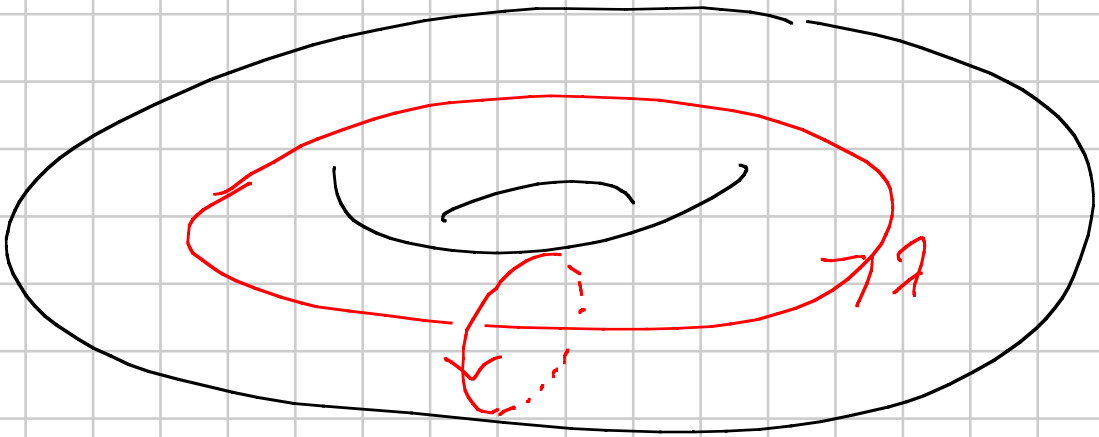
Teoria dei nodi 23/9/16

$\#$ 3-colorez. inv. per riflessione

<u>vero</u>: vediamo che è $\# \{\rho : \pi_1(\mathbb{S}^3 \setminus K) \to D_3\}_{/\text{coj.}}$

<u>Nodi toroidali</u> (o link)

Sia $U \cong D^2 \times S^1$ toro solido banalmente immerso in S^3



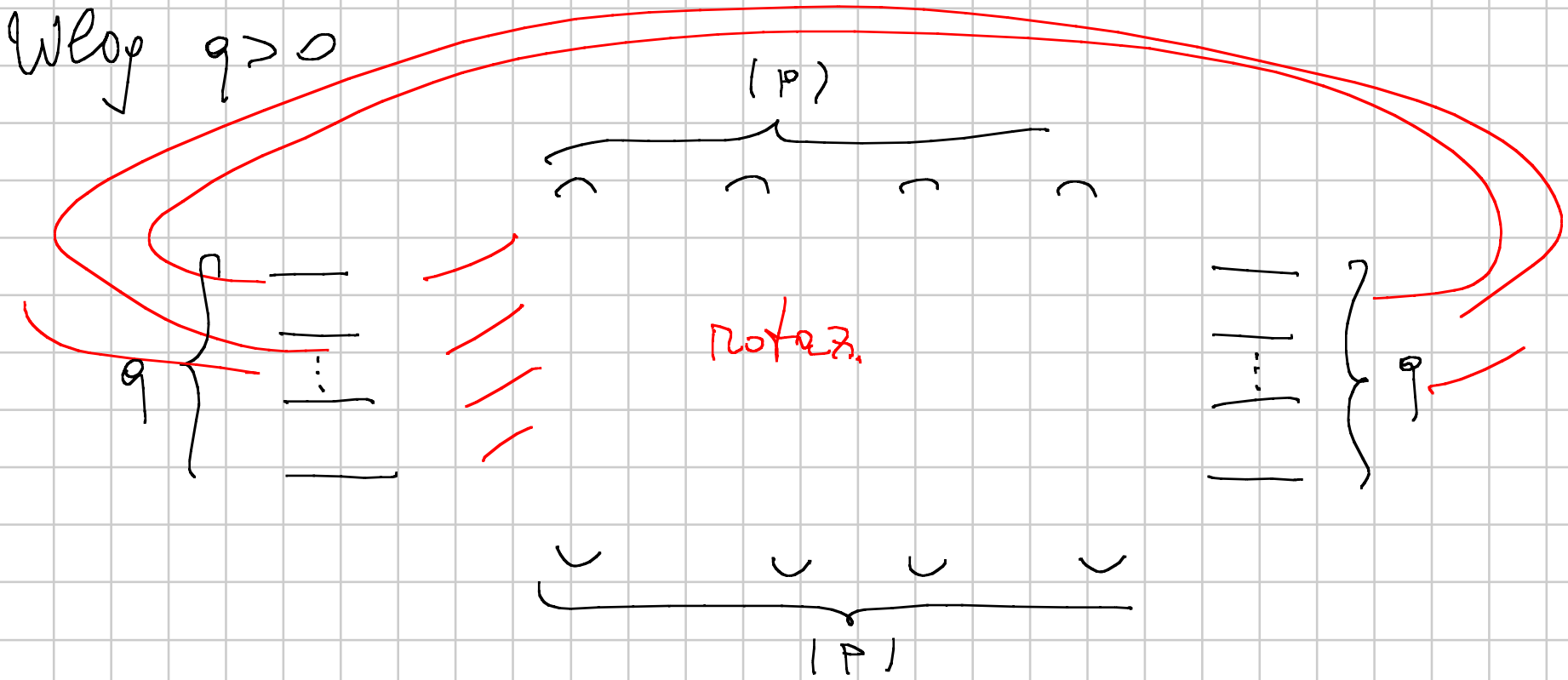
λ = curve (isotopa a longitudine su ∂U)

$\mu \subset \partial U$ meridiano t.c. (λ, μ) sia positivo.

Se $p, q \in \mathbb{Z}$ sono coprimi diamo

$K(p,q) \subset \partial U$ dato da $p\mu + q\lambda \in H_1(\partial U)$

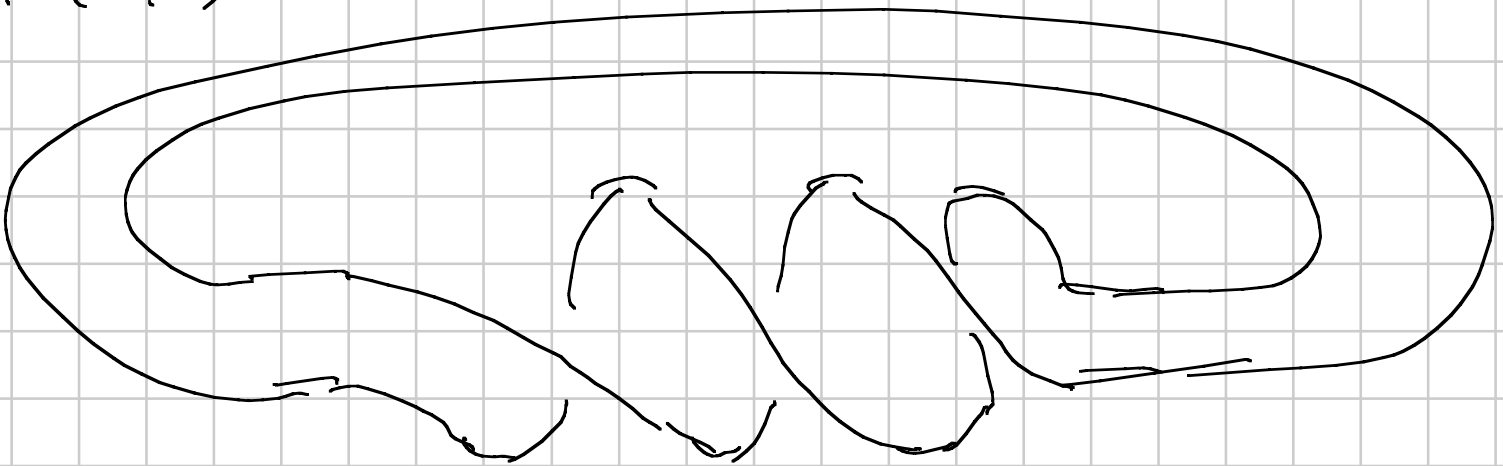
(modo orientato; se lo prendo non orientato è $\pm(p\mu + q\lambda) \in H_1(\partial U)$)



