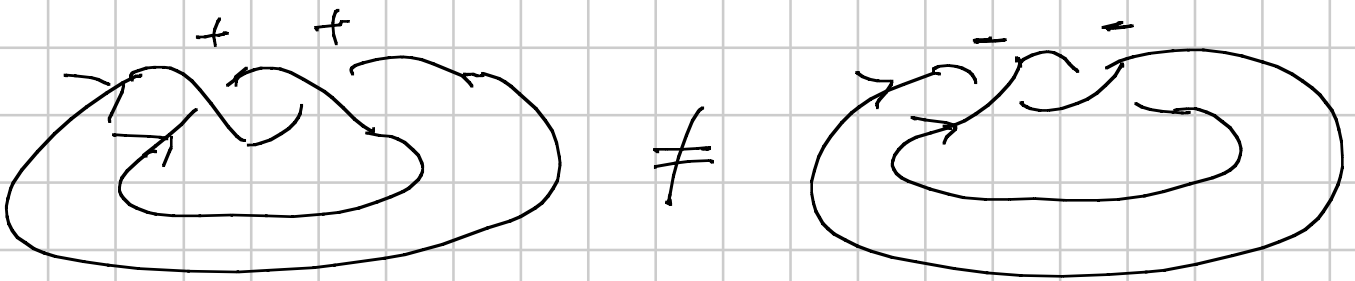
$$= \pm A^{-2n+2} \pm \underbrace{A^{-6n-2}\left(\pm A^4 \pm A^8 \cdots \pm A^{4n-4}\right)}_{\begin{array}{l}\text{grado più basso } -6n+2\\ \text{grado più alto } -6n-2+4n-4 = -2n-6\end{array}}$$

più basso $-6n+2$

più alto $-2n+2$

entrambi negativi per $n \geq 2 \Rightarrow K(n,2)$ è chirale per ogni $n \geq 2$.

