Ideal Triangulations of Hyperbolic 3-Manifolds

Carlo Petronio

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Abstract. Geodesic ideal triangulations give a very powerful method for constructing finite-volume complete hyperbolic structures on non-compact 3-manifolds, but it is not known whether the method applies in general. What is known to be always true is that partially flat ideal triangulations exist, but several points become harder to understand when one has to deal with flat ideal tetrahedra together with genuine ones. In particular, the topology of the identification space can be degenerate, and this paper carefully explains how this phenomenon can occur. Moreover, when one tries to deform a complete structure, also inverted tetrahedra arise, and in this case the paper shows that there is not even an obvious definition of the identification space itself. However, it is proved that the developing map and holonomy can be defined anyway, and a hint on how to actually construct the deformed structure is given.

Sunto. Quello delle triangolazioni geodetiche ideali è un metodo molto potente per costruire strutture iperboliche complete di volume finito su 3-varietà non compatte, ma non è noto se il metodo sia applicabile in generale. È tuttavia noto che esistono triangolazioni ideali parzialmente piatte, ma l'analisi della situazione diviene più ardua sotto diversi aspetti, quando si ha a che fare con tetraedri piatti oltre che veri tetraedri. In particolare, la topologia dello spazio di identificazione può degenerare, ed in questo lavoro tale fenomeno di degenerazione viene spiegato in dettaglio. Inoltre, quando si cerca di deformare la struttura completa, emergono anche tetraedri invertiti, ed in tal caso si mostra in questo lavoro che non c'è neppure una ovvia definizione dello stesso spazio di identificazione. Si dimostra tuttavia che la sviluppante e l'olonomia posso comunque essere definite, e si suggerisce un metodo per costruire effettivamente la struttura deformata.

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According to Thurston's geometrization conjecture, proved in the Haken case, hyperbolic geometry plays a central role in 3-dimensional topology. In particular, every closed oriented 3-manifold splits along a family of spheres and tori into canonical pieces which are conjecturally geometric and typically hyperbolic. If one concentrates on the case where the family of tori is non-empty, one is led to consider compact oriented manifolds M^3 such that ∂M is non-empty and consists of tori. In this case Thurston's hyperbolization theorem for Haken manifolds implies that under some natural topological assumptions, $\mathrm{Int}(M)$ admits a complete finite-volume hyperbolic structure. Moreover, Mostow's rigidity states that one such structure is uniquely determined by the topology of M, so every geometric invariant (such as the volume) is actually a topological invariant. Unfortunately, Thurston's proof is purely existential, so the problem naturally arises of effectively constructing hyperbolic structures. The most fruitful method so far, suggested by Thurston himself, is that of ideal triangulations. Actually, it is often conjectured that the method always works, namely that every non-compact, oriented, finite-volume complete hyperbolic M^3 can be decomposed into ideal tetrahedra. The conjecture appears to be still open, but it is known to be true if one accepts, together with genuine tetrahedra, also flat ones.

This paper surveys results obtained by the author with Jeff Weeks [10] and with Joan Porti [9] about the analysis and perturbation of the partially flat triangulations mentioned above. Proofs of theorems already published are omitted, and some original results, remarks and conjectures are provided. This is an expanded version of the talk given in Napoli at the XVI meeting of the Unione Matematica Italiana. The author is pleased to take this opportunity to thank the organizers for their excellent work, and the scientific committee of the meeting for the invitation to give the talk.

1 Hyperbolic geometry and 3-dimensional topology

One of the most natural issues in topology is that of understanding (say, classifying) manifolds (say, closed oriented ones) of a given dimension n. The classification is well-known for $n = 2$ and known to be impossible for $n \geq 4$ (even if spectacular results have been and are being proved, in particular for $n = 4$). In dimension 3 the so-called uniformization (or geometrization) conjecture of Thurston [16] states that every manifold splits along a family of spheres and tori into canonical pieces which admit a geometric structure. Here a geometric structure is a complete Riemannian metric on the interior, locally isometric to a certain homogeneous model. With some natural restrictions, it turns out that only 8 geometries exist [12], and all of them but one are well-understood and "sporadic" among 3-manifolds. The most interesting geometry is the hyperbolic one, *i.e.* the geometry of constant sectional curvature -1, and according to Thurston's conjecture this geometry plays a central role in 3-dimensional topology.

