22/5/2018

14.1.5

$$\alpha : \left[0, \frac{\pi}{2}\right] \longrightarrow \mathbb{R}^2$$

$$\alpha(t) = (\cos t, \sin t)$$

$$\int_\alpha x y^2$$

$$\alpha'(t) = (-\sin t, \cos t)$$

$$\|\alpha'(t)\| = (\sin^2 t + \cos^2 t)^{\frac{1}{2}} = 1$$

$$\int_0^{\frac{\pi}{2}} \cos t \, (\sin t)^2 \, \|\alpha'\| \, dt =$$

$$= \int_0^{\frac{\pi}{2}} \cos t \, (\sin t)^2 \, dt = \left[\frac{1}{3} \sin^3 t \right]_0^{\frac{\pi}{2}} = \frac{1}{3}$$

<u>14.1.61</u>

$$\alpha : [0,1] \to \mathbb{R}^2$$

$$\alpha(t) = (3t^2, \; 1 + t^3)$$

$$\int_\alpha \sqrt{12 + x} \quad ?$$

$$\alpha'(t) = (6t, \; 3t^2)$$

$$\|\alpha'\| = 3t\,(2^2 + t^2)^{\frac{1}{2}} =$$

$$= 3t\sqrt{4 + t^2}$$

$$\hookrightarrow \int_0^1 \sqrt{12 + 3t^2} \cdot 3t \cdot \sqrt{4 + t^2} \, dt =$$

$$= 3\sqrt{3}\int_0^1 \sqrt{4+t^2}\cdot\sqrt{4+t^2}\cdot t\,dt =$$

$$= 3\sqrt{3}\int_0^1 (4t+t^3)\,dt = 3\sqrt{3}\left[2t^2+\frac{1}{4}t^4\right]_0^1 =$$

$$= 3\sqrt{3}\left(2+\frac{1}{4}\right) = \frac{27}{4}\cdot\sqrt{3}$$

$\longmapsto$

<u>14.1.11</u>

$$G = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, e^{3x-y}+\sin(x+2y)=1\right\}$$

F

$(0,0)\in G$: $e^0+\sin 0 = 1$,

$1 + 0 = 1$ ✓

C_1 è una curva vicino a $(0,0)$:

$$\operatorname{grad} F = \left(3e^{3x-y} + \cos(x+2y),\ -e^{3x-y} + 2\cos(2x+y)\right)$$

$$\operatorname{grad} F_{(0,0)} = (4, 1) \neq (0,0)$$

per il teorema del Dini, C_1 è una curva vicino a $(0,0)$.

$\underline{14.1.2}$ $\quad C_1 = \{(x,y) \in \mathbb{R}^2 \mid 2xy^2 + 3x^2y = 1\}$ F

esistono punti di C_1: $(-1, 4)$ è

tangente a C?

Cerchiamo P: - $P \in C$ (1)

- $\text{grad}\, F_P \neq (0,0)$ (2)

- $-1 \cdot \frac{\partial F}{\partial x}(P) + 4 \frac{\partial F}{\partial y}(P) = 0$ (3)

$$\text{grad}\, F = \left(2y^2 + 6xy,\ 4xy + 3x^2\right)$$

(3) $-2y^2 - 6xy + 16xy + 12x^2 = 0$

$$y^2 - 5xy - 6x^2 = 0$$

$$t^2 - 5t - 6 = (t-6)(t+1)$$

Se $t = \frac{y}{x}$, $\quad y^2 - 5xy - 6x^2 =$

$$= (y-6x)(y+x)$$

(3) $\iff$ $y = -x$, oppure $y = 6x$

Se $y = -x$, (1) $2x^3 - 3x^3 = 1$

$$x^3 = -1$$

$$\Rightarrow \quad x = -1, \quad y = 1$$

$$\text{grad } F_{(-1,1)} = (-4, -1) \neq (0,0)$$

$\square$

14.2.41 (b) $\alpha(t) = (4t - t^5, \; 4t^2 + t^4)$, $t \in \mathbb{R}$

$t_0 = 2$. $\quad$ $k(t_0)$?

t vett. tg. a α in t_0.

n vett. normale è tale che: t, n

è una base ⊥ normale di $\mathbb{R}^2$

orientata positivamente. $\quad$ curvatura.

