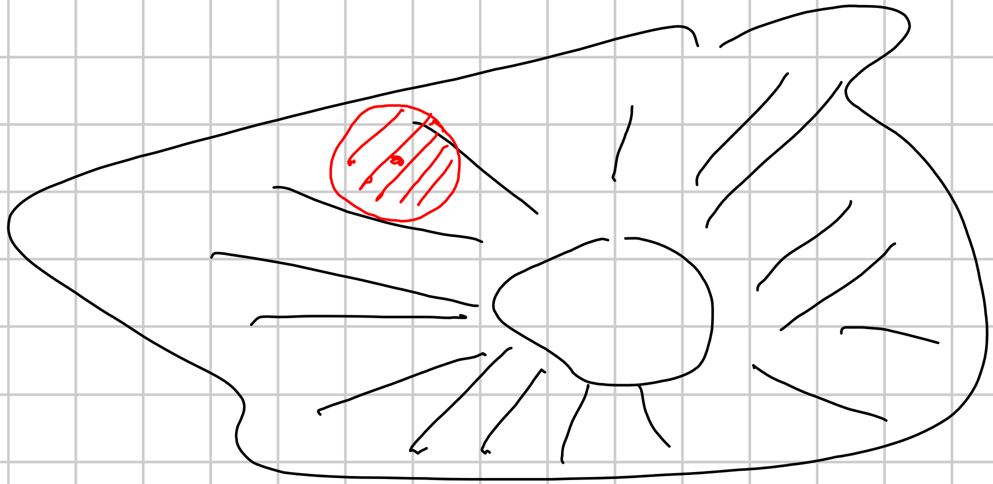
Geometria 16/5/15

$\Omega \subset \mathbb{R}^n$ è aperto se $\forall x \in \Omega$ $\exists r$ t.c.
$B(x, r) \subset \Omega$



$g, h : \Omega \to \mathbb{R}$ $\quad \omega = g\,dx + h\,dy$ $\quad$ 1-forme sur Ω

$\alpha : [a,b] \to \Omega$

$$\int_\alpha \omega = \int_a^b \left(g(\alpha(t)) \cdot X'(t) + h(\alpha(t)) \cdot Y'(t) \right) dt$$

$U : \Omega \to \mathbb{R}$ $\quad dU = \frac{\partial U}{\partial x} \cdot dx + \frac{\partial U}{\partial y} \cdot dy$

$$\Rightarrow \int_\alpha dU = U(\alpha(b)) - U(\alpha(a)) .$$

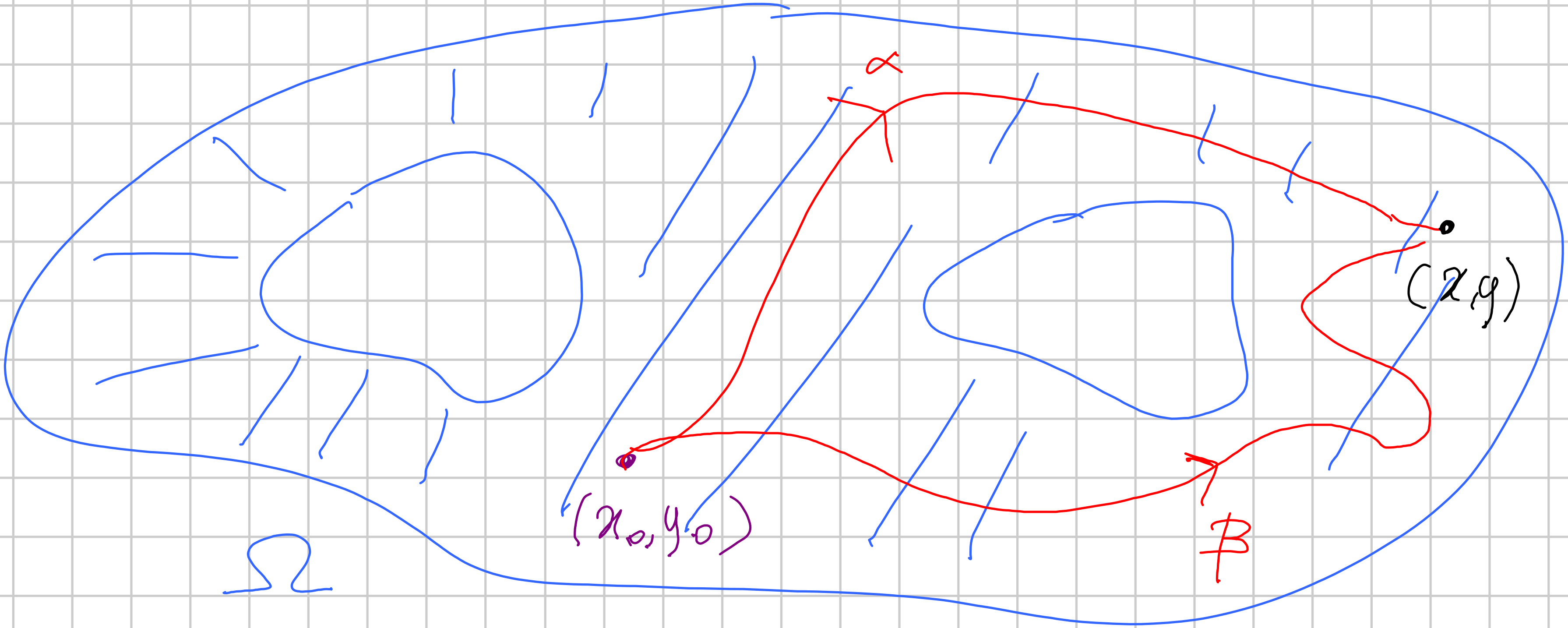
Def: ω è esatta se $\exists U$ t.c. $\omega = dU$.