rotaz. verso dx se p>0, verso sin se p<0

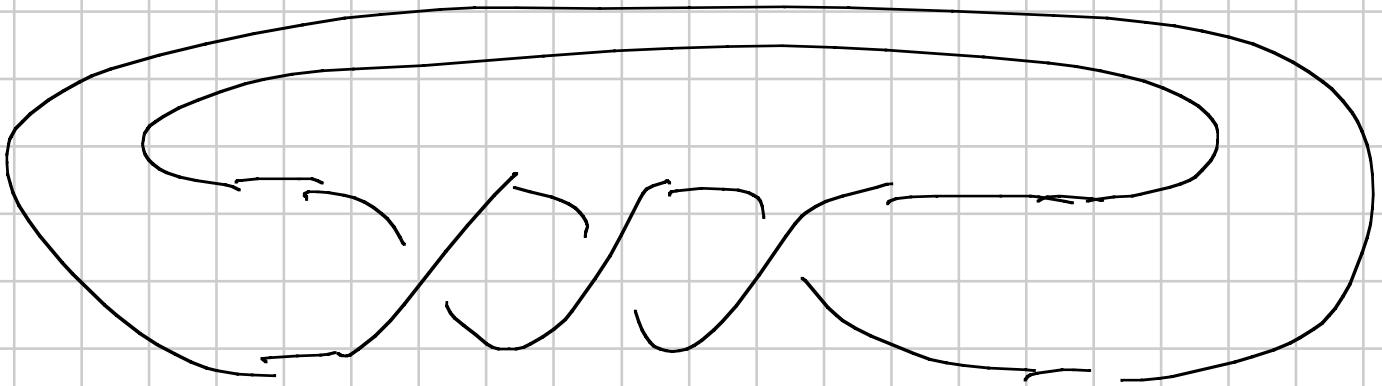
Es 1

$K(3,2)$



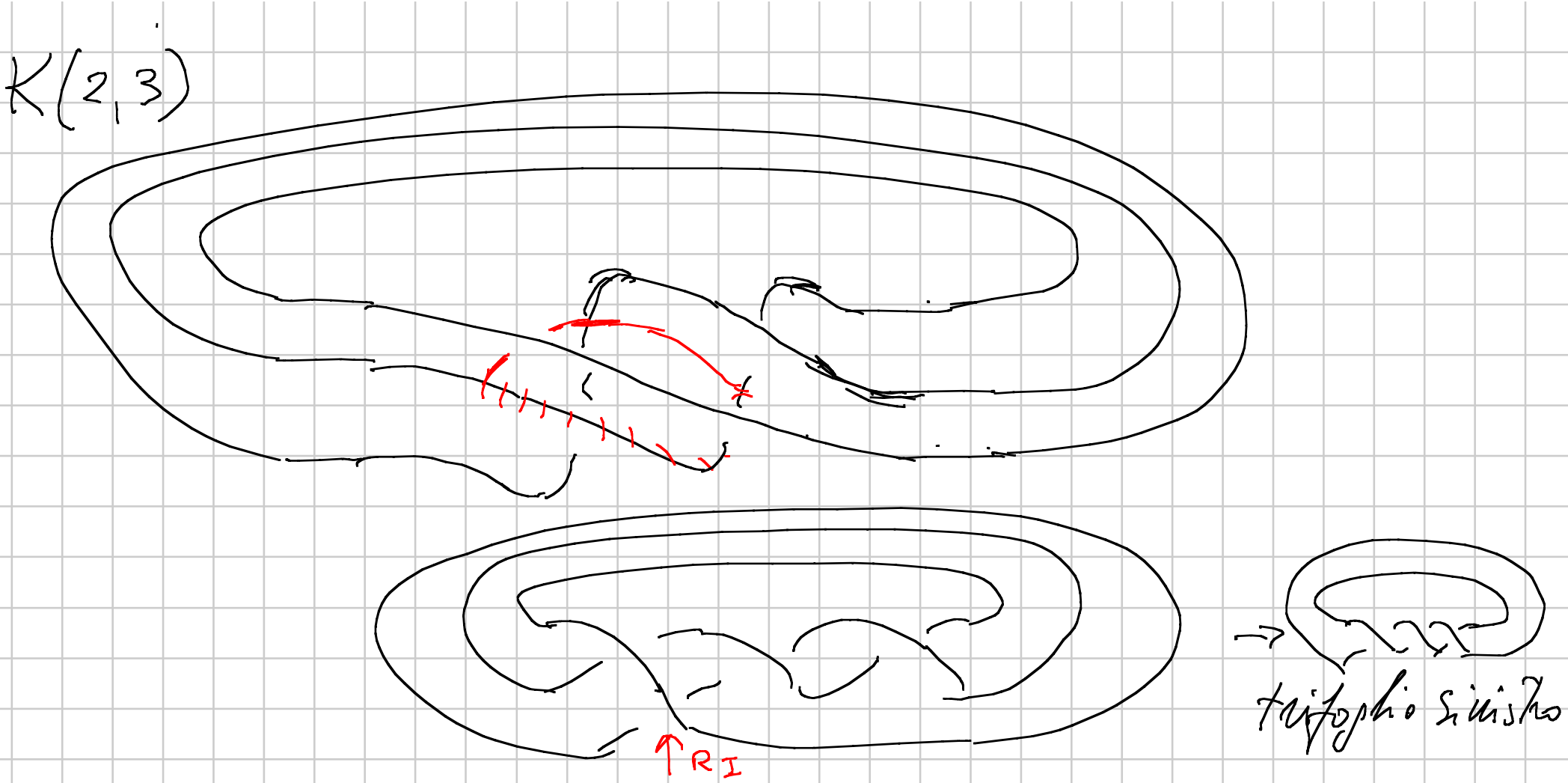
trifoglio sinistro

$K(3,-2) = K(-3,2)$



trifoglio dx

In generale $K(p,-q) = \sigma\left(K(p,q)\right)$


K(2,3)
R_I
trifoglio sinistro

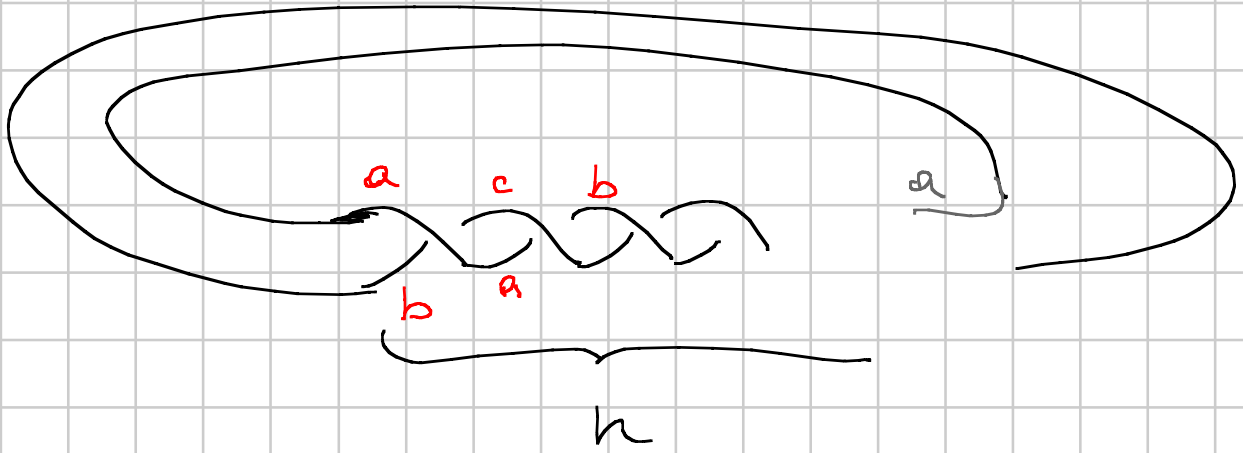
In generale $K(p,q) = K(q,p)$ poiché $\tilde{S}^3 \setminus U = \tilde{U}$ toro solido con $\tilde{\mu} = \lambda$ e $\tilde{\lambda} = \mu$.