Vale: $\quad t' = k(t)\, n$

Formula : $K(t) = \dfrac{\det(\alpha', \alpha'')}{\|\alpha'\|^3}$

$$\alpha'(t) = (4 - 5t^4, \; 8t + 4t^3)$$

$$\alpha''(t) = (-20t^3, \; 8 + 12t^2)$$

$$\alpha'(2) = (-76, \; 48) \qquad \|\alpha'(2)\| = 4\left((19)^2 + 144\right)^{\frac{1}{2}}$$

$$\alpha''(2) = (-160, \; 56)$$

$$K(2) = \frac{\det\begin{pmatrix} -76 & -160 \\ 48 & 56 \end{pmatrix}}{4(505)^{\frac{3}{2}}}$$

(d) $\alpha(t) = \begin{pmatrix} \sin(3t)+t^2 \\ t - e^{-4t} \end{pmatrix}$ $t \in \mathbb{R}$

$t_0 = 0$

$k(0)$?

$\alpha'(t) = \left(3\cos 3t + 2t, \quad 1 + 4e^{-4t}\right)$

$\alpha''(t) = \left(-9\sin(3t) + 2, \quad -16 e^{-4t}\right)$

$\alpha'(0) = (3, 5)$ $\|\alpha'(0)\|^2 = (9+25)^{\frac{1}{2}} = \sqrt{34}$

$\alpha''(0) = (2, -16)$

$$k(0) = \frac{\det \begin{pmatrix} 3 & 2 \\ 5 & -16 \end{pmatrix}}{34\sqrt{34}} = \frac{-58}{34\sqrt{34}}$$

$\mapsto$ $\underline{14 \cdot 3 \cdot 3|}$ (b)

$$\alpha(s) = (1+2s+s^2-s^3,\ e^{2s},\ \sin s),\quad s \in \mathbb{R}$$

$s_0 = 0$

Trovare riferimento di Frenet e curvatura e torsione in s_0.

Riferimento di Frenet è dato: t, n, b

che sono una base $\perp$ normale, orientata positivamente di $\mathbb{R}^3$. Inoltre:

$$\operatorname{Span}(t) = \operatorname{Span}(\alpha')$$

$$\operatorname{Span}(t, n) = \operatorname{span}(\alpha', \alpha'')$$

Vale $t'(s) = k(s)\, n(s)$. $n(s)$ è scelto in modo che $k(s)$ sia > 0

$$b = t \wedge n$$

$$\alpha'(s) = (2 + 2s - 3s^2,\ 2e^{2s},\ \cos s)$$

$$\alpha''(s) = (2 - 6s,\ 4e^{2s},\ -\sin s)$$

$$\alpha'''(s) = (-6,\ 8e^{2s},\ -\cos s)$$

$$\alpha'(0) = (2, 2, 1), \quad \alpha''(0) = (2, 4, 0),$$

$$\alpha'''(0) = (-6, 8, -1)$$

$$\|\alpha'(0)\| = 3 \qquad t = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$\alpha''(0) - \langle \alpha''(0), t \rangle t = \alpha''(0) - \langle \alpha''(0), \alpha'(0) \rangle \frac{\alpha'(0)}{\|\alpha'(0)\|^2}$$

$$= (2,4,0) - \frac{12}{9}(2,2,1) = (2,4,0) - \left(\frac{8}{3}, \frac{8}{3}, \frac{4}{3}\right)$$

$$= \left(-\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) \leftarrow v$$

$$n = \frac{v}{\|v\|} = \frac{1}{3}(-1, 2, -2) = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

$$b = t \wedge n = \frac{1}{9}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} =$$

$$= \frac{1}{9}\begin{pmatrix} -6 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = b$$

$t' = \kappa(s)\, n \quad , \quad n' = -\kappa(s)\, t + \tau(s)\, b$

<u>Formule</u>:

$$\kappa(s) = \frac{\| \alpha'(s) \wedge \alpha''(s) \|}{\| \alpha'(s) \|^3}$$

$\kappa(0) =$

$$\alpha'(0) \wedge \alpha''(0) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$$

$$\| \alpha'(0) \wedge \alpha''(0) \| = 6$$

$$k(0) = \frac{6}{27} = \frac{2}{9}$$

Torsione:

$$\tau(s) = \frac{\langle \alpha' \wedge \alpha'' \mid \alpha''' \rangle}{\| \alpha' \wedge \alpha'' \|^2}$$

$$\tau(0) = \frac{(-4, 2, 4)\begin{pmatrix} -6 \\ 8 \\ -1 \end{pmatrix}}{36} = \frac{24 + 16 - 4}{36} = 1$$