Prop: ω esatta $\Longleftrightarrow$ $\int_\alpha \omega$ dipende solo dagli estremi di α.

Dim: $\Longrightarrow$ visto sopra.

$\Longleftarrow$ Devo costruire un potenziale U.

Fisso $(x_0, y_0) \in \Omega$ e stabilisco che $U(x_0, y_0) = 0$.

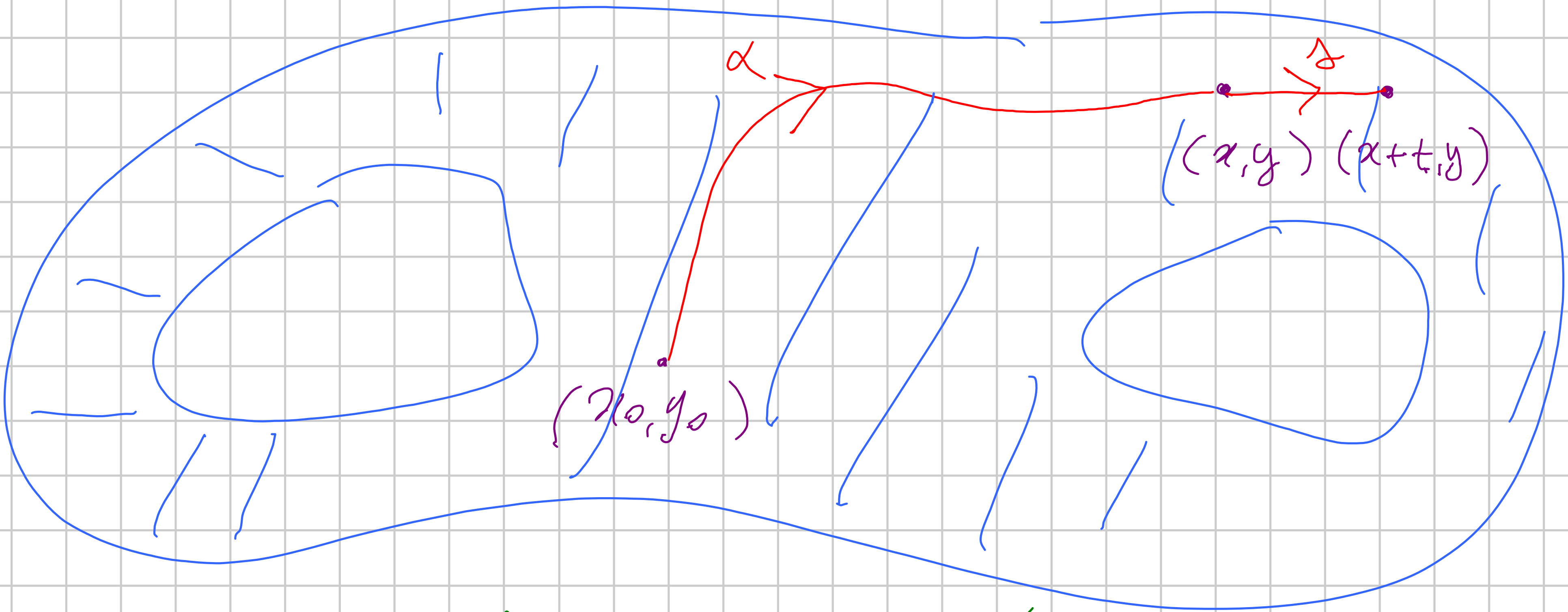


Per $(x,y) \in \Omega$ qualsiasi definisco $U(x,y)$ come

$$U(x,y) = \int_\alpha \omega \qquad \alpha = \text{qualsiasi curva che va da } (x_0, y_0) \text{ a } (x, y)$$

U ben def grazie a ipotesi.

Resta da veder che $dU = \omega$, cioè se $\omega = g\,dx + h\,dy$ che $\frac{\partial U}{\partial x} = g \qquad \frac{\partial U}{\partial y} = h$.

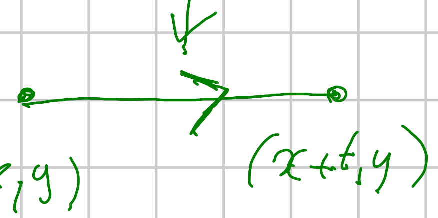


$$\frac{\partial U}{\partial x}(x,y) = \lim_{t\to 0} \frac{U(x+t,y) - U(x,y)}{t}.$$

$$= \lim_{t\to 0} \frac{1}{t}\left(\cancel{\int_\alpha \omega} + \int_\gamma \omega - \cancel{\int_\alpha \omega} \right)$$