Per $n=2$

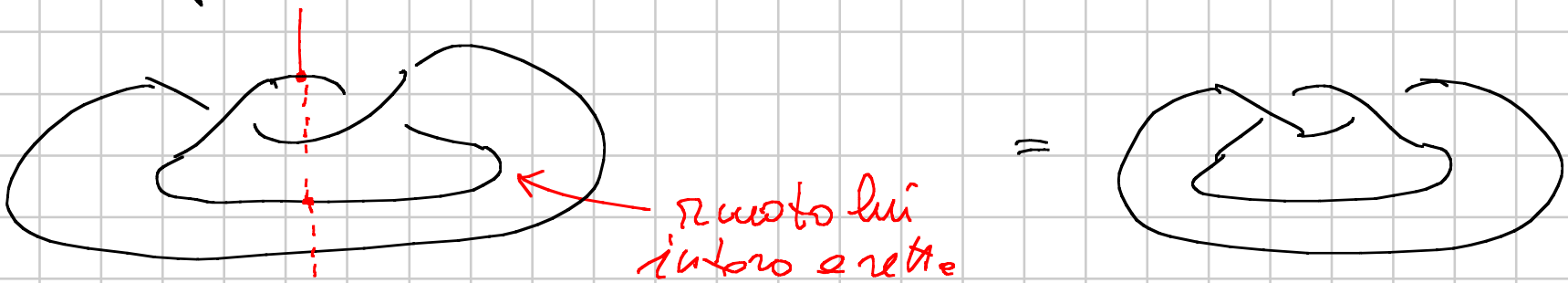


Hopf $\neq$ $\overline{\text{Hopf}}$

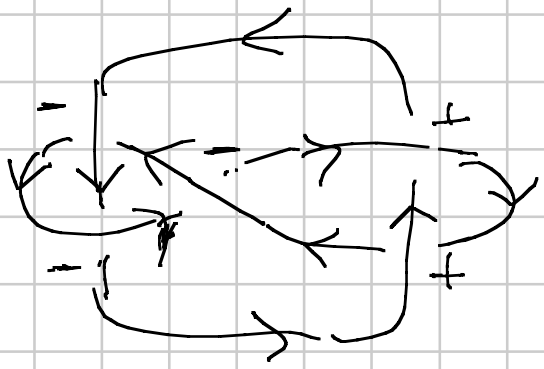
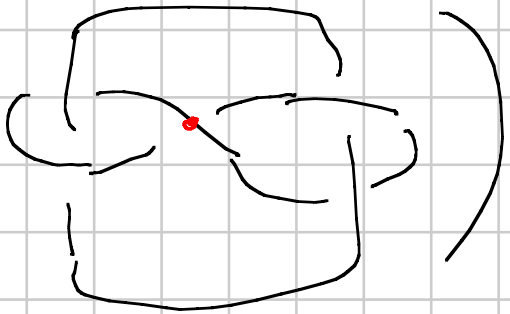
$\text{Hopf} \neq \overline{\text{Hopf}}$ come link orientati.

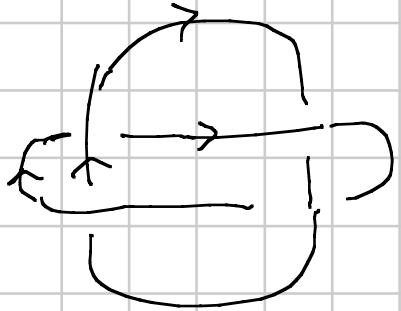
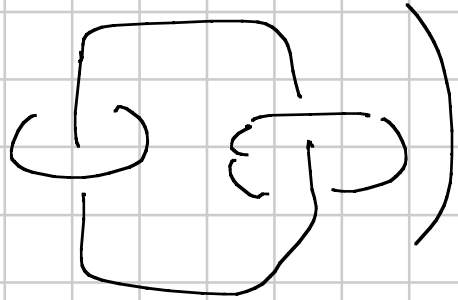
Però sono uguali come link non orientati:

$\overline{\text{Hopf}}$



Whitehead link :

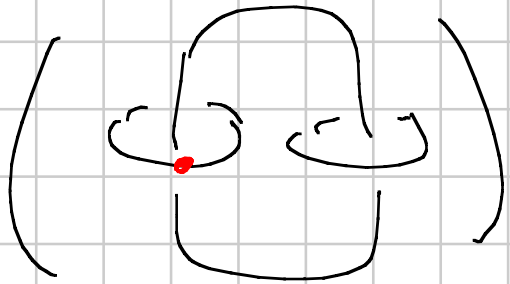
$P\Big($  $\Big) = (-A^3) \cdot B\Big($  $\Big)$

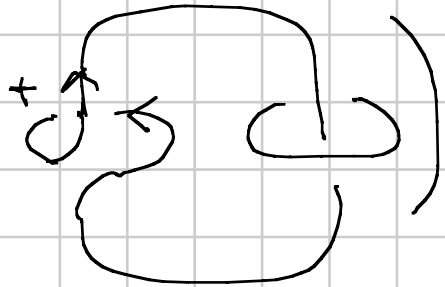
$= (-A^3) \cdot A \cdot B\Big($  $\Big) + (-A^3) \cdot A^{-1} \cdot B\Big($  $\Big)$

$K(4,2)$

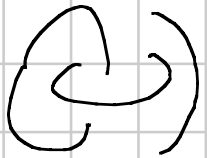
$$= \left(-A^4\right)\cdot\left(-A^6 + A^{-14}\left(-A^6 + A^8 - A^{12}\right)\right)\Big|_{A^{-1}} - A^2 \cdot B\left(\quad\right)$$

$$B\left(\quad\right) = A\cdot B\left(\quad\right) + A^{-1}\cdot B\left(\quad\right)$$





$$= \left(A^4 + A^{-4}\right)\cdot B\left(\quad\right) = -\left(A^4 + A^{-4}\right)^2$$



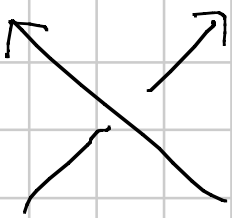
$K(2,2)$

$$P(\text{Whitehead}) = A^{-2} + A^{14} - A^{10} + A^{6} - A^{8} - 2 - A^{-8}$$

$\Rightarrow$ non banale e chirale

Se in D (orientato) ho un incrocio in rilievo indico

P_+



P_-

P_0

Thm: $A^4 \cdot P(D_+) - A^{-4} \cdot P(D_-) = (\vec{A} - A^2) \cdot P(D_0)$

Dim: $P(D_+) \underset{\text{loc}}{=} (-A)^{-3} \cdot B(D_+) = (-A^{-3}) \cdot \left(A \cdot B\left(\right)\left(\right) + A^{-1} \cdot B\left(\right) \right)$

