Geometria 17/5/18

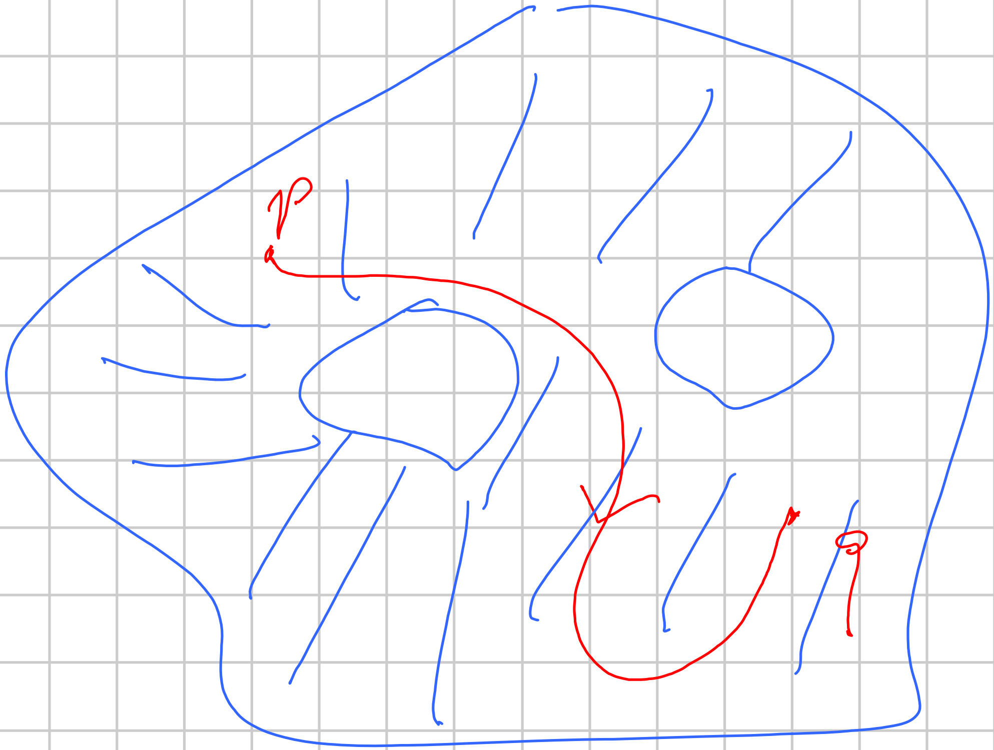
$\omega = g\,dx + h\,dy$ su Ω aperto connesso

un solo pezzo

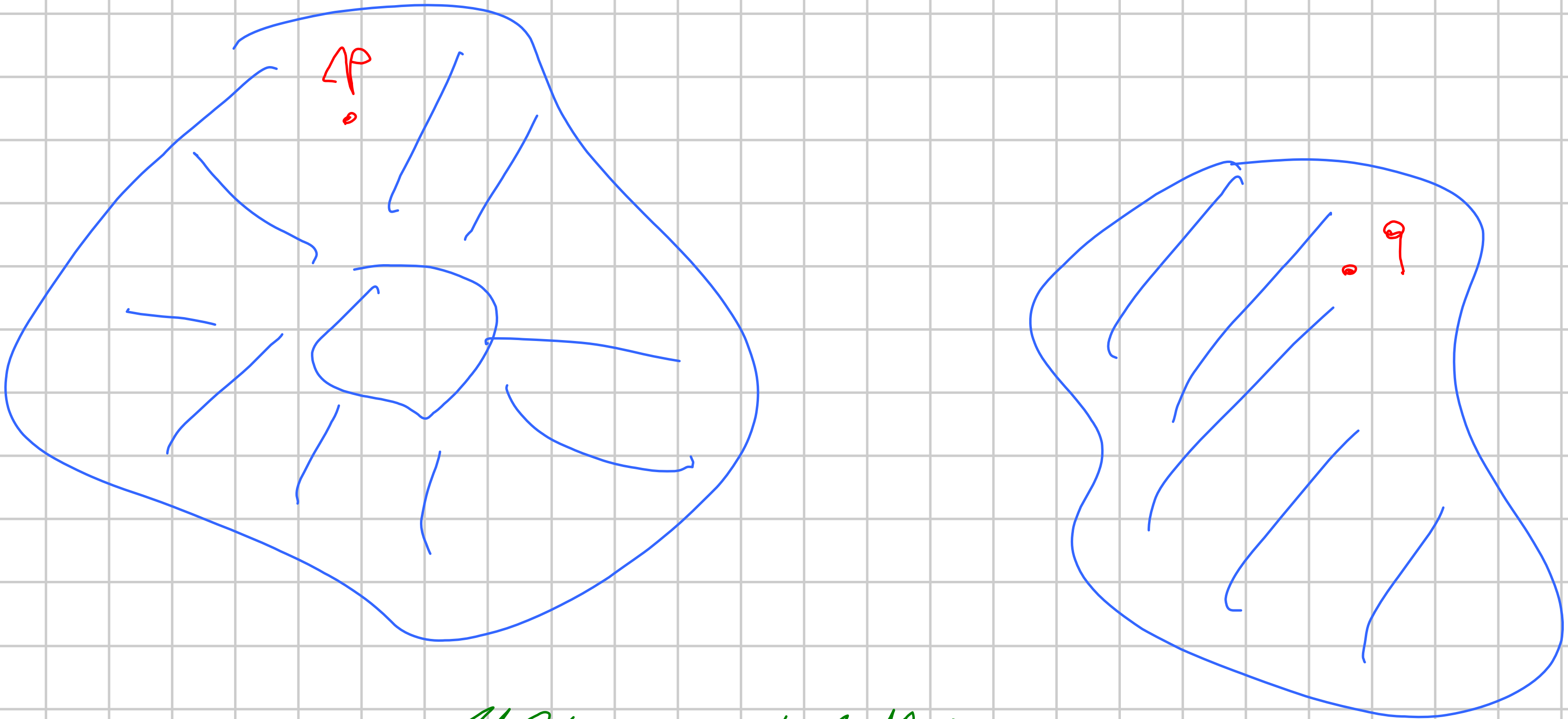
$\forall p, q \in \Omega$

$\exists \alpha : [a,b] \to \Omega$

$\alpha(a) = p \quad \alpha(b) = q$



Connesso



non connesso

$\omega = g\,dx + h\,dy$ è esatta se $\exists\, U : \Omega \to \mathbb{R}$ t.c.