$u \longmapsto (x+u, y)$

$0 \leq u \leq t$



$$= \lim_{t\to 0} \frac{1}{t} \int_0^t \Big(g(x+u,y)\cdot 1 + h(x+h,y)\cdot 0 \Big)\, du$$

$$= \lim_{t\to 0} \frac{1}{t} \int_0^t g(x+u,y)\, du = g(x,y).$$

Analogamente $\partial U/\partial y = h.$ $\blacksquare$

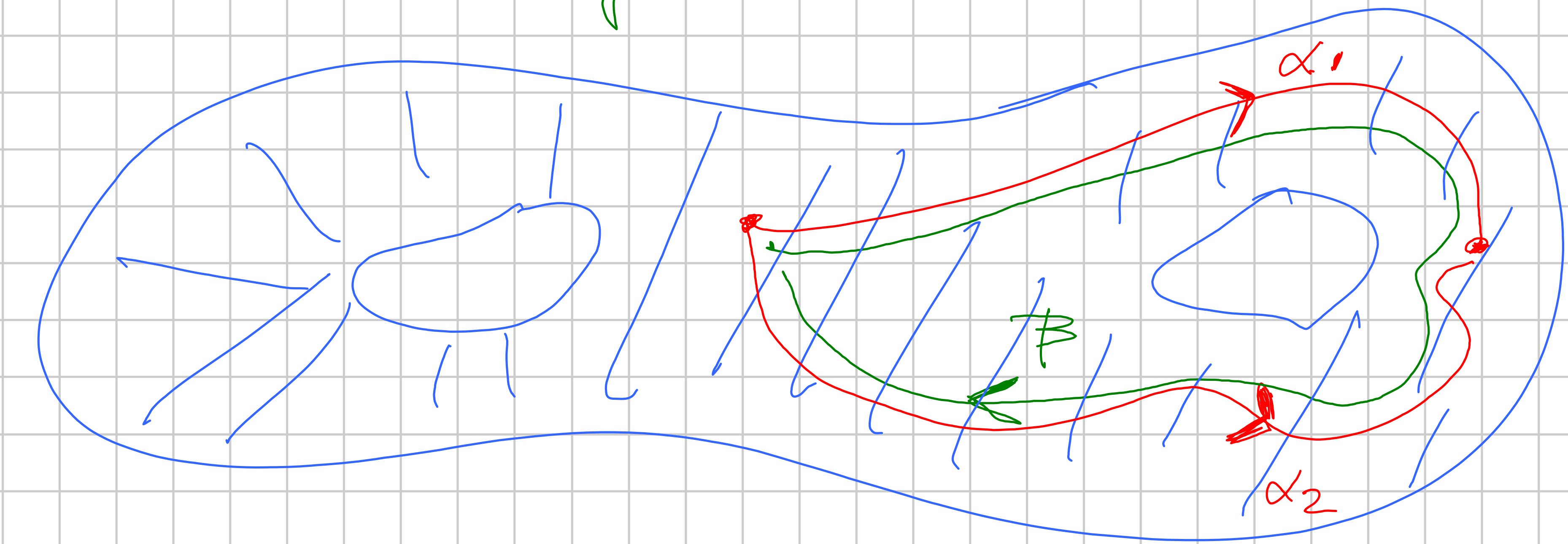
Visto:

$$\omega \text{ esatta } \iff \int_\alpha \omega \text{ dipende solo dagli estremi di } \alpha$$

<u>Cor</u>: ω esatta $\iff \displaystyle\int_\beta \omega = 0 \quad \forall \beta$ chiusa.

<u>Dim</u>: $\Longrightarrow$: $\omega = dU \Longrightarrow \displaystyle\int_\beta \omega = U(\beta(b)) - U(\beta(a)) = 0$

$\Leftarrow$: basta vedere che $\int_\alpha \omega$ dipende solo dagli estremi di α :



$\beta =$ "α_1 seguito da α_2 al contrario" è chiusa

$$\Longrightarrow \int_\beta \omega = 0 \Longrightarrow \int_{\alpha_1} \omega - \int_{\alpha_2} \omega = 0 . \quad \blacksquare$$

<u>Oss</u>: impossibile verificare che $\int_\alpha \omega$ dipende solo da estremi

o che $\int_\beta \omega = 0$ con β chiusa prendendo ogni α o ogni β.

$$\omega = g\,dx + h\,dy$$

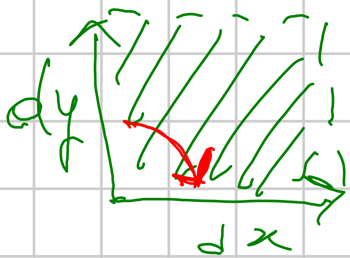
"Inventiamo" $d\omega$:

$$d\omega = dg \cdot dx + g \cdot \underbrace{ddx}_{0} + dh \cdot dy + h \cdot \underbrace{ddy}_{0}$$

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y} \cdot dy \qquad dh = \frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}\,dy$$

non dipendono dal punto

$$= \frac{\partial g}{\partial x} \underbrace{dx\, dx}_{0} + \frac{\partial g}{\partial y}\, dy\, dx + \frac{\partial h}{\partial x}\, dx\, dy + \frac{\partial h}{\partial y} \underbrace{dy\, dy}_{0}$$



el d'area

$$= -\,dx\,dy$$

Def: $d(g\,dx + h\,dy) = \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right) dx\,dy$

Dirò che ω è chiusa se $d\omega = 0$.

Prop: ω esatta $\Longrightarrow$ ω chiusa.

Dim: ω esatta: $\exists U$ t.c. $\omega = dU$

cioè se $\omega = g\,dx + h\,dy$ ho $g = \frac{\partial U}{\partial x}$, $h = \frac{\partial U}{\partial y}$.