Se p, q non sono primi fra loro

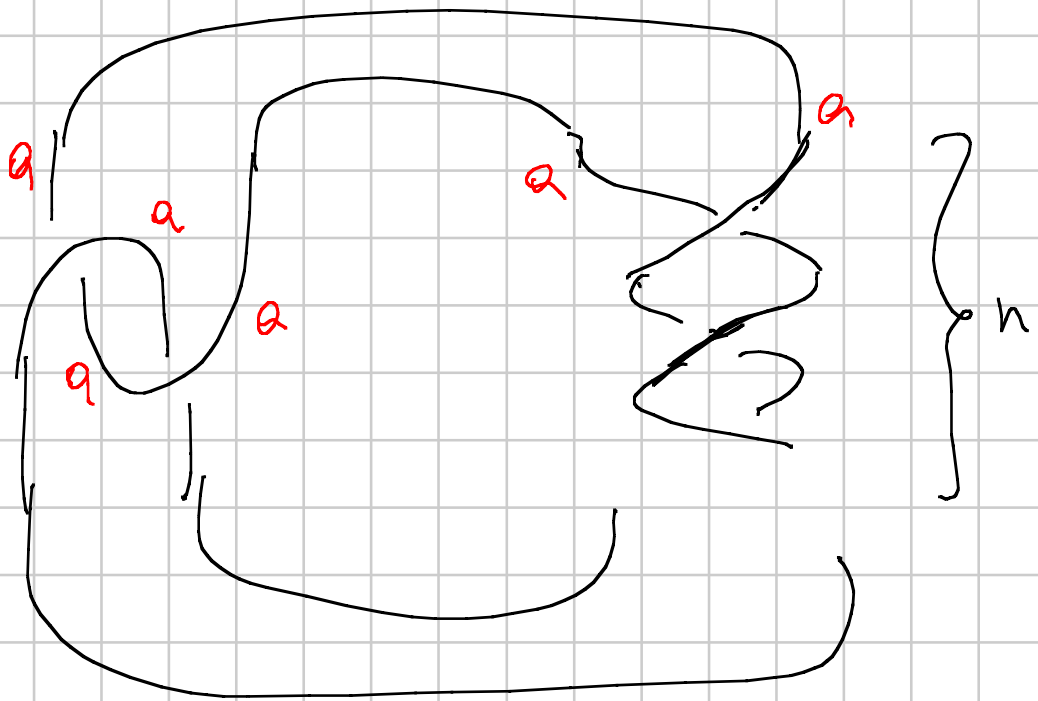
$L(p,q) =$ d copie parallele di $K(p',q')$ dove

$d = MCD(p,q)$ $\quad p' = p/d \quad q' = q/d$.

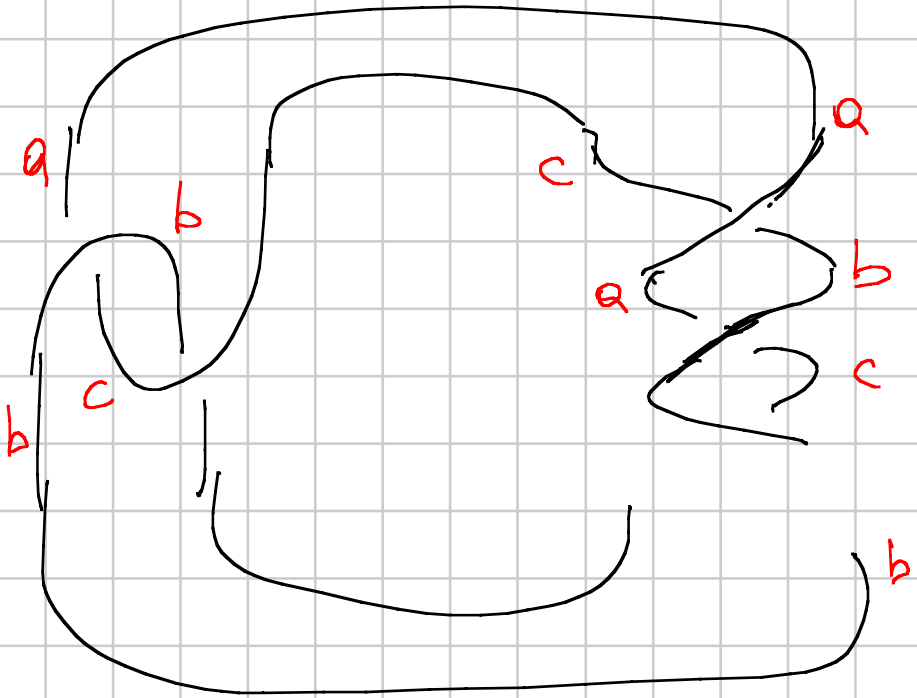
3-colorazioni di $K(2,n)$



$$= \begin{cases} 1 & \text{se } n \equiv 0 \ (3) \\ 0 & \text{altrimenti} \end{cases} \qquad n \in \mathbb{Z}$$

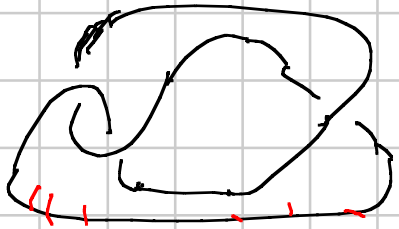


$n \in \mathbb{Z}$



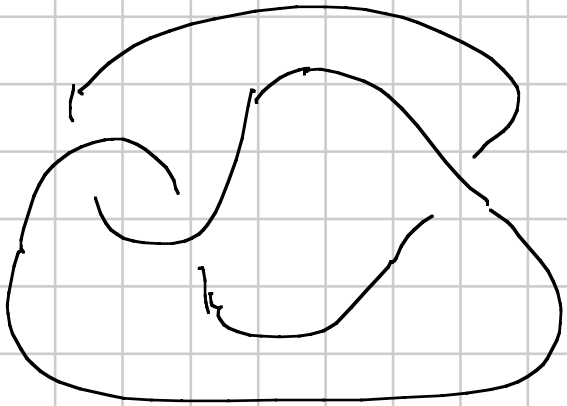
$$= \begin{cases} 1 & n \equiv 1 \ (3) \\ 0 & \text{altrimenti} \end{cases}$$