$P(D_-) \underset{\text{loc}}{=} (-A)^{3} \cdot B(D_-) = (-A^{3}) \left(A \cdot B\left(\right) + A^{-1} \cdot B\left(\right)\left(\right) \right)$

$A^4 \cdot P(D_+) - A^{-4} \cdot P(D_-)$

$= -A^2 \cdot B(D_0) - B\left(\right)$

$$+A^{-2}\,B(D_0) \qquad + B\left(\;\smile\;\right)$$

$$= \left(A^{-2} - A^{2}\right) B(D_0) = \left(A^{-2} - A^{2}\right) \cdot P(D_0). \qquad \blacksquare$$

Def: chiamo polinomio di Jones

$$V(L) = P(L)\Big|_{A = q^{-1/4}}.$$

Con: $q^{-1} \cdot V(D_+) - q \cdot V(D_-) = \left(q^{1/2} - q^{-1/2}\right) \cdot V(D_0)$

Oss: questa formula consente il calcolo di $\tilde{V}$ essendo noto quello del link banale a n componenti

SUPPONENDO CHE SIA BEN DEF COME INV. DI LINK ORIENTATI

$$\left(-A^2 - A^{-2}\right)^{n-1} = \left(-q^{-1/2} - q^{1/2}\right)^{n-1}$$

: per ricorrenza su $w(D) =$ writhing number di D:

(congiunta $m < D$ e)

se $w(D) = 0$ ho $\left(-q^{-1/2} - q^{1/2}\right)^{|L|-1}$

se $w(D) = p > 0$ prendo uno dei p incroci che vanno invertiti per rendere D diagramma del link banale:

In tale incrocio ho $D = D_+$ o $D = D_-$

se $D = D_+$ $\quad V(D_+) = * \cdot \underbrace{V(D_-)}_{u(D_-) = p-1} + * \cdot \underbrace{V(D_0)}_{c(D_0) = c(D)-1}$

oppure

se $D = D_-$ $\quad V(D_-) = * \underbrace{V(D_+)} + * \cdot \underbrace{V(D_0)}$

stesso

Proprietà di P e V:

Prop: $P(L_1 \# L_2) = P(L_1) \cdot P(L_2)$

$V(L_1 \# L_2) = V(L_1) \cdot V(L_2)$

(Oss: $L_1 \# L_2$ è tra link orientati ma non ben def. se hanno più di una componente).

Dimo: $w(L_1 \# L_2) = w(L_1) + w(L_2)$

$\Rightarrow (-A^3)^{-w(L)}$ è moltiplicativo.

Dunque basta vedere che è molt. B:

per induzione su $c(L_2)$

- se $c(L_2) = 0$ chiaro

- se $c(L_2) > 0$ uso relazione skein su incrocio di L_2.