Devo vedere che $d\omega = 0$:

$$d\omega = \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right) dx\,dy$$

$$= \left(\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 U}{\partial y \partial x}\right) dx\,dy = 0. \quad \blacksquare$$

ES: $\omega(x,y) = \cos(x-3y)\,dx + 7e^{x^2y^3}\,dy$

$$\Longrightarrow d\omega = \underbrace{\left(7\cdot 2x\cdot e^{x^2y^3} - \left((-3)\cdot(-\sin(x-3y))\right)\right)}_{\text{non è } \equiv 0} dx\,dy$$

$\Longrightarrow$ ω non è chiusa

$\Longrightarrow$ ω non è esatta.

Sarà vero che ω chiusa $\Longrightarrow$ ω esatta?

$\boxed{\text{No}}$

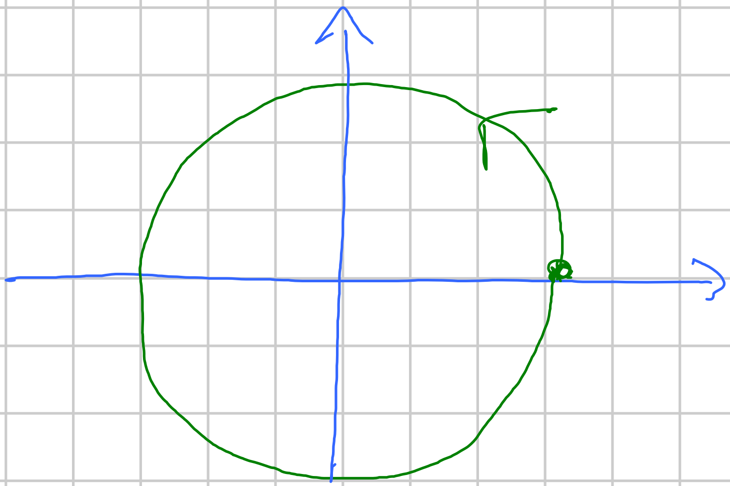
$$\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$$

$$\omega(x,y) = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

ω chiusa:
$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = 0$$

ω non esatta :

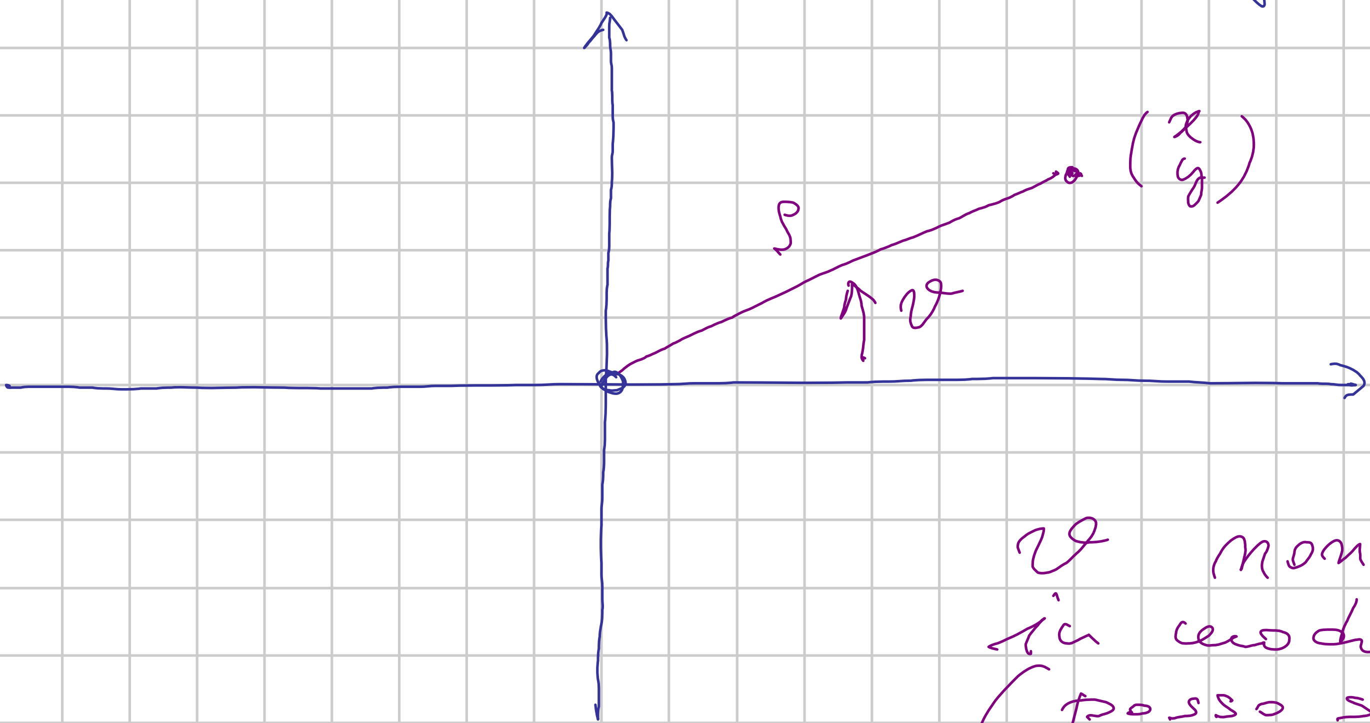


$$\alpha(t) = (\cos(t), \sin(t))$$

$$\int_\alpha \omega = \int_0^{2\pi} \frac{-\sin(t)\cdot(-\sin(t)) + \cos(t)\cdot\cos(t)}{\cos^2(t)+\sin^2(t)}\,dt$$