Oss: $n = 1$



trifoglio

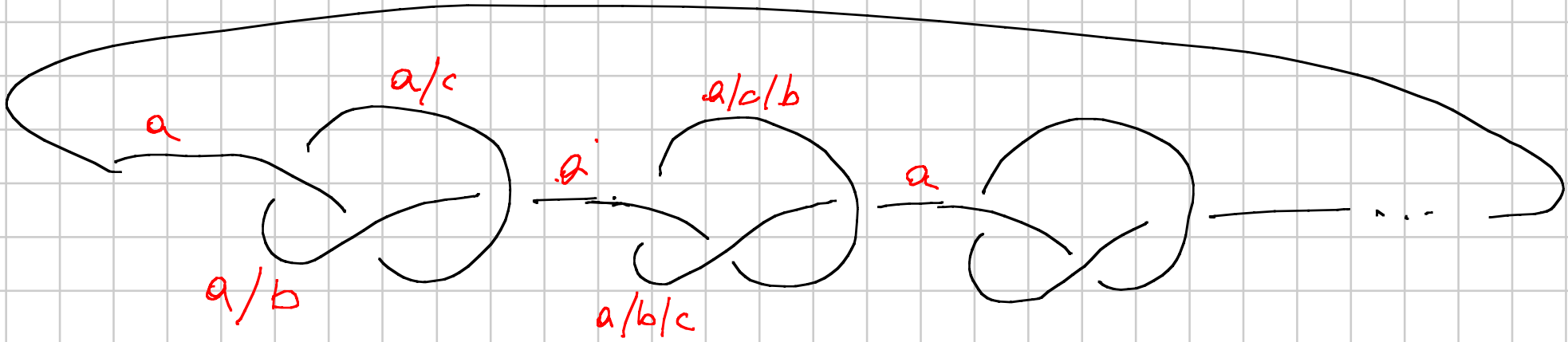
$n = -1$



banda

Prop: esistono infiniti nodi non equivalenti

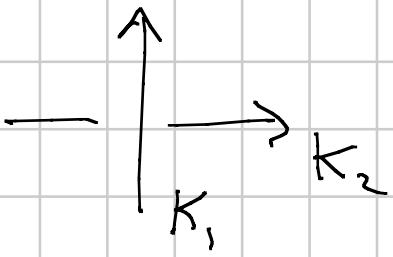
Dim:



$\Rightarrow \quad 2 \cdot 3^{n-1} - 1$ ∎

Linking number $L = K_1 \sqcup K_2$ orientiert

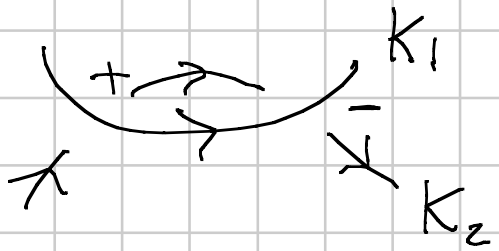
$$lk(K_1, K_2) = \sum \begin{cases} +1 \\ -1 \end{cases}$$

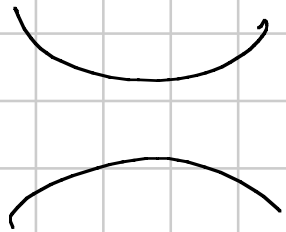


<u>Lem</u> : $lk(K_1, K_2)$ invariante par isotopie de link orientés.

Dim: R_I ✓

R_{II}



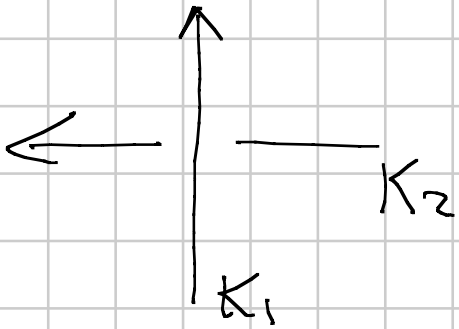


R_{III} ...

∎

Prop: $lk(K_2, K_1) = lk(K_1, K_2)$

Dim: Provo che $lk(K_1, K_2) - lk(K_2, K_1)$ è invariante per scambio di incroci. (Basta.)



$lk(K_1, K_2) = +1$

$lk(K_2, K_1) = 0$

$+1 - 0$

$+1$

$lk(K_1, K_2) = 0$

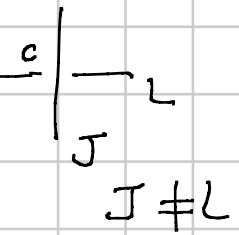
$lk(K_2, K_1) = -1$

$0 - (-1)$

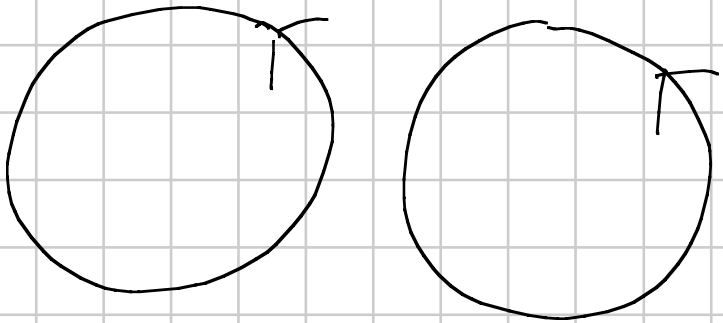
$+1$

Consequence:

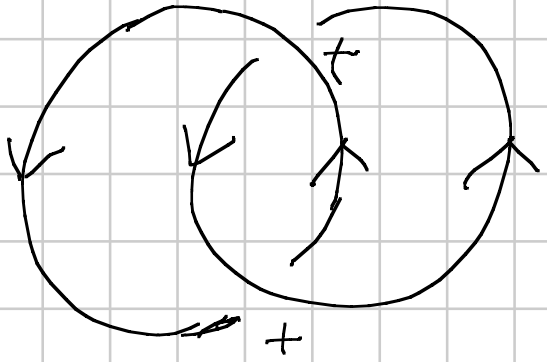
- $lk(L) = lk(K_1, K_2) = lk(K_2, K_1) = \sum_{\substack{c \\ J \neq L}} sgn(c)$



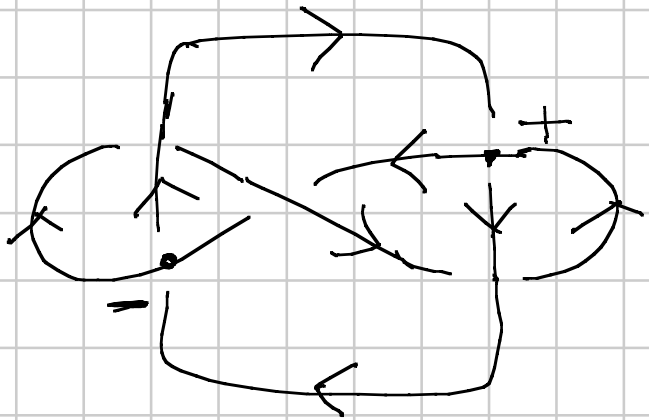
- $lk(\sigma(L)) = -lk(L)$. ☒



$lk = 0$



$lk = 1 \quad (\pm 1)$



$lk = 0$

tuttavia non è ○○

Link (o nodi) con framing

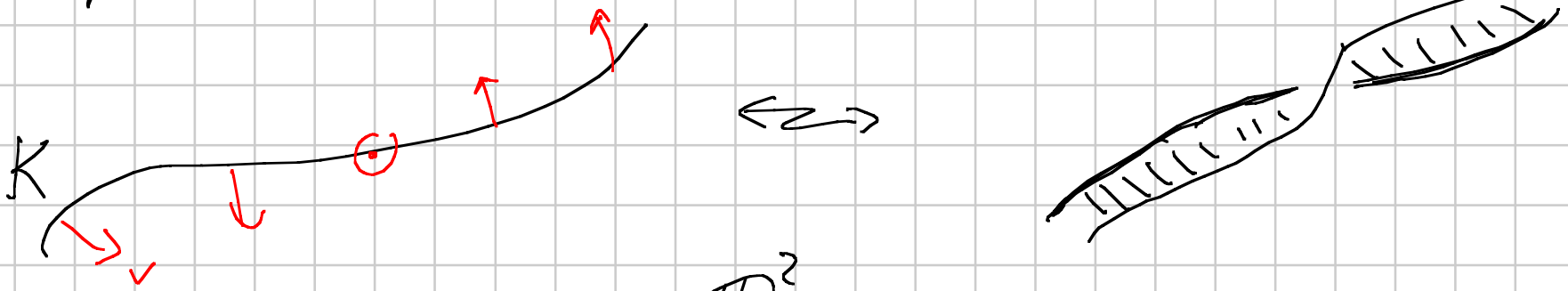
K nodo chiamo framing per K uno dei seguenti (equivalenti):