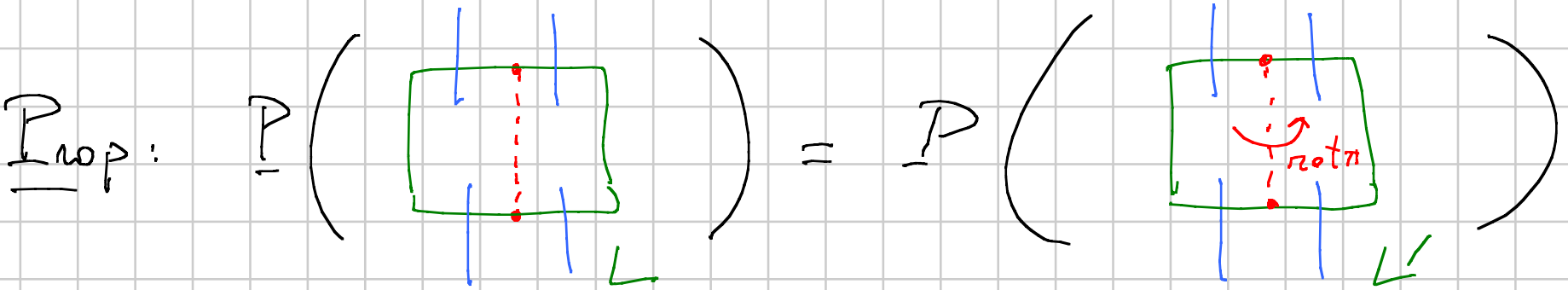
∎

Prop: $P(L_1 \amalg L_2) = (-A^2 - A^{-2}) \cdot P(L_1) \cdot P(L_2)$

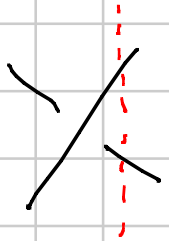
(segue da $P(L) = \sum_n A^{\ell(n)} \cdot A^{-n(n)} \cdot (-A^2 - A^{-2})^{c(n)-1}$)

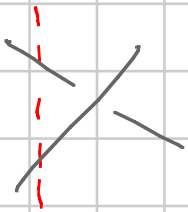
$V(L_1 \amalg L_2) = -(q^{1/2} + q^{-1/2}) \cdot V(L_1) \cdot V(L_2)$

Prop:

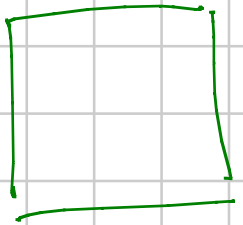


Dim: $w(L) = w(L')$.

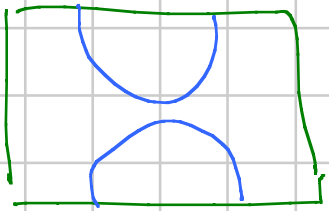


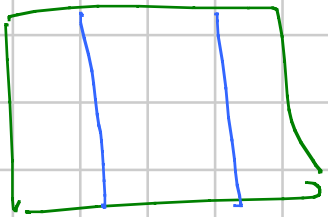


Una semplificazione completa di L con archi A (sin) e A' (dx) ne induce una corrisp. su L'; devono

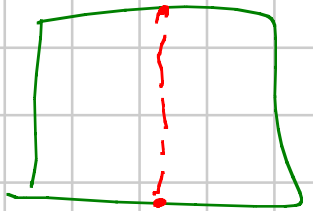


per L posso avere solo





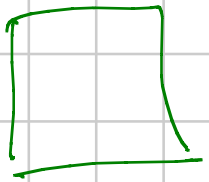
invariante rispetto alla rotaz. intorno a



$\Rightarrow$ su L' ho la stessa figura

$\Rightarrow$ $C(A) = C'(A')$ -

Per L "abbastanza complicato" sia dentro sia fuori



ho $L' \neq L$ $\Rightarrow$ P non è invariante completo

Problema aperto: $\exists$ nodi non banali con $P = 1$?

Prop: $|L|$ dispari $\Rightarrow V(L) \in \mathbb{Z}[q^{\pm 1}]$

$|L|$ pari $\Rightarrow q^{1/2} \cdot V(L) \in \mathbb{Z}[q^{\pm 1}]$

Dim: per induzione su $n_{c(L)}(L)$.

Per $n(L) = 0$ ho $V(L) = (-q^{1/2} - q^{-1/2})^{|L|-1}$ vero.