I state now two fundamental results of hyperbolic geometry in dimension 3, needed below. Proofs may be found in [1].

Theorem 1.1. *If a* 3*-manifold admits a complete finite-volume hyperbolic structure then it admits only one such structure up to isometry.*

This result is known as Mostow's rigidity theorem, and it holds in all dimensions $n \geq 3$. For $n = 2$ a hyperbolic metric corresponds to a Riemann surface structure, so there are continuous moduli.

Theorem 1.2. *Let M be an orientable* 3*-manifold which admits a complete finite-volume hyperbolic structure. Then:*

1. *M is the interior of a compact* 3*-manifold $\overline{M}$ with boundary, where $\partial\overline{M}$ consists of a finite family of tori;*

2. *M is irreducible (every embedded S^2 bounds an embedded D^3) and atoroidal (every subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ is conjugate to the fundamental group of a component of $\partial \overline{M}$).*

This result is a consequence of Margulis' lemma, and it has an analogue in all dimensions. Concerning the definition of "atoroidal" manifold, I mention that there are several variations on this definition, and their mutual relationships have been investigated until quite recently [6], [13], [3]. I do not want to get into this matter here.

It is a great achievement of Thurston, and an important step towards the geometrization conjecture, that a 3-manifold M satisfying the necessary conditions of Theorem 1.2, with certain obvious exceptions, actually is hyperbolic, *provided* either $\partial M \neq \emptyset$ or M is Haken (*i.e.* it contains a two-sided π_1-injective surface). In particular all link exteriors in S^3, with some well-understood families of exceptions, are hyperbolic [15]. Adding this fact to Lickorish's theorem (every closed 3-manifold is a Dehn-filled link exterior —see *e.g.* [11]) and to the hyperbolic Dehn filling theorem stated below, one gets a heuristic confirmation that a closed 3-manifold typically is hyperbolic. The same is true also for a manifold bounded by tori, and this paper describes a method for constructing and analyzing its hyperbolic structure.

2 Topological and hyperbolic ideal triangulations

In order to endow a manifold with a hyperbolic structure, one needs to start with some concrete presentation of the manifold itself. Since a hyperbolic structure is a Riemannian metric with $K \equiv -1$, one may expect differential equations to arise. Remarkably enough, there is a combinatorial presentation for 3-manifolds which allows to translate the search for a hyperbolic structure into an algebraic question, thus bypassing analysis altogether. This section describes this combinatorial presentation and its hyperbolic version.

Topological triangulations Let M be a compact, connected and oriented 3-manifold with non-empty boundary. I will call *topological ideal triangulation* of M a finite collection $\Delta_1, \ldots, \Delta_n$ of copies of the standard tetrahedron, together with a set of instructions for glueing together in pairs the triangular faces of the Δ_i's, in such a way that the space obtained by performing the glueings and removing the vertices is homeomorphic to $\mathrm{Int}(M)$. The glueings are of course simplicial and orientation-reversing. A more accurate picture is obtained, rather than by removing the vertices, by *truncating* the tetrahedra (formally, by removing the open stars of vertices in the second barycentric subdivision), because the whole of M is recovered, not only $\mathrm{Int}(M)$. Figure 1 illustrates an example in one dimension less. The set of combinatorial data defining the ideal triangulation will be denoted by $\mathfrak{T}$.

Geodesic ideal tetrahedra Assume now that ∂M consists of tori, so, by Theorem 1.2, it makes sense to try and construct a hyperbolic structure on $\mathrm{Int}(M)$. One

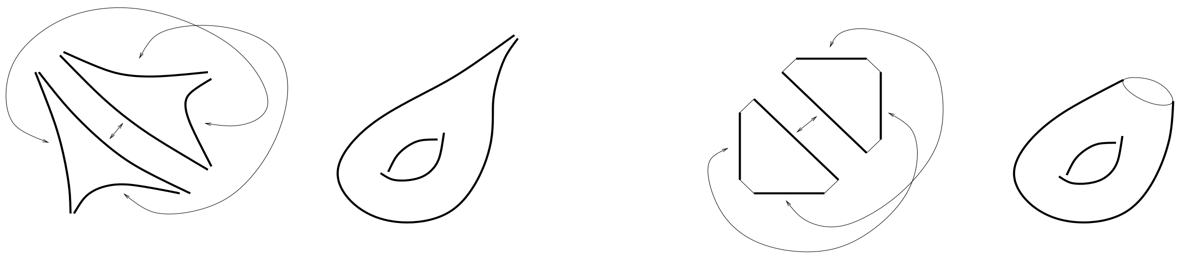

Figure 1: An ideal triangulation in dimension 2: ideal version (left) and truncated version (right).