- campo di vettori mai tangenti a K lungo K
- embedding in S^3 di $S^1 \times [0,1]$ con K immagine di $S^1 \times \{0\}$

(entrambi a meno di isotopie tra oggetti della

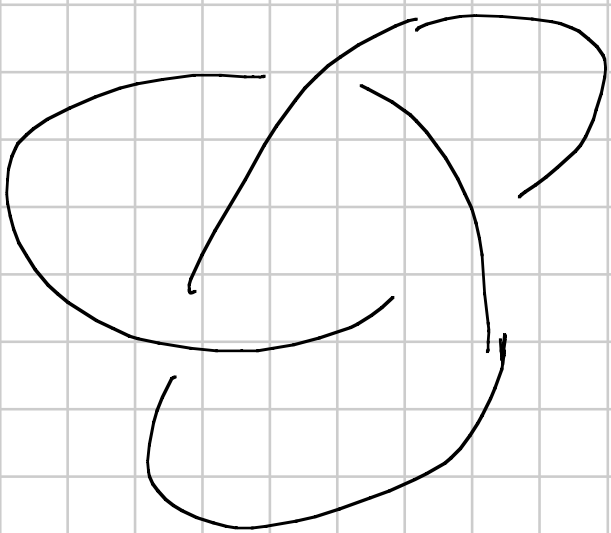
stesso tipo).

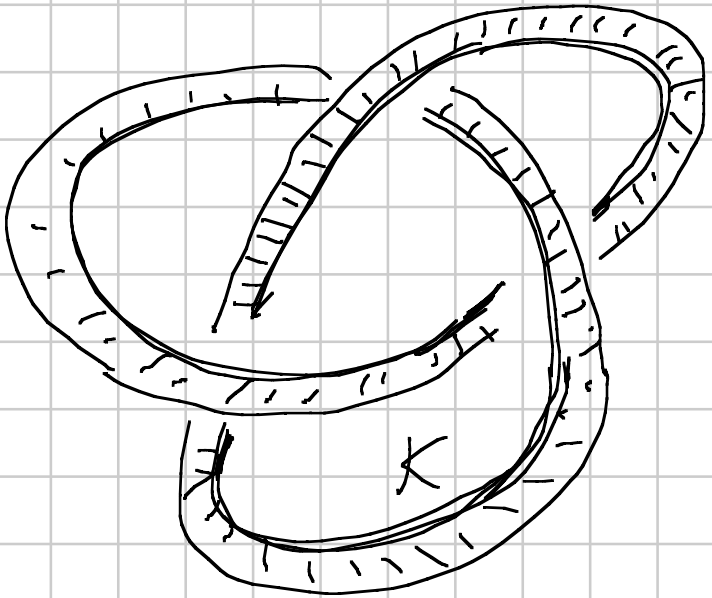
Corrispondenza:



(πR^2).

Oss: un diagramma di K fornisce un spazio per K (diagrammatico):





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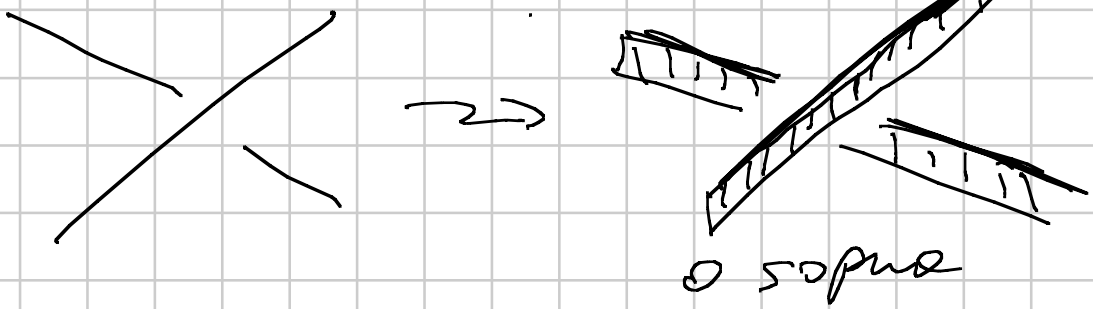
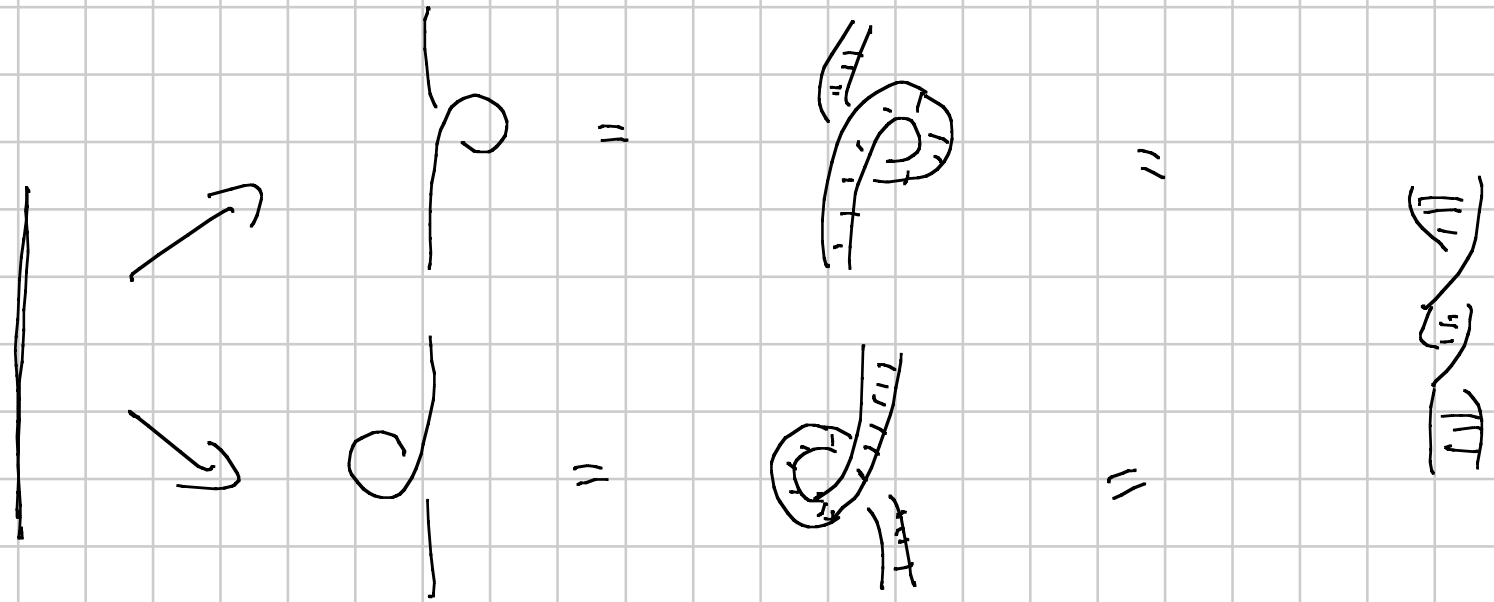


о зорро

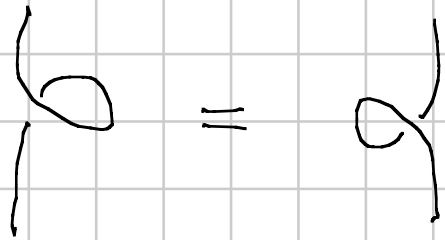
come vettore: $\mathrm{rot}_{\pm\pi/2}(K')$ oppure costante $(0,0,\pm 1)$.

