Compact 3-Manifolds of Negative Curvature

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Experimental evidence strongly suggests that $\Omega_{\rm matter} \simeq 0.3$. Until recently, many cosmologists favored a vanishing cosmological constant $\Lambda = 0$, and there was considerable interest in models of a hyperbolic (that is, negatively curved) universe. In particular, cosmologists investigating compact, multiply connected models worked in the realm of compact hyperbolic 3-manifolds. Interest in these models has recently diminished because studies of distant supernovae [1], [2] and the microwave background [3], [4] suggest $\Omega_\Lambda \simeq 0.7$ and $\Omega_{\rm total} \simeq 1.0$, implying a flat universe. On the other hand, particle physicists investigating higher-dimensional models of spacetime now have an interest in compact hyperbolic manifolds, so the basic mathematics is relevant in a new context.

The set of all compact hyperbolic 3-manifolds is far richer than the set of all flat or spherical 3-manifolds, and remains an area of intense research. Besides a specific contribution by Sandro Costa about some examples connecting quantum cosmology to nontrivial topology, the Marcel Grossman IX session on Compact 3-Manifolds of Negative Curvature consisted of a series of talks explaining the structure of, and the relations between, such geometric manifolds. We will not reproduce the details here, but instead will provide references to the literature, and invite those wishing to know more to contact the authors directly.

Readers in this field must be careful, because geometers and cosmologists use different terminology. Geometers call a positively curved space *spherical*, a space of zero curvature *flat*, and a negatively curved space *hyperbolic*, while cosmologists use the terms *closed*, *critical*, and *open*, respectively. To a geometer, *closed* means compact without boundary. Thus the 3-sphere (that is, the hypersurface of a 4-dimensional ball) is closed, as are many multiply connected flat and hyperbolic manifolds. For the remainder of this article we will continue to use the geometer's terminology.

Since the late nineteenth century it has been known that every closed 2-manifold admits a metric of constant curvature. Two of them (the sphere and the real projective plane) admit a spherical metric, two others (the torus and the Klein bottle) admit a flat metric, and all the others (infinitely many) admit a hyperbolic metric. For most of the twentieth century it was thought that the situation in three dimensions was different. Many spherical and flat 3-manifolds were known, but hyperbolic 3-manifolds were rare. In fact, the first example was not discovered until 1931 [5]. (It is a credit to Alexander Friedmann's imagination that his 1924 model of an expanding hyperbolic universe [6] explicitly allowed for the possibility of a closed hyperbolic universe, in spite of the fact that no closed hyperbolic 3-manifolds were known at the time.) It was thought that only very simple or symmetrical 3-manifolds could admit a metric of constant curvature, and that typical 3-manifolds could not. Thus it was a big surprise when Bill Thurston, in the mid 1970's, discovered that typical closed 3-manifolds admit a hyperbolic metric.

Thurston's Geometrization Conjecture provides an all-encompassing view of closed 3-manifolds. Roughly speaking, it says that every closed hyperbolic 3-manifold may be split into pieces in a canonical way, and each piece admits a homogeneous metric of one of eight types. More precisely, the splitting is the connected sum decomposition (splitting along nontrivial 2-spheres and projective planes) followed by a splitting along nontrivial tori and Klein bottles. The eight possible geometries are the three isotropic ones (spherical, flat, and hyperbolic) as well as five more types that are homogeneous but anisotropic. Weeks' *The Shape of Space* [7] includes an intuitive illustrated introduction to the eight homogeneous geometries. In [8] one can find a more technical treatment.

The best introduction to the Geometrization Conjecture is Thurston's survey article *Three-dimensional manifolds, Kleinian groups, and hyperbolic geometry* [9]. For more detail, see Thurston's lecture notes [10]. Thurston is polishing his notes and publishing them as the multivolume *Three-Dimensional Geometry and Topology.* The first volume is now available [11]. Boileau's recent [12] contains an updated report on the state of the conjecture and also an ample bibliography. Benedetti and Petronio's book *Lectures on Hyperbolic Geometry* [13] contains a careful exposition of the fundamental structural properties of hyperbolic manifolds and fills in some of the details missing from Thurston's original notes. The computer program SnapPea [14] provides a practical tool for creating hyperbolic 3-manifolds and studying their properties.

The influence of Thurston's program on research in three-dimensional topology has been and continues to be formidable. The program's main achievements have been the understanding of geometric, non-hyperbolic 3-manifolds with infinite fundamental group [15], [16], and the hyperbolization theorem for compact *hyperbolic-like* 3-manifolds which are *Haken.* Here "hyperbolic-like" means that the universal cover is homeomorphic to ordinary Euclidean 3-space, and the fundamental group is infinite and does not contain any subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. A "Haken" manifold is roughly speaking one which can be decomposed, by a splitting along a nontrivial surface, into pieces of lower "complexity," and the proof of the hyperbolization theorem consists of an (extremely sophisticated) induction argument on this complexity. Completely written proofs of this very remarkable theorem are finally available: see [17] and [18] (which are recommendable also for the very complete lists of references), and [19].

The apparently very difficult open parts of the Geometrization Conjecture concern the case of 3-manifolds with finite fundamental group (a solution of which would imply, in particular, the truth of the celebrated *Poincaré Conjecture*), and the case of non-Haken hyperbolic-like 3-manifolds.

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