$\omega = dU$ cioè $g = \dfrac{\partial U}{\partial x}$, $h = \dfrac{\partial U}{\partial y}$

ω è chiusa se $d\omega = 0$ cioè $\dfrac{\partial h}{\partial x} - \dfrac{\partial g}{\partial y} = 0$

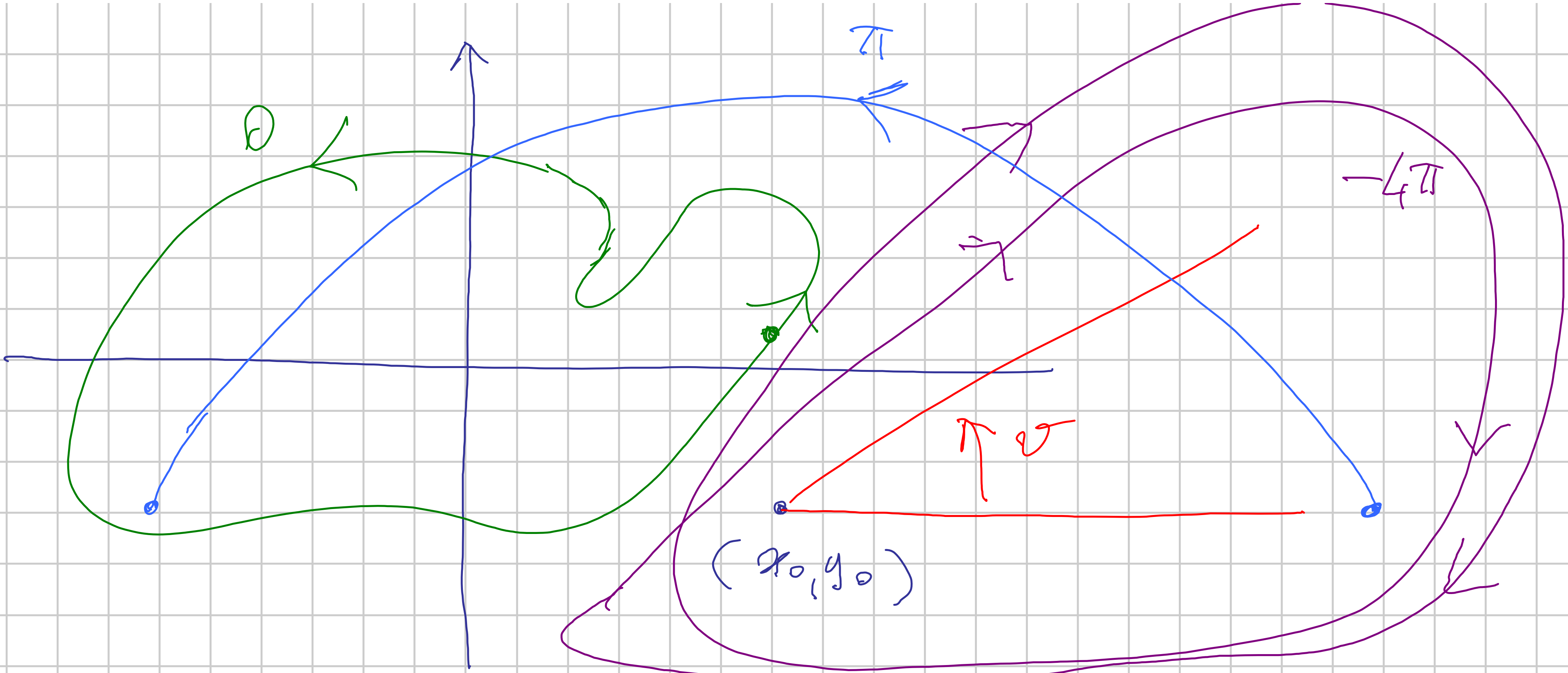
ω esatta $\Longleftrightarrow$ $\int_\alpha \omega$ dipende solo da estremi di α $\forall \alpha$ $\Longleftrightarrow$ $\int_\beta \omega = 0$ $\forall \beta$ chiusa

ω esatta $\Longrightarrow$ ω chiusa

ω chiusa $\not\Rightarrow$ ω esatta

$$\omega = \frac{-(y-y_0)\,dx + (x-x_0)\,dy}{(x-x_0)^2+(y-y_0)^2} \qquad \text{su } \mathbb{R}^2 \setminus \{(x_0,y_0)\}$$