Oss: R_{II}, R_{III} mantengono il framing diag.

Invece RI non lo mantiene; può



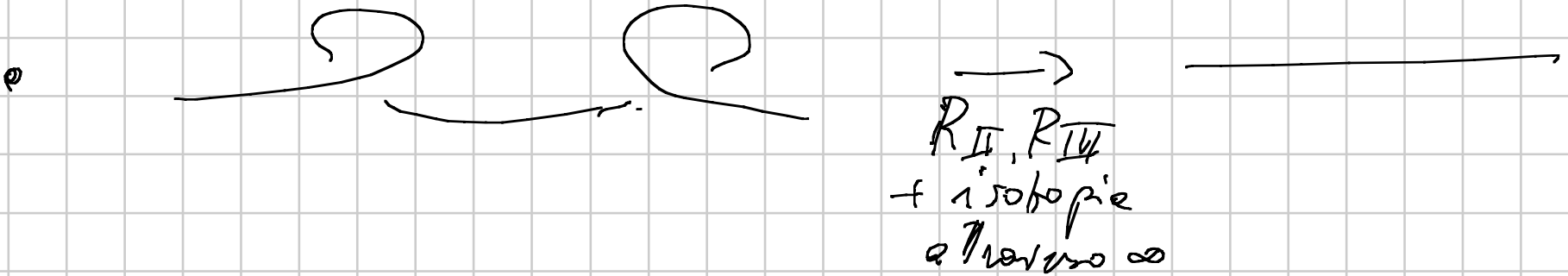
analog:



come framing.

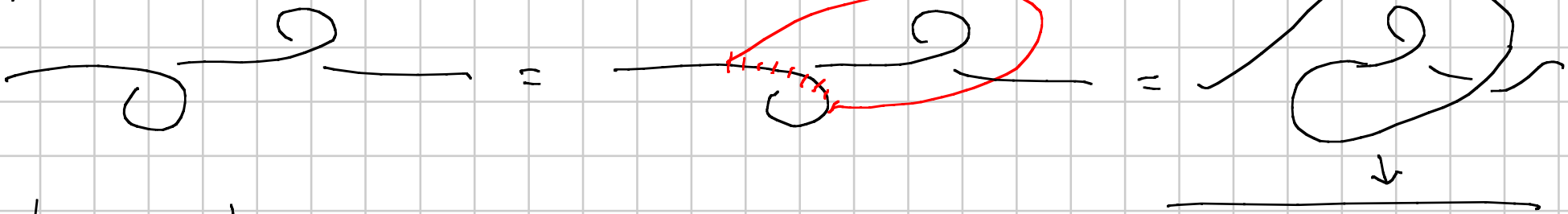
Q: poiché $\beta \to \alpha$ mantiene il fascio γ, è generato da R_{II}, R_{III}?

Sì se ammetto isotopie attraverso ∞:

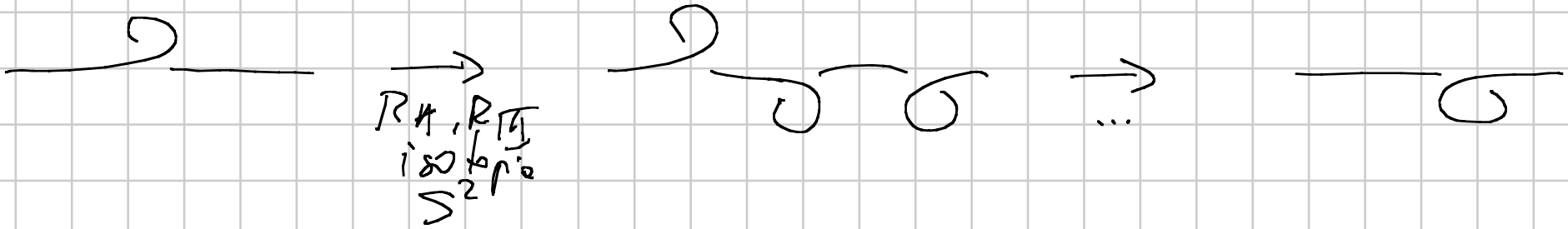


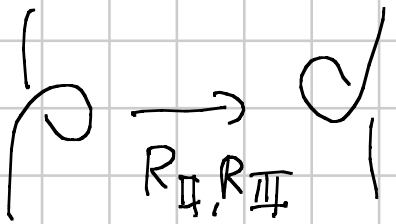
$R_{\square}, R_{\square}$

Quotient



Combinando:



Q: vero su $\mathbb{R}^2$ che  .

A: NO. Se parametrizzo K tramite $\alpha : S^1 \to \mathbb{R}^3$

$\beta : S^1 \to \mathbb{R}^2$
proiez. ortog. di α

$\longrightarrow \boxed{\deg\left(\frac{\beta'}{\|\beta'\|}\right) : S^1 \to S^1}$

è ben def. e invariante tramite R_{II} e R_{III}

E ho:

$$\left| \deg \frac{F'}{\|F'\|} \right| = 2 \qquad\qquad 4$$

Q: quanti sono i framing su un nodo K?

Fisso un framing n vettore normale a K.

Scelgo una parametrizzazione α di K (due modi)
e il riferimento $t = \frac{\alpha'}{\|\alpha'\|}$, n, $b = t \wedge n$.

Sia v un campo = vettore normale a K (unitario)

$$\Rightarrow v(z) = \cos(\vartheta(z)) \cdot n(z) + \sin(\vartheta(z)) \cdot b(z)$$

È definito $i(v, n) = \deg\left(S^1 \ni z \mapsto e^{i\vartheta(z)} \in S^1\right)$.

↑ devo provare che è indipendente da α:

$$\bar{\alpha}(z) = \alpha(\bar{z})$$

$$\bar{t}(z) = -t(\bar{z})$$

$$\bar{n}(z) = n(\bar{z})$$

$$\bar{b}(z) = -b(\bar{z})$$

$$v(z) = \cos\nu(z)\cdot n(z) + \sin\nu(z)\cdot b(z)$$

$$\Rightarrow \bar{v}(z) = \cos(\nu(z))\cdot\bar{n}(z) - \sin(\nu(z))\cdot\bar{b}(z)$$

$$\Rightarrow \bar{v}(z) = -v(\bar{z})$$

Nel calcolo usando $\bar{\alpha}$ ho

$$\deg\left(z \mapsto e^{-i\nu(\bar{z})}\right) = \deg\left(z \mapsto e^{i\nu(z)}\right).$$

Poiché le mappe $S^1 \to S^1$ / omotopie sono classificate dal grado otteniamo che $i(V, M)$ è invariante completo per V.

$\Rightarrow$ Fissato un framing preferito η ho

$$\{\text{framings su } K\} \longleftrightarrow \mathbb{Z}$$

(indipendente dall'orientazione).

Fatto: esiste una scelta di n privilegiata (=framing canonico su K).