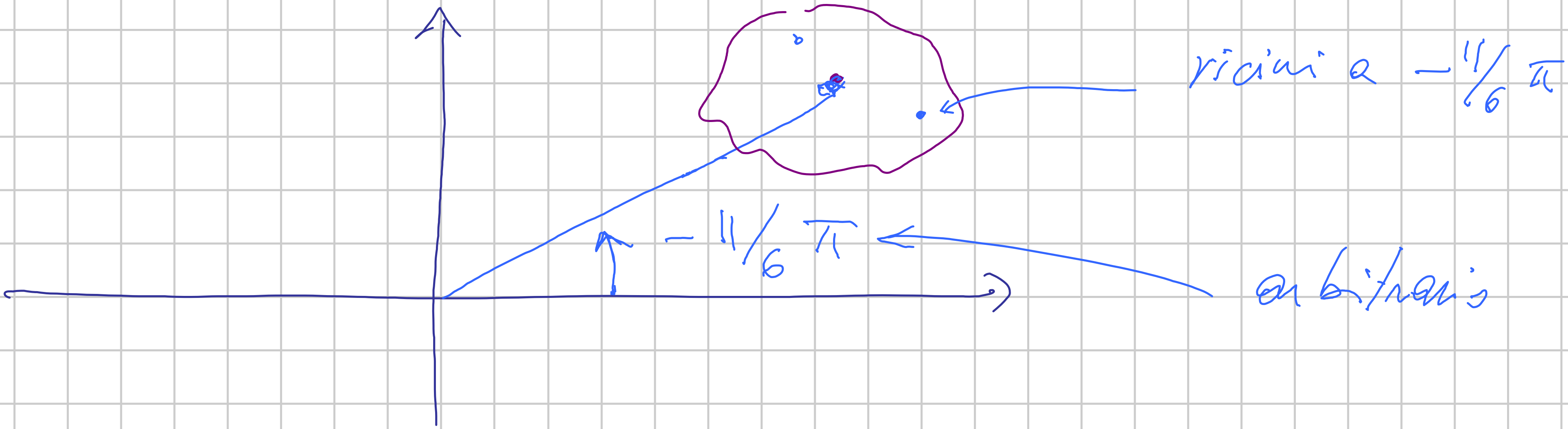
$$= \int_0^{2\pi} dt = 2\pi \neq 0$$

Da dove viene $\dfrac{-y\,dx + x\,dy}{x^2+y^2}$?

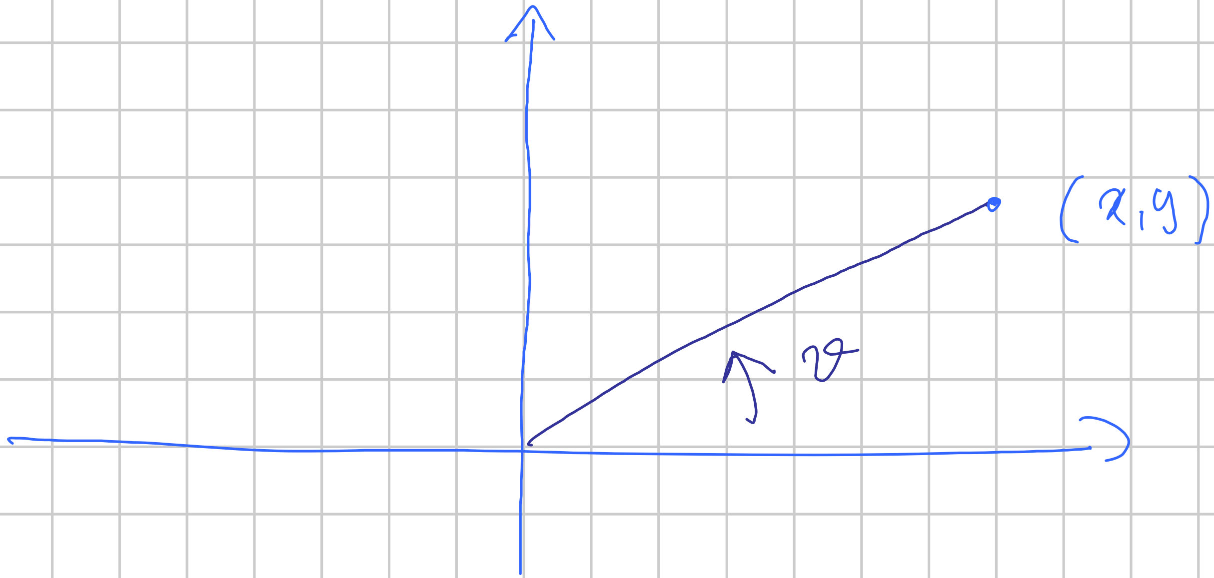


ϑ non è definito in modo unico

(posso sommare $2k\pi$, $k \in \mathbb{Z}$)



ϑ definita in modo ambiguo ma $d\vartheta$ è ben definita.