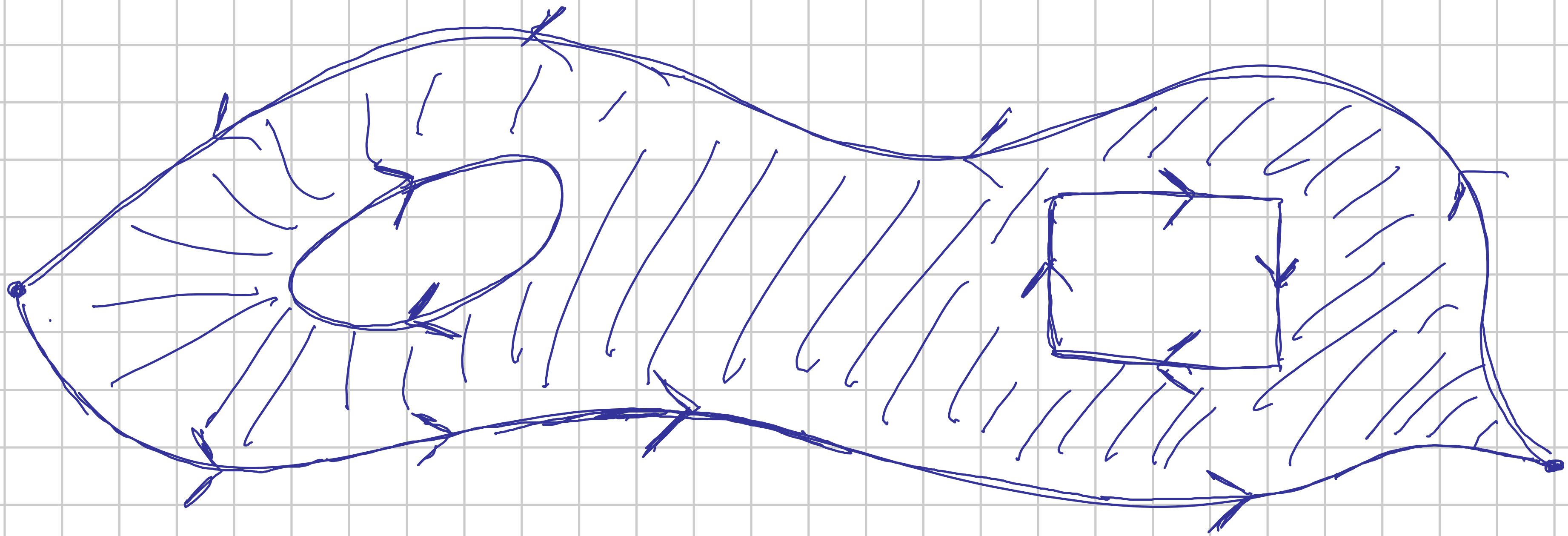
chiusa ma non esatta



θ
π
−4π
(x_0, y_0)
r
θ

Fisso $\Omega \subset \mathbb{R}^2$ aperto, connesso, limitato

con <u>bordo</u> costituito da curve:



Def: indico con $\partial\Omega$ il bordo di Ω orientato in modo da lasciare Ω a sinistra:

Ricordo:

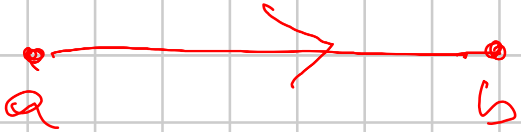
$$\omega = g\,dx + h\,dy$$

$$d\omega = \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right)\underbrace{dx\,dy}_{\text{elemento d'area nel piano}}$$

Teo (Gauss-Green): Ω come sopra; ω definita su Ω fino al bordo e un po' oltre.

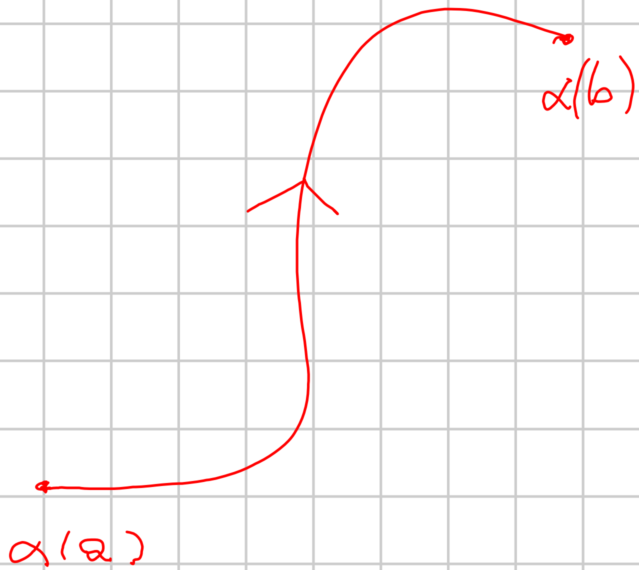
Allora
$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega .$$

Oss: TFCI:
$$\int_a^b f' = f(b) - f(a)$$



$$\int_{[a,b]} f' = f\Big|_{\partial[a,b]}$$

$$\partial[a,b] = \text{“}b\text{”} - \text{“}a\text{”}$$



Oss:

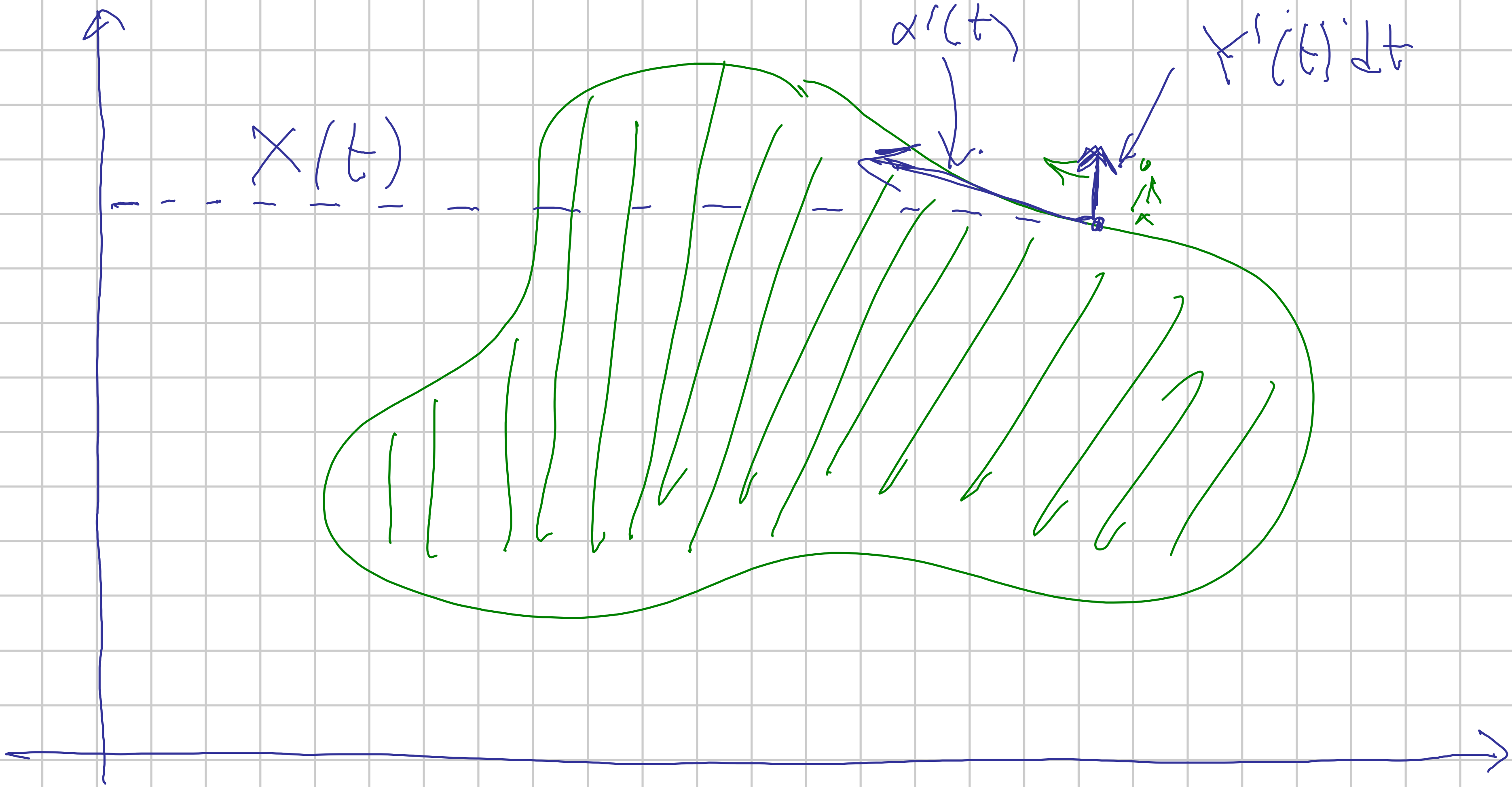
$$\int_\alpha dU = U(\alpha(b)) - U(\alpha(a))$$