Per $n(L) > 0$ uso

$$q^{-1} \cdot V(L_+) - q \cdot V(L_-) = (q^{1/2} - q^{-1/2}) \cdot V(L_0)$$

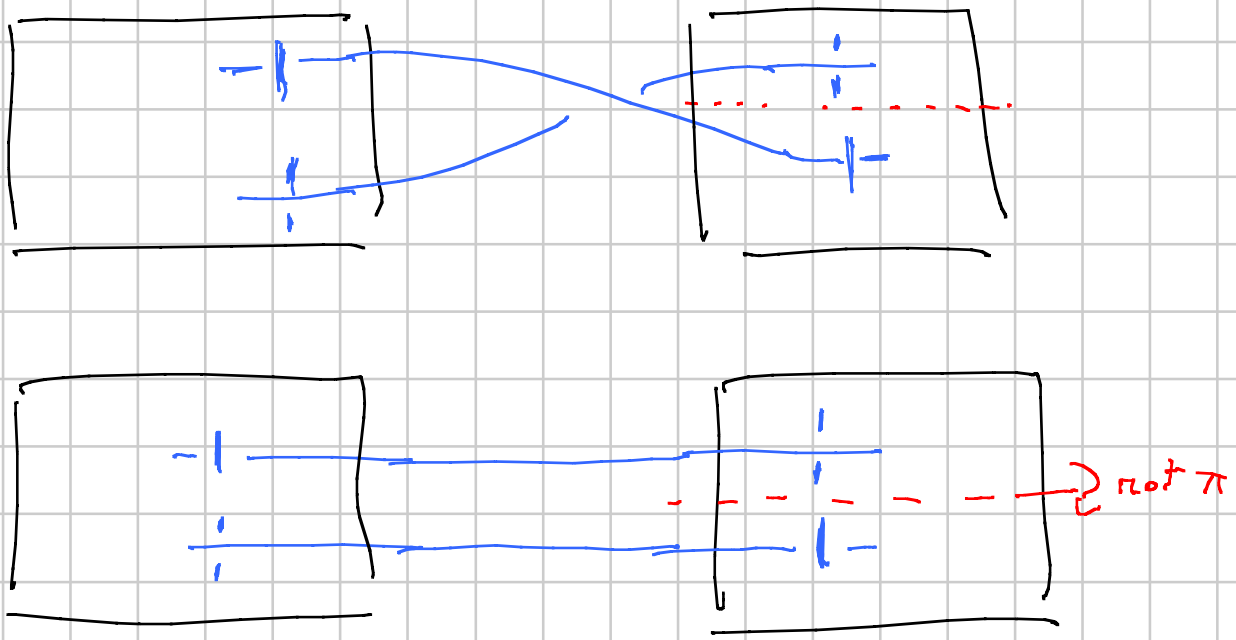
↖ uno dei due che ha il più basso

↗ che è più basso di entrambi

$\Rightarrow$ Ok. ∎

Ricordo: D è alternante se luogo ogni componente di L si alternano overcrossing e undercrossing.

Un incrocio di D si dice mugetory (sicuro) se



Def: D diagramma alternante è detto ridotto se non ha incroci scemi.

Def: chiamo $S(L)$ (S = span)
la differenza tra la potenza più alta e quella più bassa di A in $P(L)$.

Teo: se D è un diagramma alternante ridotto non-split allora $c(D) = 4 \cdot S(L)$. (qui c = crossing number)

Cor: due diagrammi alternati ridotti dello stesso link hanno lo stesso numero di incroci.

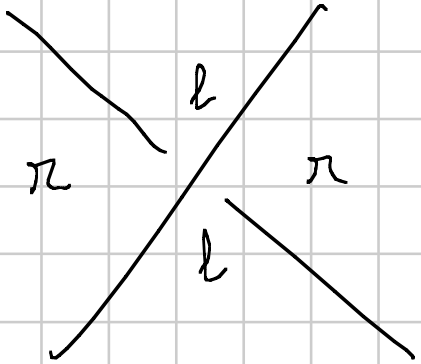
E' anzi vero che:

- tale numero di incroci è quello minimale
- D_1, D_2 diagrammi alternati ridotti rappresentano link isotopi $\Leftrightarrow$ sono isotopi come diagrammi piani

Dico: uso appaccio stato sum a P, anzi B (stesso span).

Osservo che le regioni di $S^2 \setminus D$ sono dischi (non-split).