Dato $K \subset S^3$ chiamo $U(K) \cong D^2 \times S^1$ intorno regolare di K e $E(K) = S^3 \setminus \mathring{U}(K)$

$$T(K) = \partial U(K) = U(K) \cap E(K).$$

Mayer-Vietoris

$$S^3 = U(K) \cup E(K)$$

$$H_2(S^3) \longrightarrow H_1(T(K)) \cong H_1(U(K)) \oplus H_1(E(K)) \longrightarrow H_1(S^3)$$

$$\| \qquad\qquad \| \qquad\qquad\qquad \| \qquad\qquad\qquad \|$$

$$0 \qquad\quad \mathbb{Z}^2 = \langle \mu, \lambda \rangle \qquad \mathbb{Z} = \langle \lambda \rangle \qquad\quad 0$$

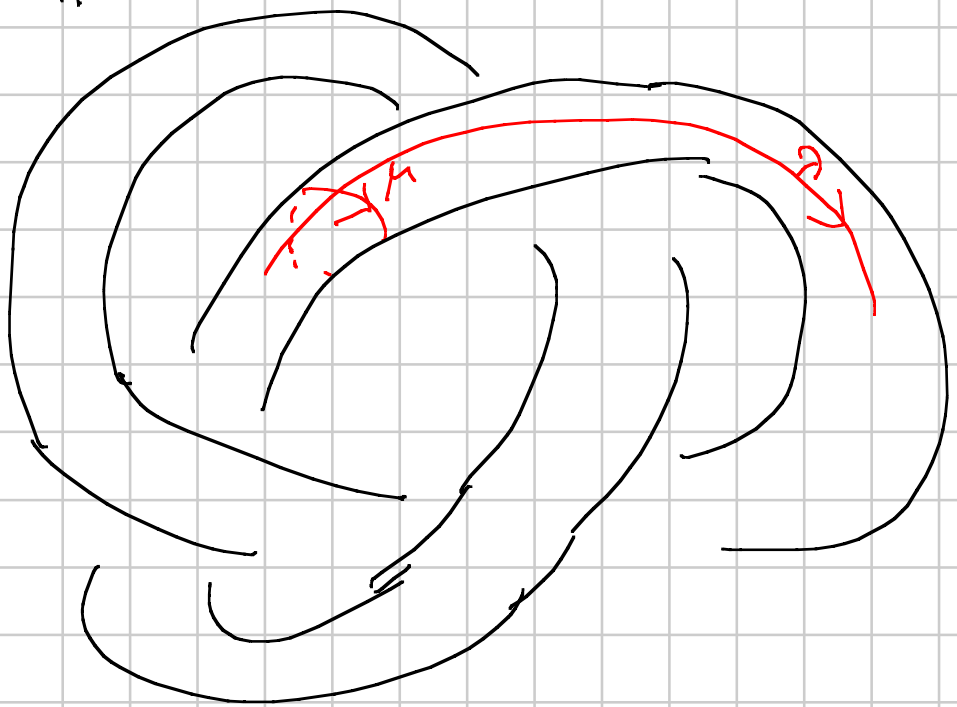
$$\Longrightarrow H_1(E(K)) \cong \mathbb{Z} = \langle \mu \rangle .$$

Ora consideriamo:

$$H_1(T(K)) \longrightarrow H_1(E(K))$$

$$\mu \longmapsto \mu$$

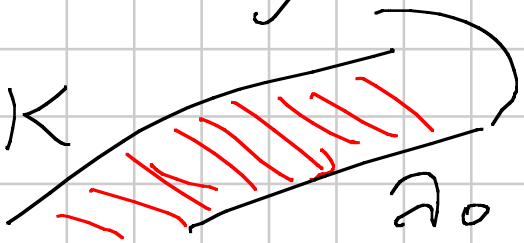
$$\lambda \longmapsto a\mu \qquad a \in \mathbb{Z}$$



$\Rightarrow \lambda_0 = \lambda - a\mu$ l'unico elemento di $H_1(T(K))$ che è nullo in $H_1(E(K))$.

Chiamo λ_0 longitudine privilegiata per K.
(Unica come curva non orientata).

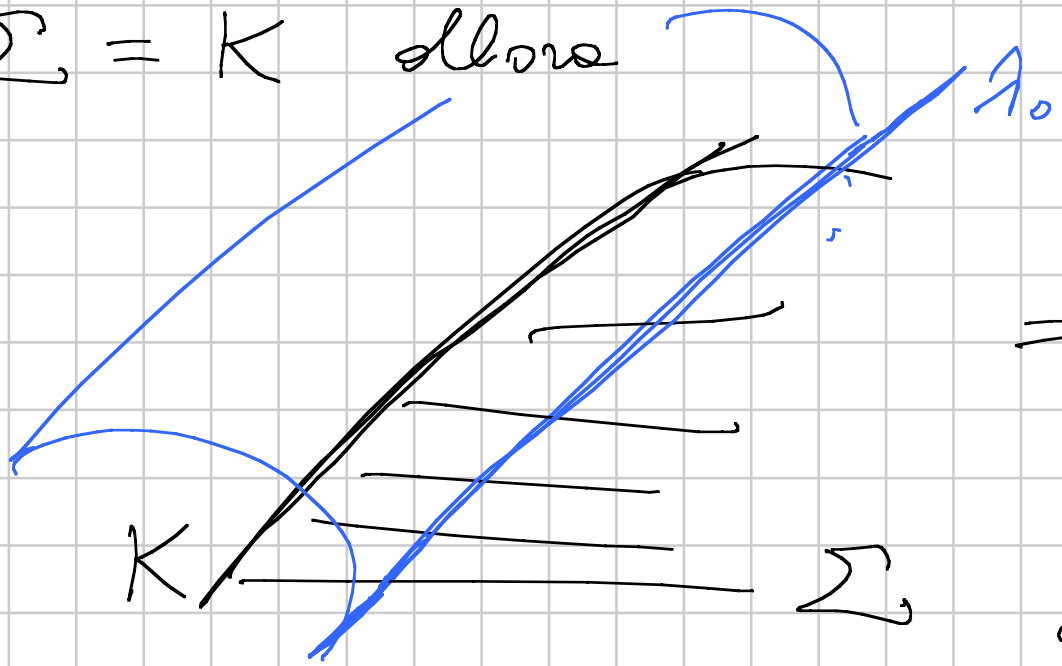
Framing privilegiato lungo K: quello che passa da K e λ_0.



Come trovare λ_0?

Oss: se trovo $\Sigma \subset S^3$ orientabile embedded t.c.

$\partial \Sigma = K$ allora



$\Rightarrow T(K) \cap \Sigma$ è λ_0 poiché $\Sigma \setminus \mathring{U}(K)$ è omotopicamente λ_0 in $E(K)$.

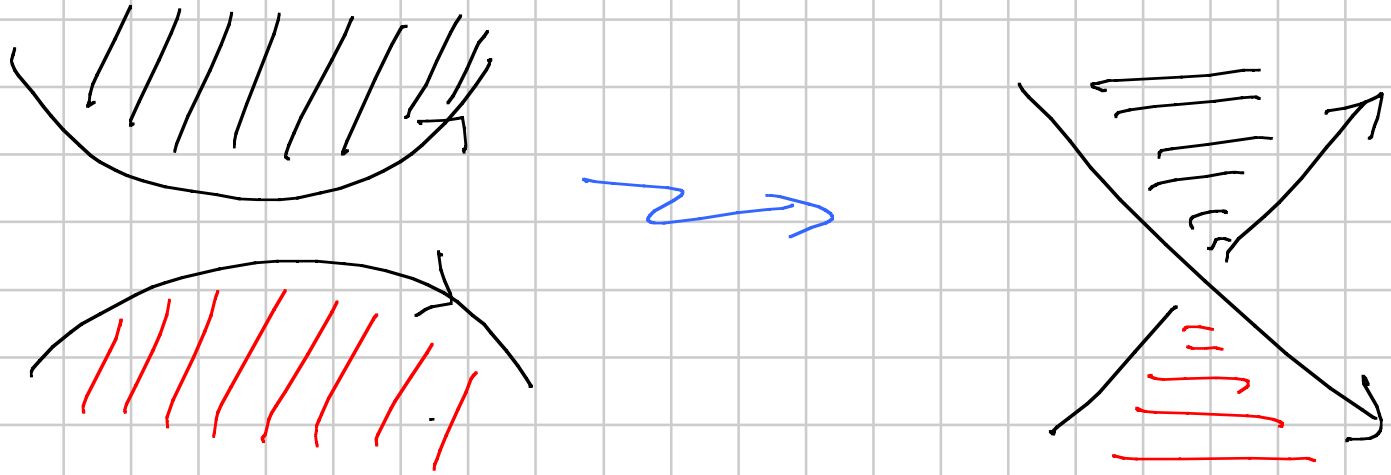
Come costruire Σ? Algoritmo di Seifert (nodi):