way to get such a structure using $\mathfrak{T}$ is to define the structure first on the Δ_i's and then try to extend it. The easiest way to put a hyperbolic structure on Δ_i is to realize it as a *geodesic ideal tetrahedron* in $\mathbb{H}^3$, *i.e.* as the convex envelope of 4 points on $\partial\mathbb{H}^3$. Since the isometry group of $\mathbb{H}^3$ acts in a triply transitive way on $\partial\mathbb{H}^3$, three vertices of Δ_i can be chosen to be $0, 1, \infty$ in the half-space model. Taking into account the orientation one sees that the fourth vertex is a complex number z_i in the upper half-plane π_+, and it determines Δ_i up to isometry, so it is called the modulus of Δ_i. More precisely, z_i is attached to the edge having ends 0 and ∞ and to the opposite one. The other pairs of opposite edges have moduli $z_i' = 1/(1-z_i)$ and $z_i'' = 1 - 1/z_i$.

For later purpose I note that by taking horospherical sections at ∞ one gets a correspondence between geodesic ideal tetrahedra up to isometry and Euclidean triangles up to similarity. The upper half-plane π_+ is the moduli space for both objects.

Compatibility equations Given a choice of moduli $z_1, \ldots, z_n \in \pi_+$ for the $\Delta_1, \ldots, \Delta_n$ the question arises whether the structure defined on the Δ_i's induces one on M. First of all, using again the fact that the isometry group of $\mathbb{H}^3$ acts in a triply transitive way on $\partial\mathbb{H}^3$, one sees that all the pairings can be realized by isometries, so the structure extends to the triangular faces. Now, if one focuses on an edge, one easily sees that the structure extends along that edge if and only if the tetrahedra cyclically arranged around the edge close up to give a portion of hyperbolic space. This translates into the condition that the product of the moduli around the edge should be 1 and the sum of the corresponding arguments should be 2π, yielding a system $\mathcal{C}_{\mathfrak{T}}^*$. Actually, using the fact that $\chi(\partial M) = 0$, one sees that there are exactly n edges and that the angle equations are implied by the moduli equations (see [1] for details), so $\mathcal{C}_{\mathfrak{T}}^*$ can be reduced to a smaller system $\mathcal{C}_{\mathfrak{T}}$. Summing up, the following holds:

Proposition 2.1. *Consider a topological ideal triangulation $\mathfrak{T}$ of a manifold M bounded by tori, involving tetrahedra $\Delta_1, \ldots, \Delta_n$. Then the combinatorics of $\mathfrak{T}$ determines a system $\mathcal{C}_{\mathfrak{T}}$ of n rational equations with integer coefficients in n variables, with the property that for $z_1, \ldots, z_n \in \pi_+$ the structure defined by the z_i's on the Δ_i's extends to M if and only if $(z_1, \ldots, z_n)$ is a solution of $\mathcal{C}_{\mathfrak{T}}$.*

Completeness equations In the same setting as above, one has to face the question whether the structure induced on M by $z_1, \ldots, z_n$ is complete. It is a known fact (see *e.g.* [5]) that the boundary tori of M have a similarity structure induced by the hyperbolic structure on $\mathrm{Int}(M)$. In the particular situation under consideration this may be seen directly, because the boundary tori are naturally triangulated by the horospherical sections of the ideal tetrahedra. These sections are Euclidean triangles, and they are glued along similarities. Moreover, M is complete if and only if the tori are actually Euclidean. (This may also be seen directly, but not so easily, see [1] for details.)