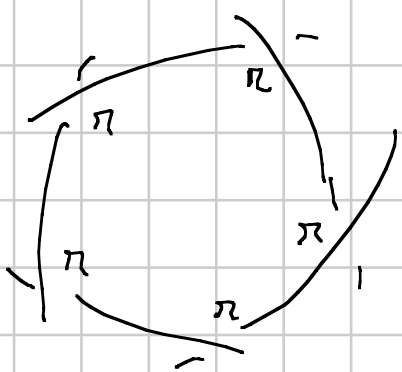
Marco gli angoli con l e r così:

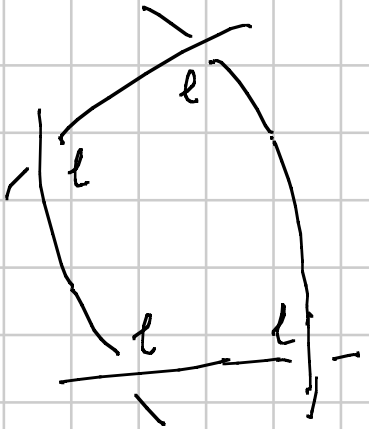


dunque uno stato left o quello incaccio significa fondere tra loro le due regioni l

$r \overset{l}{\frown} r$ left $\qquad r \underset{l}{\frown} r$ right

Nota: D alternante $\Rightarrow$ ogni regione ha tutti angoli ℓ o tutti π:





Oss: D come prof. ha $c(D)$ vertici, quadrivalenti $\Rightarrow 2c(D)$ lati

$\Rightarrow \quad c(D) - 2c(D) + \#\text{regioni} = \chi(S^2) = 2$

$\Rightarrow$ # regioni $= c(D) + 2$.

Supponiamo che ci siano L regioni con angoli ℓ
R " " " r .

$$B(D) = \sum_{s} A^{\ell(s)} \cdot A^{-r(s)} \cdot \left(-A^2 - A^{-2}\right)^{|s|-1}$$

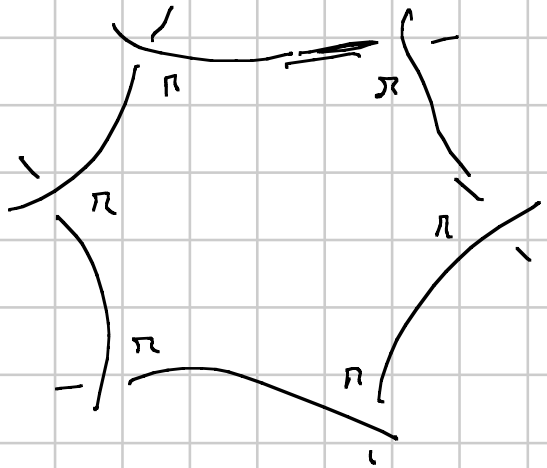
Cerchiamo esponente più alto:

$|s|$ = # componenti banali ottenute con semplificazione

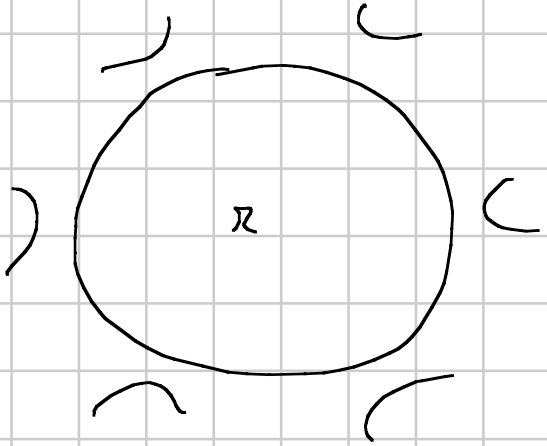
Se s contiene tutti i ($n := c(D)$) ho

$$A^{m} \cdot A^{-o} \cdot \left(-A^{2} - A^{-2}\right)^{|A|-1}$$

$\Rightarrow$ esponente $\quad m + 2|A| - 2$



$s = \text{tubi}$



$\Rightarrow |A| = R$

Claim: se cambio qualche 1 in x da qualsiasi stato s trovando s' la potenza più alta di A nel corrispondente addendo per B non può crescere e se s è tutti 1 decresce.

Questo comporta che la potenza più alta in $A \cdot B(D)$ è proprio $n+2R-2$.

Analogamente: la potenza più bassa di A è

$$-n-2L+2$$

$$\Rightarrow S(L) = n+2R-2-(-n-2L+2)$$

$$= 2n+2(R+L)-4$$

$$= 2n+2(\#\text{regioni})-4$$

$$= 2n+2(n+2)-4 = 4n$$

Resta da provare il Claim. ▨