$$\| $$

$$\int_{\partial\alpha} U$$

$$\boxed{\int_{\Omega} d\omega = \int_{\partial\Omega} \omega}$$

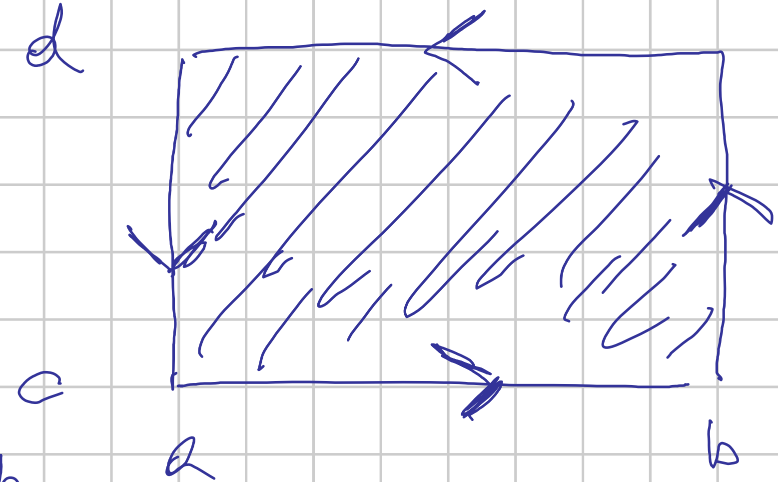
<u>Cor</u>: $\displaystyle\int_{\partial\Omega} x\,dy = \int_{\Omega} dx\,dy = \text{area}(\Omega)$



$X(t)$
$\alpha'(t)$
$Y'(t)\,dt$

$$\boxed{\int_{\Omega} d\omega = \int_{\partial\Omega} \omega}$$

Dimostro per $\Omega = [a,b] \times [c,d]$



$$\omega = g\,dx + h\,dy$$

$$d\omega = \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right) dx\,dy =$$

$$= \int_c^d \left(\int_a^b \frac{\partial h}{\partial x}(x,y)\,dx \right) dy - \int_a^b \left(\int_c^d \frac{\partial g}{\partial y}(x,y)\,dy \right) dx$$