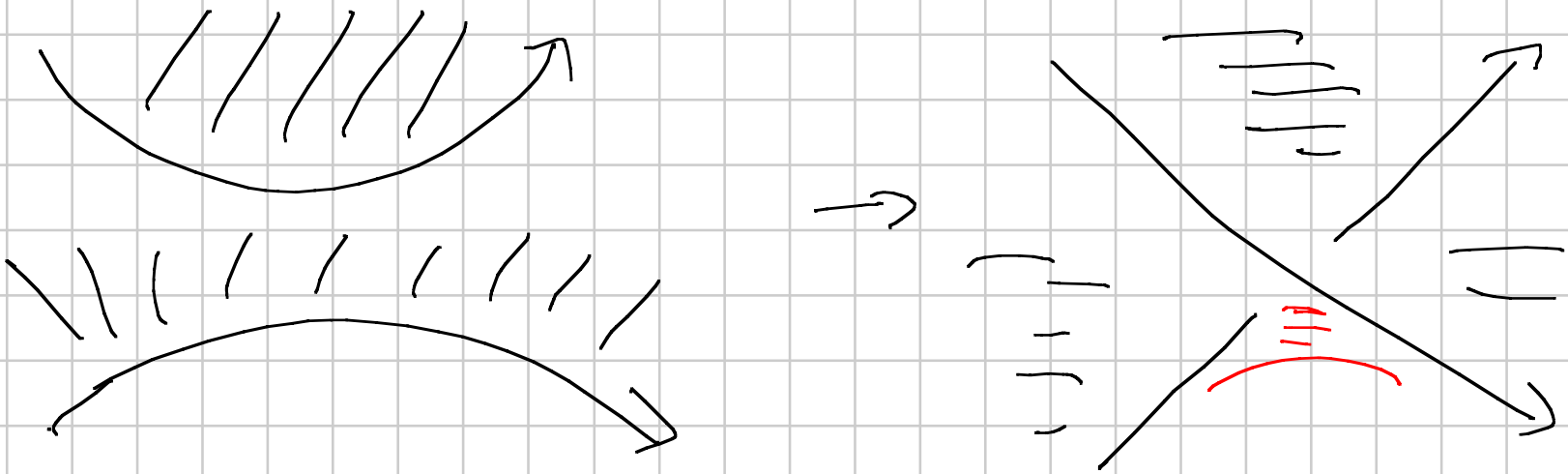
1. Oriento a caso il nodo
2. Sciolgo gli incroci usando l'orientazione:

3. Ho collezione di S^1 orientati nel piano;

le metto ad altezze diverse e introduco dischi orientati che sono bordati da esse

4. Ricostruisco K e Σ aggiungendo "quadrati twistati" lungo bordo dei dischi





Def: chiamo self-linking number di un diagramma di nodo la somma algebrica dei segni dei suoi incroci rispetto a un'orientazione qualsiasi.

Ben def:

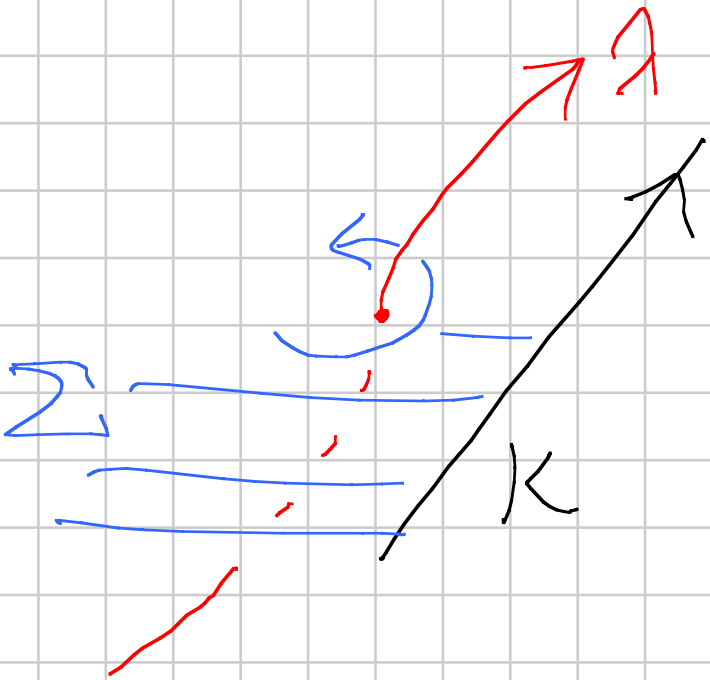
Teo: l'indice del framing diagrammatico di un diagramma rispetto al framing privilegiato è il self-linking number del diagramma.

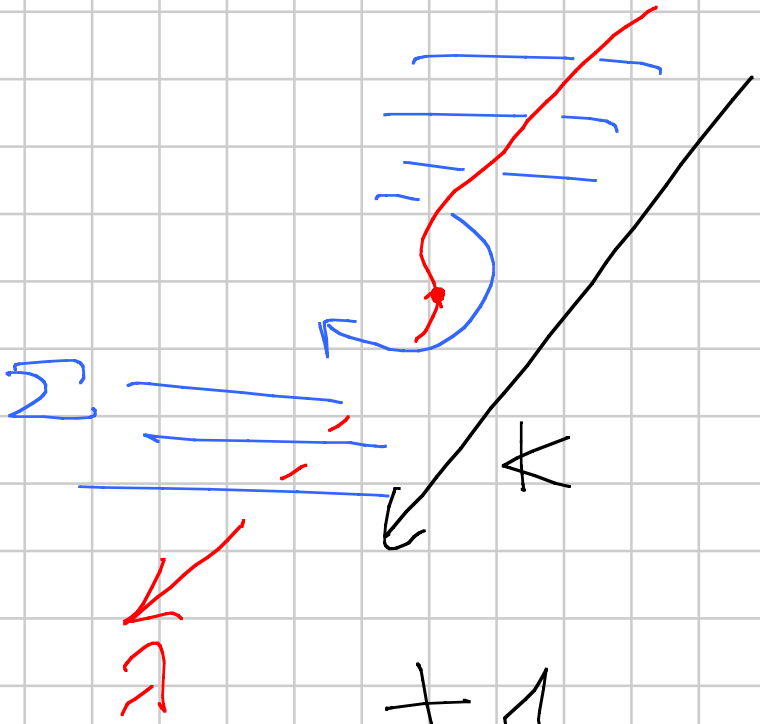
Dimo: (1) Vu parende per $\lambda = \lambda_0 + p \cdot \mu$

$$\Rightarrow p = \#_{alg}(\lambda \pitchfork \Sigma)$$

Σ sup. di Seifert.

(Att.: indipendente da orientazione
devo orientare K per orientare Σ
e orientato λ parallelamente a K.
Se inverto l'orientaz. iniziale cambio tutto ma





$+1$ $\quad$ $+1$ $\quad$ $\pm \#_{\text{alg}}(\gamma \cap \Sigma)$

(lo devo provare).

(2) λ Young diagram

$\Longrightarrow$ self-linking number of diagram $= \#_{alp}(\lambda \cap \Sigma)$.

$\blacksquare$