From this fact one is lead to the question whether a similarity structure on a triangulated torus is Euclidean. The necessary and sufficient condition for this to happen is that the holonomy of the structure should consist of translations. Taking a pair of simplicial generators of the fundamental group of the torus the condition can be rephrased by requiring that the dilation components of the holonomy of these generators should be 1. Now, the dilation component of the holonomy of a simplicial loop can be computed as the product of all moduli found on one of the sides (left or right) of the loop, so again the result is a rational function of the $z_1, \ldots, z_n$. Summing up one has:

Proposition 2.2. *Under the assumptions of Proposition 2.1, suppose that M has k boundary tori. Then the combinatorics of $\mathfrak{T}$ determines a system $\mathcal{M}_{\mathfrak{T}}$ of $2k$ rational equations with integer coefficients in n variables, with the property that the structure on M defined by a solution $z_1, \ldots, z_n \in \pi_+$ of $\mathcal{C}^*_{\mathfrak{T}}$ is complete if and only if $(z_1, \ldots, z_n)$ is also a solution of $\mathcal{M}_{\mathfrak{T}}$.*

As a consequence of Mostow's rigidity (Theorem 1.1) one can show that a simultaneous solution z of $\mathcal{C}_{\mathfrak{T}}$ and $\mathcal{M}_{\mathfrak{T}}$ with $z_i \in \pi_+$, if any, is unique.

The triangulation conjecture The method described above is a very effective one for constructing hyperbolic structures, and it is wonderfully implemented by the software SnapPea [18]. However it is still an open question whether this method is a general one or not. On one hand, it is very easy to see that, even if M is complete hyperbolic, there are plenty of triangulations $\mathfrak{T}$ for which the system $\{\mathcal{C}_{\mathfrak{T}}, \mathcal{M}_{\mathfrak{T}}\}$ does *not* have a solution in π_+. On the other hand, it is commonly conjectured that at least one $\mathfrak{T}$ exists such that $\{\mathcal{C}_{\mathfrak{T}}, \mathcal{M}_{\mathfrak{T}}\}$ has a solution. As a matter of fact, the way SnapPea works is roughly as follows. It starts with some $\mathfrak{T}$ and it tries to solve the corresponding system. If it finds a solution, M is hyperbolic, and SnapPea can compute the volume and several other invariants. If it does not find a solution, then it checks whether the combinatorics of $\mathfrak{T}$ implies the existence of some topological obstructions to hyperbolicity. If not, then it changes the triangulation and it starts over again. On a counterexample to the conjecture stated above, SnapPea would never halt, which is not known to have happened yet. (SnapPea is actually smarter than I have just described, so it does stop in some cases even if it does not find a genuine solution, but the geometric conjecture has independent interest.)

I conclude this section by rephrasing the triangulation conjecture in a more direct geometric way, which does not require to mention equations:

Conjecture 2.3. *If M is a non-compact, finite-volume and complete hyperbolic 3-manifold then M admits a topological ideal triangulation which can be isotoped to a geodesic one.*

The process of isotoping a triangulation to a geodesic one is best described as 'straightening.' I refer to [8] for some remarks on this process.

3 Partially flat triangulations

The following weaker version of Conjecture 2.3 has been established in [4]:

Theorem 3.1. *If M is a non-compact, finite-volume and complete hyperbolic 3-manifold then M can be obtained by pairing the faces of finitely many geodesic ideal* polyhedra *in $\mathbb{H}^3$.*

Cutting polyhedra into tetrahedra Since an ideal polyhedron can be easily cut into ideal tetrahedra, one may believe (and many of us actually did for quite a while) that Theorem 3.1 implies Conjecture 2.3. This is actually not true, because while cutting the faces of the polyhedra one has to be careful and do it coherently with the face-pairings, which is not at all clear to be possible (see [17] for a sufficient condition recently established). Figure 2 shows how the difficulty can arise. The same figure also suggests how to generalize the notion of geodesic ideal triangulation in order to get a general existence result. I will call *partially flat* a triangulation in which there are some genuine ideal tetrahedra and some tetrahedra which have flattened out into quadrilaterals with distinct vertices. One sees from Fig. 2 that Theorem 3.1 implies that a partially flat triangulation always exists.

Partially flat solutions Having decided to consider also flat tetrahedra one is now forced to enlarge the space of moduli, by considering $\overline{\pi_+} \setminus \{0, 1, \infty\}$ rather than π_+ only (moduli $0, 1, \infty$ are associated to wilder degenerations than just flattening, so I do not consider them). The fact that for every finite-volume, non-compact and complete hyperbolic 3-manifold M a partially flat triangulation $\mathfrak{T}$ exists can now be expressed by the condition that the system $\{\mathcal{C}_{\mathfrak{T}}, \mathcal{M}_{\mathfrak{T}}\}$ has a partially flat solution $z_1, \ldots, z_n$, namely one with the z_i's in $\overline{\pi_+} \setminus \{0, 1, \infty\}$, but not all in $\partial\pi_+$. Actually, if there are flat tetrahedra, the angle equations are no more implied by the edge equations, so I reintroduce them and consider $\mathcal{C}^*_{\mathfrak{T}}$ rather than $\mathcal{C}_{\mathfrak{T}}$ only.