$$= \int_c^d \left(h(b,y) - h(a,y)\right) dy$$

$$- \int_a^b \left(g(x,d) - g(x,c)\right) dx$$

$$= \int_c^d h(b,y)\, dy$$

$$\text{2}$$

$$- \int_c^d h(a,y)\, dy - \int_a^b g(x,d)\, dx$$

$$\text{4} \qquad \text{3}$$

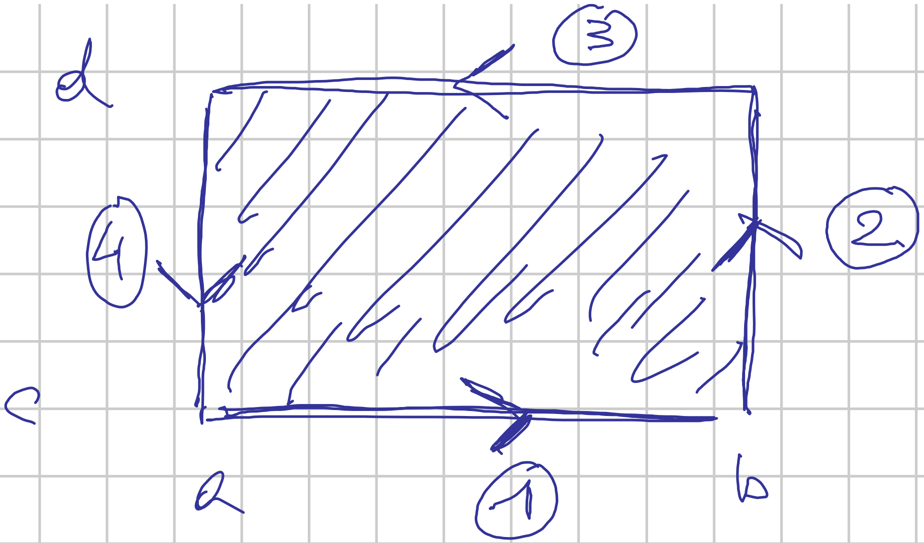
$$+ \int_a^b g(x,c)\, dx$$

$$\text{1}$$

Fatto: $\displaystyle\int_{[a,b]\times[c,d]} F(x,y)\,dx\,dy$

$$= \int_a^b \left(\int_c^d F(x,y)\, dy \right) dx$$

$$= \int_c^d \left(\int_a^b F(x,y)\, dx \right) dy$$



$$\omega = g\,dx + h\,dy$$

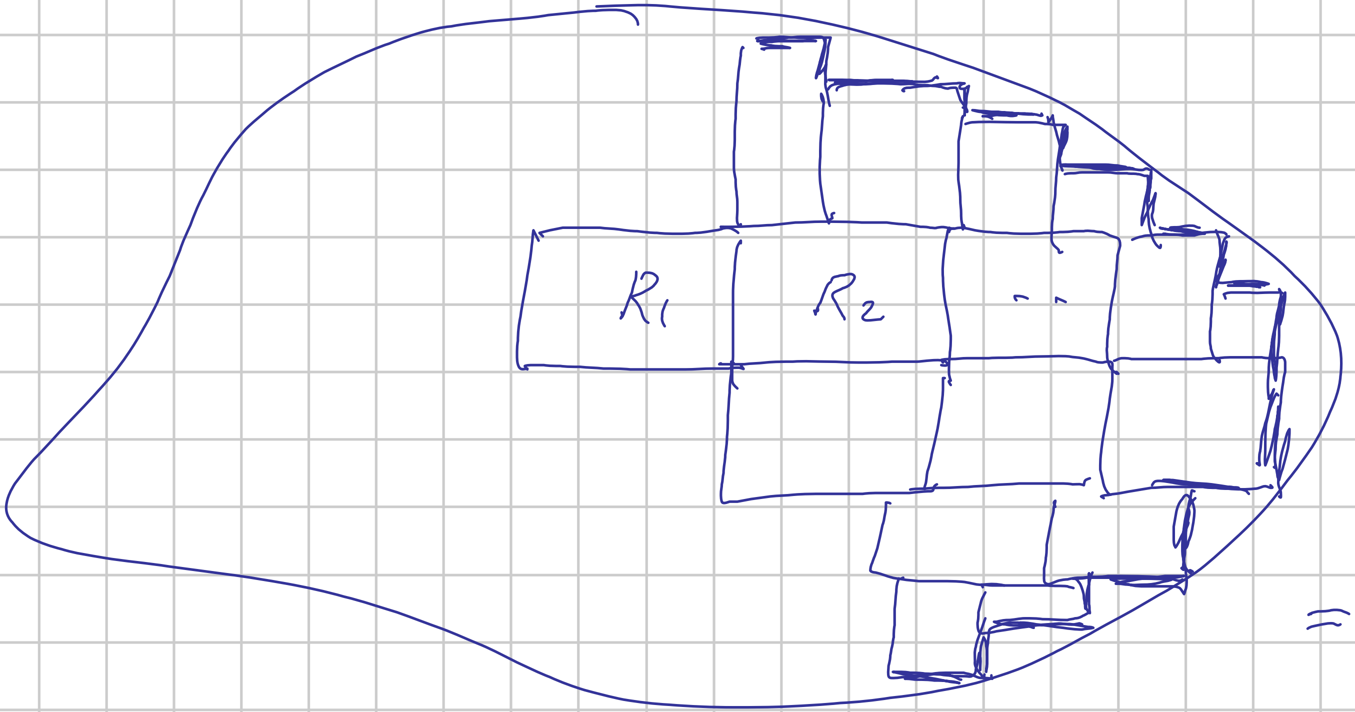
su ① $dx = 1 \quad dy = 0$

su ② $dx = 0 \quad dy = 1$

su ③ $dx = -1 \quad dy = 0$

...

Ω qualsiasi: lo approssimo tramite rettangoli:

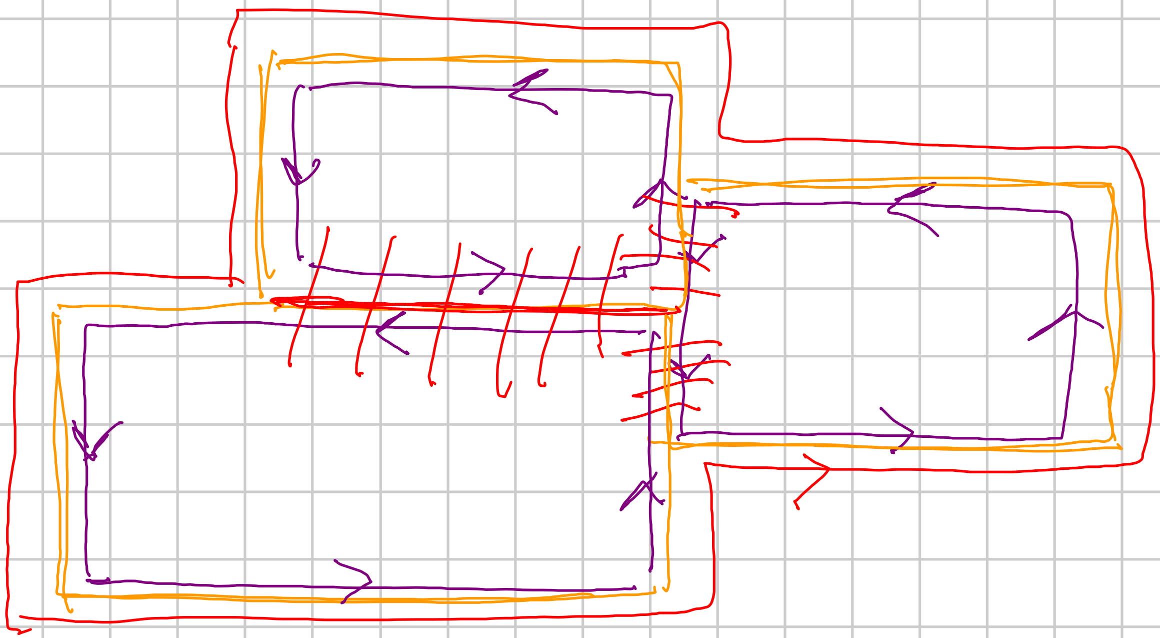


$$\int_{\Omega} d\omega \cong \int_{\cup R_i} d\omega = \sum_{R_i} \int_{R_i} d\omega$$