15-1-1 (b) $\quad \alpha : [0, \theta] \longrightarrow \mathbb{R}^2, \quad \alpha(t) = (\cos t, \sin t)$

$$\omega = x\,dy - y\,dx$$

$$\int_\alpha x\,dy - y\,dx = \int_0^\theta [\cos t \cdot \cos t - \sin t(-\sin t)]\,dt$$

$$= \int_0^\theta 1\,dt = \theta$$

(d) $\alpha : [0,1] \longrightarrow \mathbb{R}^3$, $\alpha(t) = (t - e^t, 1 - e^{2t}, t^2 + 3e^t)$

$$\omega = 2y^2 z\,dx - 3xz^2\,dy$$

$$\int_{\alpha} \omega = \int_0^1 2\left(1-e^{2t}\right)^2\left(t^2+3e^t\right)\left(1-e^t\right)dt \; +$$

$$+ \int_0^1 -3\left(t-e^t\right)\left(t^2+3e^t\right)^2\left(-2e^{2t}\right) dt$$

$$\cdots$$

15.2_1/ (b) calcolare il differenziale.

$$f(x,y) = \tan\left(\frac{1-x^2y^3}{y-2\sin(3x-y)}\right)$$

$$df(x,y) = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy$$

$$\frac{\partial f}{\partial x} = \frac{1}{\cos^2\left(\frac{1-x^2y^3}{y-2\sin(3x-y)}\right)} \cdot \frac{-2xy^3(y-2\sin(3x-y)) - (1-x^2y^3)(-2\cos(3x-y)\cdot 3)}{(y-2\sin(3x-y))^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{\cos^2\left(\frac{1-x^2y^3}{y-2\sin(3x-y)}\right)} \cdot \frac{-3y^2x^2(y-2\sin(3x-y)) - (1-x^2y^3)(1+2\cos(3x-y))}{(y-2\sin(3x-y))^2}$$

<u>15.2.21</u>

$$\omega(x,y) = f(x,y)\,dy$$

per quali f questa forma è chiusa?

Se $\omega = A\,dx + B\,dy$

$$d\omega = \left(-\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x}\right) dx \wedge dy$$

ω si dice <u>chiusa</u> se $d\omega = 0$, cioè

se $$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

Nel nostro caso: ω è chiusa $\Longleftrightarrow$

$$\frac{\partial f}{\partial x} \equiv 0 \iff$$ f non dipende dalla

variabile x.

$\underline{13 - 2 - 3}$ dire per quali $K \in \mathbb{R}$ la 1-forma ω ammette un potenziale su $\mathbb{R}^2$:

ω ammette potenziale $\iff \exists f : \mathbb{R}^2 \to \mathbb{R}$

: $\omega = df$ ("ω è esatta")

In generale: ω esatta $\Rightarrow$ ω chiusa

ma non sempre vale il viceversa

Le 2 condizioni sono equivalenti se il dominio della forma è semplicemente connesso (ad esempio, è $\mathbb{R}^2$)

(a) $\omega(x,y) = (x + ky^2)\,dx + (xy - k^2y^2)\,dy$

ω è chiusa se

$2ky = y \iff (2k-1)y = 0$

$\iff k = \frac{1}{2}$

(f) $$\omega(x,y) = \cos\left(3x^{k(k+1)} - 2y^{k(k+2)}\right)(x\,dx - y^2 dy)$$

$$= A\,dx + B\,dy$$

dobbiamo vedere se $\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}$

$$\frac{\partial A}{\partial y} = \sin(- \quad \ldots \quad)\, x\,(-2)\,y^{k(k+2)-1}\,(k+2)k$$

$$\frac{\partial B}{\partial x} = \sin(- \quad \ldots \quad)\, y^2\left(3k(k+1)x^{k(k+1)-2}\right.$$

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y} \iff -2\,x\,y^{k(k+2)-1}\,\cancel{k}(k+2) = 3\,\cancel{k}(k+1)\,y^{2}\,x^{k(k+1)-1}$$

$$\begin{cases} -2(k+2) = 3(k+1) \\ 1 = k(k+1)-1 \\ k(k+2)-1 = 2 \end{cases}$$