The converse of the existence result just stated was proved in [10]:

Theorem 3.2. *Consider a topological ideal triangulation $\mathfrak{T}$ of a 3-manifold M bounded by tori. Then a partially flat solution of $\{\mathcal{C}^*_{\mathfrak{T}}, \mathcal{M}_{\mathfrak{T}}\}$ defines a finite-volume and complete hyperbolic structure on M.*

The subtle point in the proof of this result is that the scheme valid for genuine solutions cannot be applied, because a partially flat solution of $\mathcal{C}^*_{\mathfrak{T}}$ only does not allow

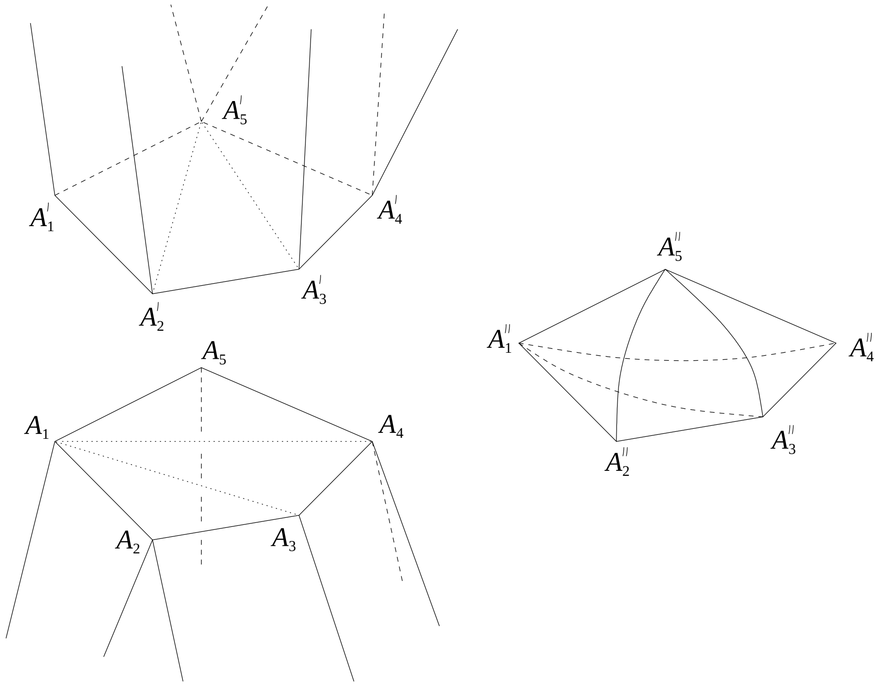

Figure 2: If two paired pentagonal faces $(A_1, \ldots, A_5)$ and $(A_1', \ldots, A_5')$ as in the figure are subdivided by the dotted lines shown, the "flat" tetrahedra $(A_1'', A_2'', A_3'', A_5'')$ and $(A_1'', A_3'', A_4'', A_5'')$ must be inserted

to construct a hyperbolic structure on M. Quite surprisingly, there are cases where z solves $\mathcal{C}^*_{\mathfrak{T}}$ but the space obtained by pairing the genuine and flat tetrahedra corresponding to z is not homeomorphic to M. The argument in [10] shows how to use *both* $\mathcal{C}^*_{\mathfrak{T}}$ and $\mathcal{M}_{\mathfrak{T}}$ to show that a partially flat solution z of *both* yields a hyperbolic structure on M.

The next section is devoted to a careful explanation of the phenomenon of degeneration of topology for partially flat solutions of $\mathcal{C}^*_{\mathfrak{T}}$ only.

4 Degeneration of partially flat solutions

I start by considering the 2-dimensional Euclidean (rather than 3-dimensional hyperbolic) setting, both because it serves as a good model and because it naturally appears on the boundary of the 3-dimensional hyperbolic setting. Note in particular that hyperbolic compatibility corresponds exactly to similarity compatibility on the boundary, and hyperbolic completeness corresponds to flatness of the similarity structure. I will always use the fact that an oriented similarity structure is the same as a complex affine structure, and I will consider the group $\mathrm{Aff}(\mathbb{C})$ of complex-affine automorphisms of the plane.