1) $\vartheta = \operatorname{arctg}\frac{y}{x} + k\pi$

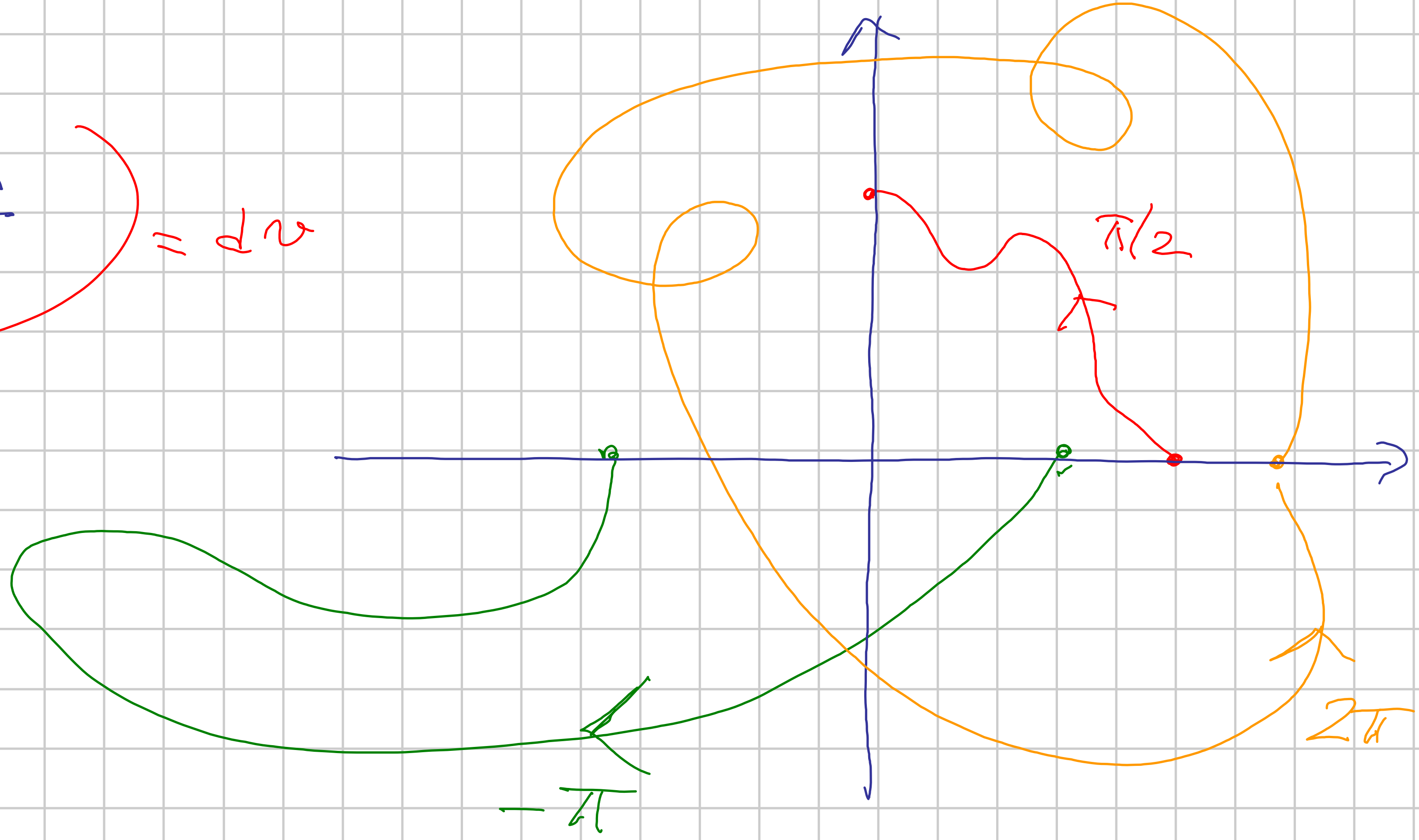
2) $\vartheta = \operatorname{arcctg}\frac{x}{y} + k\pi$

1) $$d\vartheta = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\,dx + \frac{1}{x}\,dy\right) = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

2) $$d\vartheta = -\frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\,dx - \frac{x}{y^2}\,dy\right) = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

Calcolo di

$$\int_{\gamma} \frac{-y\,dx + x\,dy}{x^2+y^2} = d\vartheta$$

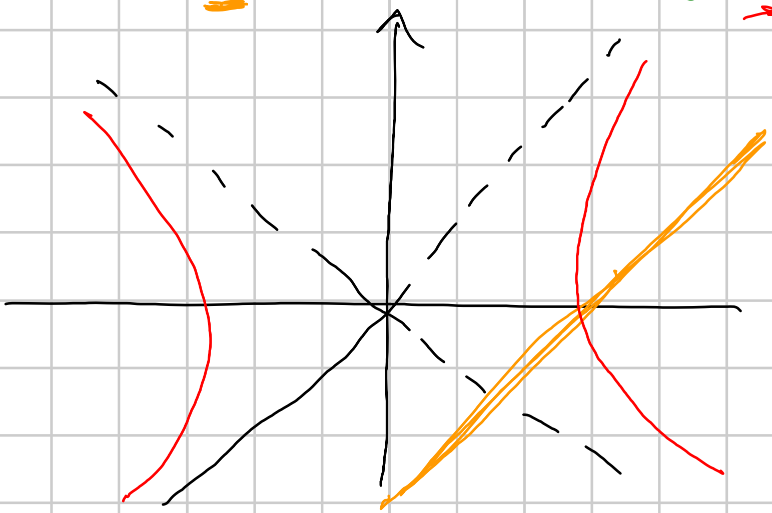


$\boxed{12.2.10}$ Trovare i pti all'∞ di

$$x^3+y^3-x^2y-xy^2+x^2-y^2-x+y+1=0$$

I

$$(x-y-1)(x^2-y^2-1)=0$$



$\Longrightarrow$ $[1:1]$ (doppio)