$$= \sum_{R_i} \int_{\partial R_i} \omega = \int_{\partial(\cup R_i)} \omega$$

spiego

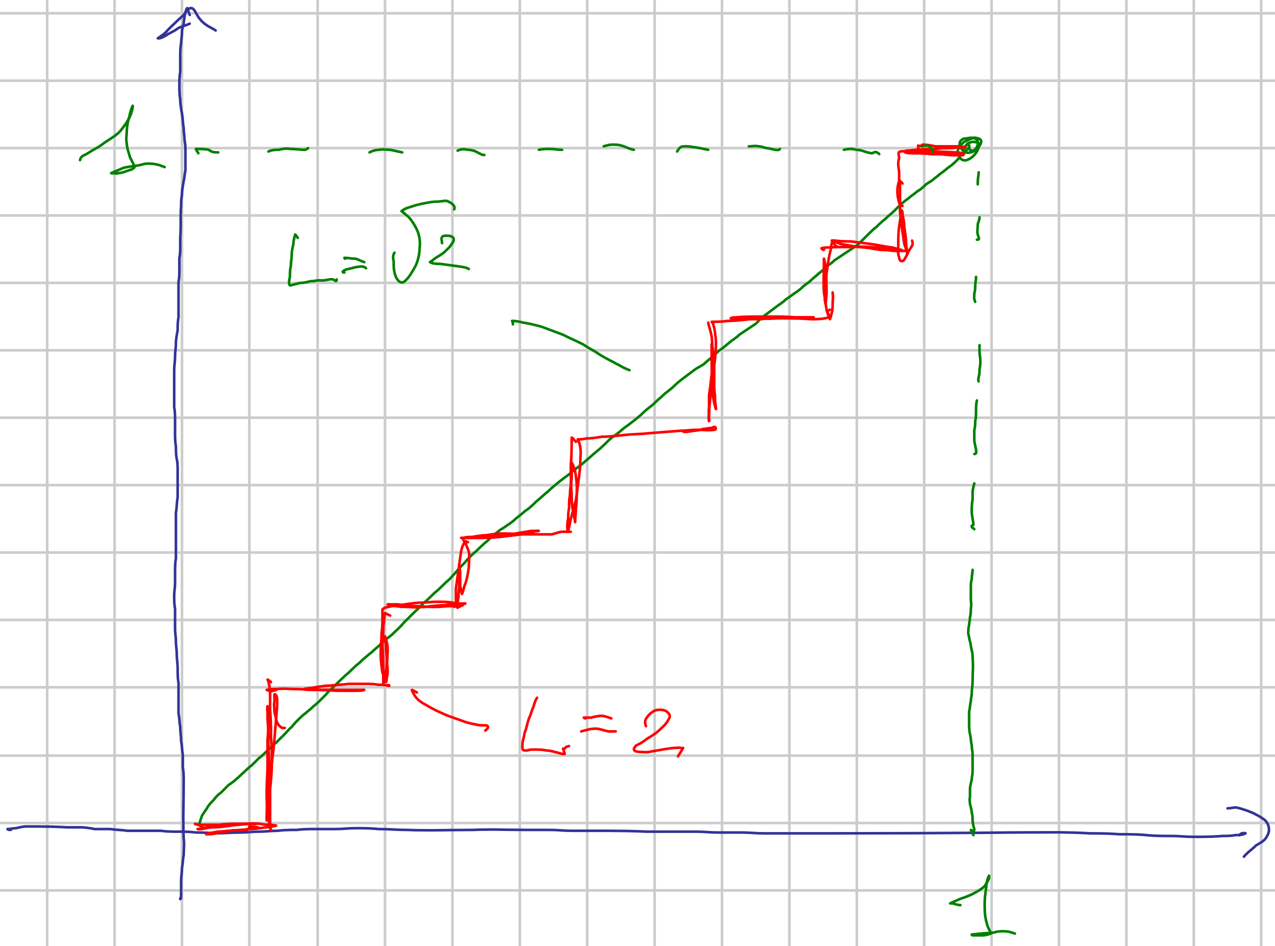
: i contributi interni si elidono



Ora $\displaystyle\int_{\partial(\cup R_i)} \omega \simeq \int_{\partial\Omega} \omega$

non ovvio

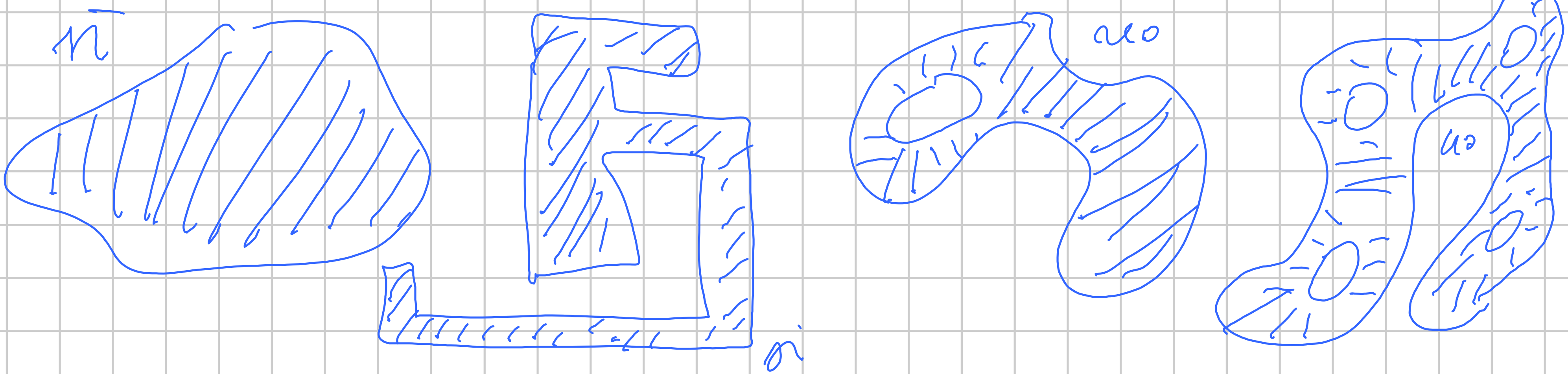
Esempio: se approssimo una curva con spezzate di segmenti orizz/vert. non approssimo bene la lunghezza:



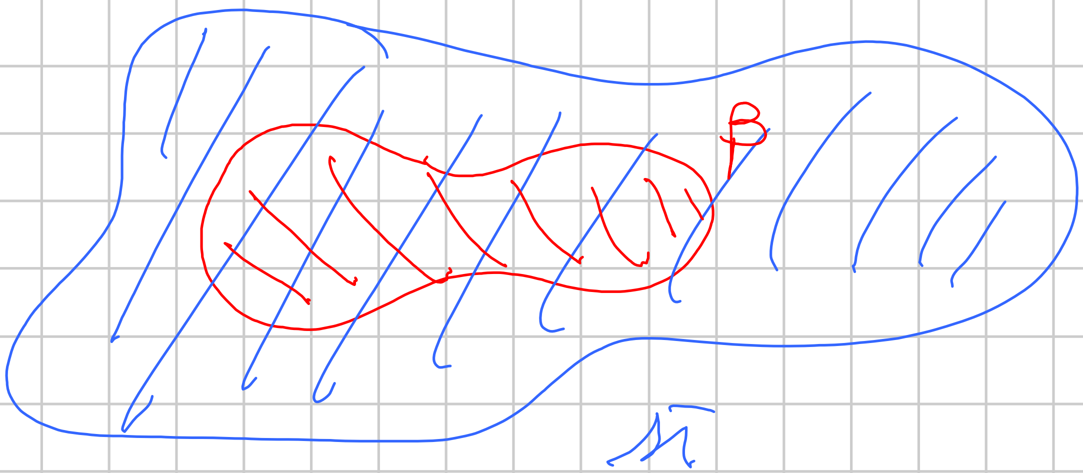
1
L = √2
L = 2
1

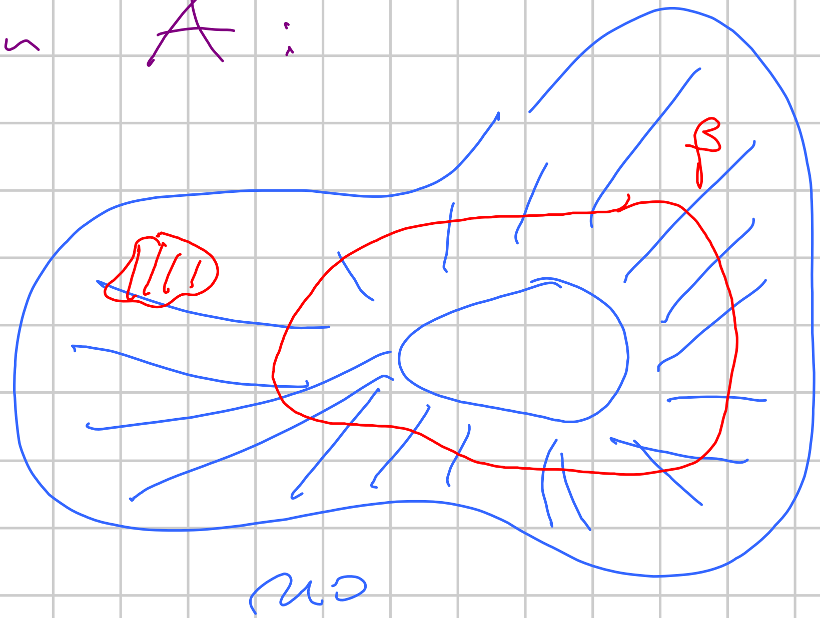
$$\boxed{\int_{\Omega} d\omega = \int_{\partial\Omega} \omega}$$

Def: $A \subset \mathbb{R}^2$ aperto connesso si dice semplicemente connesso se "non ha buchi":



Formalmente: è semplic. connesso se $\forall$ curva β semplice chiusa in A l'insieme B limitato con bordo β è contenuto in A:





Prop: se ω è una forma chiusa definita su A semplicemente connesso allora ω è esatta su A.

Dim: basta vedere che $\int_\beta \omega = 0$ $\forall \beta$ chiusa in A.

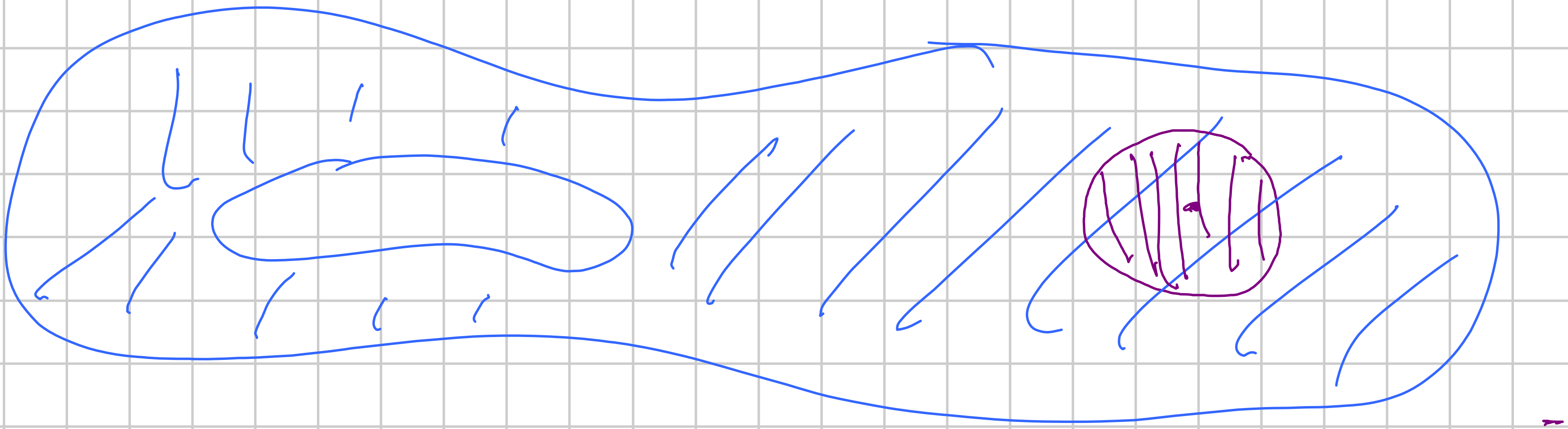
$\beta = \partial B$ con $B \subset A$ dunque:

$$\int_{\beta} \omega = \int_{\partial B} \omega = \int_{B} d\omega = \int_{B} 0 = 0 .$$

(In realtà dovrei considerare anche le β non semplici:

) $\blacksquare$

Con: una forma chiusa su A "con buchi" può non ammettere un potenziale globale ma ammette potenziali locali.



$\boxed{13.1.3}$ Trovare pti a ∞ e tipo affine conica.

(b) $x^2 + xy + 5y^2 - x + 2y + 3 = 0$

a ∞ : $x^2 + xy + 5y^2 = 0$

$x^2 + xy + \frac{y^2}{4} + \frac{19}{4} y^2 > 0$

$\left(x + \frac{y}{2}\right)^2$

$\emptyset$

$$\begin{pmatrix} 1 & 1/2 & -1/2 \\ 1/2 & 5 & 1 \\ -1/2 & 1 & 3 \end{pmatrix}$$

$d_1 > 0 \quad d_2 > 0$

$$d_3 = 15 - \frac{1}{4} - \frac{1}{4} - \frac{5}{4} - \frac{3}{4} - 1 > 0 \quad \cancel{\bigcirc}$$

(d) $25x^2 - 10xy + y^2 - 10x + 2y - 3 = 0$

Q ∞ : $25x^2 - 10xy + y^2 = 0 \qquad (5x - y)^2 = 0$

$\Longrightarrow [1 : 5 : 0]$ un punto

$$\begin{pmatrix} 25 & -5 & -5 \\ -5 & 1 & 1 \\ -5 & 1 & -3 \end{pmatrix} \quad d_2 = 0 \quad d_3 = -75 + 25 + 25 - 25 + 75 - 25 = 0$$

$$(5x-y)^2-2(5x-y)-3=0$$

$$(5x-y-3)(5x-y+1)=0 \quad \text{due rette } //$$

(f) $2x^2+xy-21y^2+4x-y+5=0$

a ∞ : $2x^2+xy-21y^2=0$ $\quad y=0$ NO

$y=1 \quad 2x^2+x-21=0$ $\quad \dfrac{-1\pm\sqrt{1+168}}{4} = \begin{cases} 3 \\ -\frac{7}{2} \end{cases}$