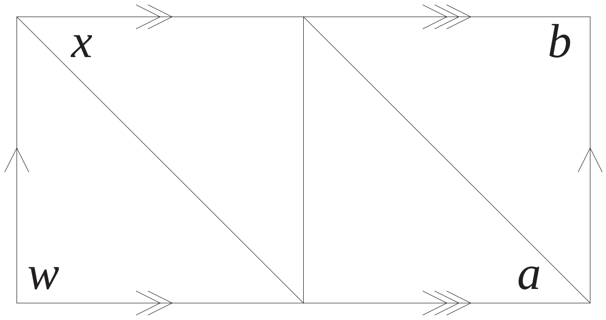

Figure 3: A counterexample to compatibility in 2 dimensions

Example of degeneration The following example is taken from [10]. It must be enclosed as it is referred to below.

Proposition 4.1. *For every $a, b \in (0,1)$ and $w \in \pi_+$, setting $x = (a(1-b)(1-w))^{-1}$, one has that $x \in \pi_+$ and the compatibility equations of the triangulation of the torus described by Fig. 3 are satisfied. However, the corresponding identification space is non-Hausdorff.*

The identification space appearing in this statement is defined as the quotient space of the geometric versions of the triangles (so, two genuine "fat" ones and two which have flattened out to segments), under the only isometric edge-pairings compatible with the combinatorial ones described in Fig. 3. To explain the degeneration better I define this space more carefully.

Identification space, developing map and holonomy Consider the torus T and a topological triangulation $\mathfrak{T}$ of T, possibly with self-adjacencies and multiple adjacencies. I denote by Δ_i the triangles of $\mathfrak{T}$ and consider a partially flat solution $z_1, \ldots, z_n$ of the compatibility equations of $\mathfrak{T}$. By replacing each Δ_i by a (fat or flat) triangle with modulus z_i, and then performing the same simplicial glueings as between the Δ_i's, one gets an identification space which will be denoted by X. I am interested in understanding the cases where X fails to be homeomorphic to T.

I start by noting that z_i determines up to the action of $\mathrm{Aff}(\mathbb{C})$ a map $\psi_i : \Delta_i \to \mathbb{C}$, which maps simplicially Δ_i to a triangle with modulus z_i. Of course ψ_i is injective if and only if z_i is in π_+ rather than $\mathbb{R} \setminus \{0,1\}$. With the aim of defining the developing map and the holonomy I will now describe how to choose a "fundamental domain" Q of T, in a suitable abstract sense. Then I will modify the ψ_i's in order to make them match on Q.

I denote by $G(\mathfrak{T})$ the graph having the Δ_i's as vertices, and edges corresponding to the pairings. I note that $G(\mathfrak{T})$ can be embedded in T as the 1-skeleton of the cellularization dual to $\mathfrak{T}$. I select a maximal tree τ_0 in $G(\mathfrak{T})$ and define:

$$\tau = \tau_0 \cup \left(\bigcup \{ e : \ e \text{ edge of } G(\mathfrak{T}),\ \mathrm{Im}(H_1(\tau_0 \cup e) \to H_1(T)) = 0 \} \right).$$

Note that $\mathrm{Im}(H_1(\tau) \to H_1(T)) = 0$. I define Q_0 as the quotient space of the disjoint union of the Δ_i's modulo the glueings corresponding to edges of $G(\mathfrak{T})$ which lie in τ_0. I define Q in a similar way using τ, and I note that there is a natural projection $q : Q_0 \to Q$.

I fix now an index $\underline{i}$ and a map $\psi_{\underline{i}}$ as described above, *i.e.* I select a geometric model of $\Delta_{\underline{i}}$ in $\mathbb{C}$. Note that if Δ_i is joined to $\Delta_{\underline{i}}$ in τ_0 I can choose the geometric model of Δ_i in such a way that the glueing between $\Delta_{\underline{i}}$ and Δ_i is geometrically given by the identity. Since τ_0 is a tree I can proceed like this and arrange all the maps ψ_i's in such a way that the disjoint union of the ψ_i's induces a well-defined map $D_0 : Q_0 \to \mathbb{C}$.