$[1:-1]$

II $\quad \mathcal{J} = \{ (x,y) \in \mathbb{R}^2 : \ , \ \_\_ \ \_\_ \}$

$$\overline{\mathcal{J}} = \{ [x:y:z] \in \mathbb{P}^2(\mathbb{R}) : x^3 + y^3 - x^2 y - x y^2 + z x^2 - z y^2 - z^2 x + z^2 y + z^3 = 0 \}$$

$$\mathcal{J}_\infty = \overline{\mathcal{J}} \cap \mathbb{P}^1(\mathbb{R})_\infty$$

$$\| $$

$$\{ [x:y:0] \}$$

$$\Longrightarrow \quad \mathcal{J}_\infty : x^3 + y^3 - x^2 y - x y^2 = 0$$

$$x^2(x-y)+y^2(y-x)=0$$

$$(x^2-y^2)(x-y)=0$$

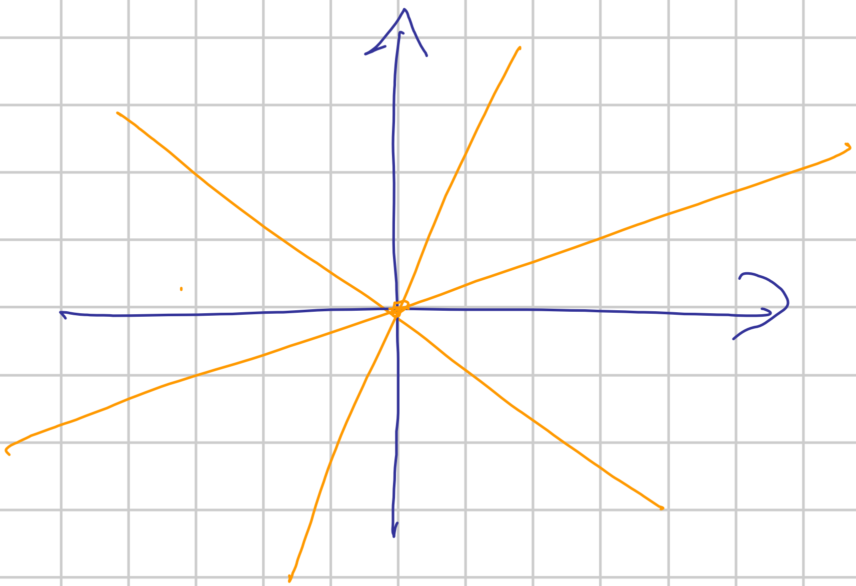
$$(x+y)(x-y)^2=0$$

$\rightarrow [1:1]$ doppio

$\rightarrow [1:-1]$

$\boxed{12.2.15}$

Trovare $X \subset \mathbb{R}^2$ definito da eq. di III grado con 3 pti a ∞ distinti che non contiene rette.