$[3:1]$, $\left[-\frac{7}{2}:1\right] = [-7:2]$

$$\begin{pmatrix} 2 & 1/2 & 2 \\ 1/2 & -21 & -1/2 \\ 2 & -1/2 & 5 \end{pmatrix} \quad d_2<0 \quad d_3 = -210 - \frac{1}{2} - \frac{1}{2} + 84 - \frac{5}{4} - \frac{1}{2} \neq 0 \text{ iperbole}$$

$\boxed{13.4.1}$ Trovare tipo affine quadrica.

(b) $x^2+3y^2+4xy-2xz-2yz-4x+3=0$.

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 2 & 3 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & 0 & 0 & 3 \end{pmatrix}$$

$d_1 = 1 > 0$

$d_2 = 3 - 4 = -1 < 0$

$$d_3 = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 0$$

parab. iperbolico purché $d_4 \neq 0$.

$$d_4 = \begin{vmatrix} 1 & 2 & -1 & -2 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & 0 & 0 & 3 \end{vmatrix} = 0 \Longrightarrow \boxed{\text{degenere}}$$

(e) $2x^2+y^2+z^2+2xy-4yz+6z=0$

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$d_1>0 \qquad d_2>0$

$d_3 = 2-1-8<0$

autoval Q: $+ + -$

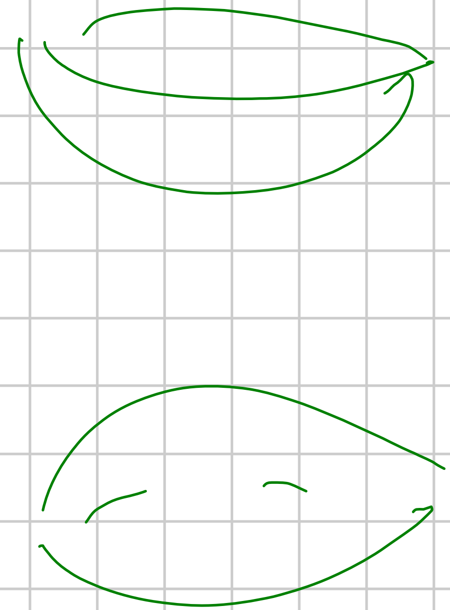
$$d_4 = -3\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & -2 \\ 0 & 0 & 3 \end{vmatrix} = -9 < 0$$

autoval A: $+ + - +$

$$x^2+y^2-z^2+1=0$$

$$z^2 = 1+x^2+y^2$$

iperb. ell. (2 falde)



(g) $x^2 + 2y^2 + 5z^2 + 2xy - 4yz - 4x + 2y = 0$

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 \\ 0 & -2 & 5 & 0 \\ -2 & 1 & 0 & 0 \end{pmatrix}$$

$d_1 > 0 \quad d_2 > 0$

$d_3 = 10 - 5 - 4 > 0$

$Q : + + +$

$$d_4 = \det \begin{vmatrix} 3 & 5 & -4 & 0 \\ 1 & 2 & -2 & 1 \\ 0 & -2 & 5 & 0 \\ -2 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 13 & 5 & -4 \\ -4 & -2 & 5 \\ 0 & 1 & 0 \end{vmatrix} < 0$$

$A : \; + \; + \; + \; -$

$$x^2 + y^2 + z^2 = 1 \qquad \text{ellissoide}$$

(f) $-x^2 + y^2 - 4xy + 6yz - 2x + 4z = 0$

$$\left(\begin{array}{ccc|c} -1 & -2 & 0 & -1 \\ -2 & 1 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ \hline -1 & 0 & 2 & 0 \end{array}\right)$$

$d_2 < 0$

$d_3 > 0$

$Q : \; + \; - \; -$

$$d_4 = \begin{vmatrix} -1 & -2 & 0 & -1 \\ -2 & 1 & 3 & 0 \\ -2 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 \end{vmatrix} = 4 - 3 + 4 > 0$$

$A: + \; - \; - \; +$

$$x^2 - y^2 - z^2 + 1 = 0$$

$$y^2 + z^2 = 1 + x^2$$

iperb. iperb. 1 falda

(i) $x^2+3y^2-3z^2+4xy-2xz-2x-2y+1=0$.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 3 & 0 & -1 \\ -1 & 0 & -3 & 0 \\ \hline -1 & -1 & 0 & 1 \end{array}\right)$$

$d_2<0$

$d_3=-9-3+12=0$

$$d_4=\begin{vmatrix} 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 0 & -3 & 0 \\ -1 & -1 & 0 & 1 \end{vmatrix}=-2+3\neq 0$$

parab. iperb.

(l) $y^2 + 3z^2 + xy + 3xz + 4yz + y = 0$

$$\begin{pmatrix} 0 & 1/2 & 3/2 & 0 \\ 1/2 & 1 & 2 & 1/2 \\ 3/2 & 2 & 3 & 0 \\ 0 & 1/2 & 0 & 0 \end{pmatrix}$$

$d_2 < 0$

$d_3 = \frac{3}{2} + \frac{3}{2} - \frac{9}{4} - \frac{3}{4} = 0$

$d_4 = 0$ degenere

$d_4 \neq 0$ Teneb. ipeb.

(m) $\quad -xy + 2xz + yz + x + 2 = 0$

$$\begin{pmatrix} 0 & -1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 1 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 2 \end{pmatrix}$$

$d_2 < 0$

$d_3 = -\frac{1}{4} - \frac{1}{4} < 0$

$Q: \; + \; - \; +$

$$d_4 = \begin{vmatrix} 0 & -1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 1 & 1/2 & 0 & 0 \\ 1/2 & 2 & -4 & 0 \end{vmatrix} = -\frac{1}{2}\left(1 + 1 - \frac{1}{8}\right) < 0$$

$A: \; + \; - \; + \; +$

$$x^2 - y^2 + z^2 + 1 = 0$$

$$y^2 = x^2 + z^2 + 1$$

iperb. ell. 2 falde.