$$xy(x-y) = 1$$

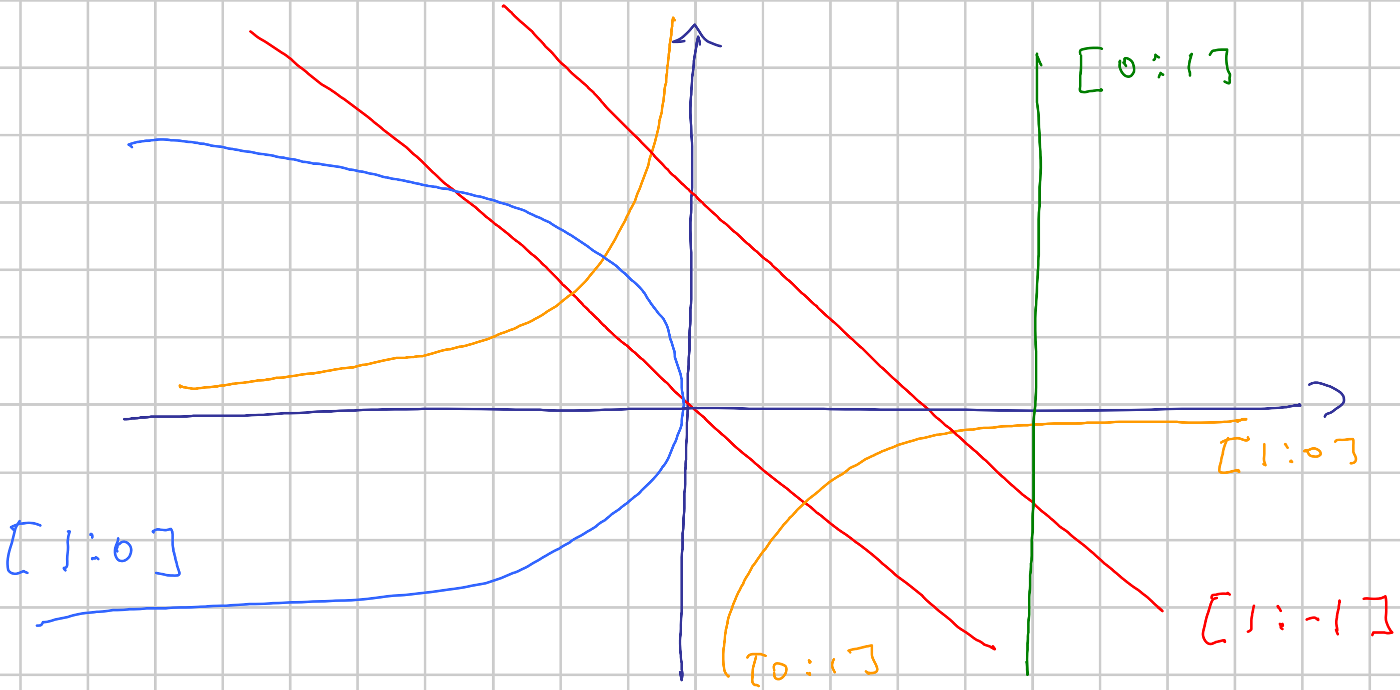
$[1:0]$ $[0:1]$ $[1:1]$

verificare che non contiene rette

$\boxed{12.2.12}$ Trovare i pti all'∞ di:

$$\left\{ x \in \mathbb{R}^2 : \left((x_1+x_2) - (x_1+x_2)^2 \right) \cdot (x_1 x_2 + 1) \cdot (x_1 - \sqrt{\pi}) \cdot (x_1^2 + 17 x_2) = 0 \right\}$$

I



3 pti a ∞

$\mathrm{II}: \quad (x_1+x_2)^2 \cdot x_1 x_2 \cdot x_1 \cdot x_1^2 = 0$

$[1:0] \quad [0:1] \quad [1:1]$

$\boxed{12.3.1\ (b)}$

Trovare le intersezioni tra ℓ e Q_∞:

$\ell : \{ [2-t : t-3 : t-1] \in \mathbb{P}^2(\mathbb{R}) : t \in \mathbb{R} \}$

$Q : 2x^2 - y^2 + 4z^2 - 5xy + 8xz$

$-7x + \sqrt{2}y + z + 3 = 0.$

$Q_\infty : 2x^2 - y^2 + 4z^2 - 5xy + 9xz = 0$

$2t^2 - 8t + 8$

$-t^2 + 6t - 9$

$+4t^2 - 8t + 4$

$+5t^2 - 25t + 30$

$-8t^2 + 27t - 18 = 0$

$t^2 - 8t + 15 = 0$

$(t-5)(t-3) = 0$

$[-3 : 2 : 4]$

$[-1 : 0 : 2]$

$\boxed{12.3.2\ (b)}$ Trovare i punti d'intersezione tra

$$X = \{ [2t : t-5 : t^2+1] : t \in \mathbb{R} \} \subset \mathbb{P}^2(\mathbb{R})$$

$$Y = \{ [2+t^2 : -t : 4+t^2] : t \in \mathbb{R} \} \subset \mathbb{P}^2(\mathbb{R})$$

SBAGLIATO: imporre $[2t : t-5 : t^2+1]$

$$= [2+t^2 : -t : 4+t^2]$$

GIUSTO : imporre $[2t : t-5 : t^2+1]$

$= [2+r^2 : -1 : 4+r^2]$

I : cerco t, r, k t.c.

$$\begin{pmatrix} 2t \\ t-5 \\ t^2+1 \end{pmatrix} = k \cdot \begin{pmatrix} 2+r^2 \\ -1 \\ 4+r^2 \end{pmatrix}$$

non conviene ...

II: cerco t, s t.c.

$$\begin{pmatrix} 2t & 2+s^2 \\ t-5 & -s \\ t^2+1 & 4+s^2 \end{pmatrix}$$

abbia rango 1

ovvero tutti i det delle sottomatrici 2x2 siano 0

$$-2ts-(t-5)(2+s^2)=0 \Rightarrow t=\frac{5(2+s^2)}{s^2+2s+2}$$

Ora

$$2t \cdot (4+\lambda^2) - (t^2+1)(2+\lambda^2) = 0$$

$$t = \frac{5(2+\lambda^2)}{\lambda^2+2\lambda+2}$$

equaz. polinomiale difficile ... trovo $\lambda = \lambda_1, \lambda_2 \dots$

e $t = t_1, t_2 \dots$

li sostituisco nella 3x2

e verifico per quali valori viene rank = 1.

$\boxed{12.3.4\ (b)}$ Per quali $t \in \mathbb{R}$ il punto di $\mathbb{P}^2(\mathbb{R})$

$[t^2-2t : t^2-4 : t-2]$ appartiene alla

retta che passa per $[1 : -1 : 3]$ e $[1 : -3 : 5]$.

Noto che $\begin{pmatrix} t^2-2t \\ t^2-4 \\ t-2 \end{pmatrix} = (t-2)\begin{pmatrix} t \\ t+2 \\ 1 \end{pmatrix}$ $t=2$ no

$\Rightarrow$ basta risolvere per $[t : t+2 : 1]$.

$\cdot\ \ell$ = proiez. in $\mathbb{P}^2(\mathbb{R})$ di Span $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$

Damit $[t : t+2 : 1] \in \ell$

$$\iff \begin{pmatrix} t \\ t+2 \\ 1 \end{pmatrix} \in \operatorname{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} \right)$$

$$\iff \det \begin{pmatrix} t & 1 & 1 \\ t+1 & -1 & -3 \\ 1 & 3 & 5 \end{pmatrix} = 0$$

$$\iff (\ldots) \quad t = 3$$

Damit $\cdot \cap \cdot = [3 : 5 : 1]$.