I can now define a map $\rho_0 : H_1(G(\mathfrak{T})) \to \mathrm{Aff}(\mathbb{C})$ as follows. I note that $H_1(G(\mathfrak{T}))$ is freely generated by the (arbitrarily oriented) edges lying in $G(\mathfrak{T}) \setminus \tau_0$. If e is one such edge which goes from Δ_{i_0} to Δ_{i_1}, I define $\rho(e)$ as the complex-affine automorphism of $\mathbb{C}$ which geometrically realizes the glueing corresponding to e, namely the glueing between $D_0(\Delta_{i_0})$ and $D_0(\Delta_{i_1})$. Before giving the first statement, I note that $H_1(T)$ is a quotient of $H_1(G(\mathfrak{T}))$, and denote by h the projection.

Proposition 4.2. *There exist well-defined maps*

$$\rho : H_1(T) \to \mathrm{Aff}(\mathbb{C}), \qquad D : Q \to \mathbb{C}$$

such that $\rho_0 = \rho \circ h$ and $D_0 = D \circ q$.

Proof of 4.2. The existence of ρ easily implies the existence of D, by definition of q. To check that ρ exists one must show that ρ_0 is the identity on the boundaries of the 2-cells of the cellularization dual to $\mathfrak{T}$. This is readily implied by the fact that $z_1, ..., z_n$ solve the compatibility equations $\mathcal{C}^*_{\mathfrak{T}}$. $\boxed{4.2}$

Geometric fundamental domain and action of the holonomy The underlying idea of the constructions described above is as follows: Q is an abstract fundamental domain for T, and $D(Q)$ is its geometric realization corresponding to $z_1, \ldots, z_n$. Moreover the $\rho(e)$'s for e edge of $G(\mathfrak{T}) \setminus \tau$ represent the geometric versions of the glueings which allow to get T from Q. This seems to suggest that the identification space X to be understood is simply the quotient of $D(Q)$ under the action of these $\rho(e)$'s, and indeed this is the case when $z_1, \ldots, z_n$ is a fat solution also of the completeness equations $\mathcal{M}_{\mathfrak{T}}$. By [10] the same is true for partially flat solutions of $\mathcal{C}^*_{\mathfrak{T}}$ and $\mathcal{M}_{\mathfrak{T}}$, but the situation changes dramatically when $\mathcal{M}_{\mathfrak{T}}$ is dropped, even if one restricts to fat solutions. The point is that on one hand $D : Q \to \mathbb{C}$ can be far from injective in this case, and on the other hand $\mathrm{Im}(\rho)$ may be non-discrete.

Example 4.3. *Consider the similarity structure $\mathfrak{s}$ on T obtained by identifying T to $\mathbb{C}_*/\mathbb{Z}$, where $\mathbb{Z}$ acts as $n(w) = 2^n \cdot w$. If one considers a triangulation of T made of triangles which are straight in $\mathfrak{s}$, then $D(Q)$ will typically be an annulus (whereas Q is a disc), so D cannot be injective. However, it is true in this case that $T = D(Q)/\mathrm{Im}(\rho)$. But, by considering a (suitable) double cover $T \to T$ and the pull-back $\tilde{\mathfrak{s}}$ of $\mathfrak{s}$ under this cover, one gets a case where the typical map $D : Q \to \mathbb{C}$ will be 2-to-1 almost everywhere, and $D(Q)/\mathrm{Im}(\rho)$ does not define on T the right similarity structure*

Example 4.4. *Let $t \in (0,1)$ be an irrational real number. By glueing together the opposite edges of the quadrilateral in $\mathbb{C}$ with vertices 1, 2, $2\mathrm{e}^{it\pi}$ and $\mathrm{e}^{it\pi}$ one gets the torus T with a similarity structure $\mathfrak{s}$, and it is clear that for $\mathfrak{s}$ the $\mathrm{Im}(\rho)$-orbit of every point in $D(Q)$ is non-discrete in $D(Q)$.*

These examples show that it may not be possible to recover X directly from $D(Q)$ and $\mathrm{Im}(\rho)$. However my conjecture is that $D(Q)$ and $\mathrm{Im}(\rho)$ determine whether $X \cong T$ or not. Before stating the conjecture precisely, I need a result:

Proposition 4.5. *The group $\mathrm{Im}(\rho)$ consists either of translations or of transformations all having a common fixed point w_0. In the former case X is homeomorphic to T and $z_1, \ldots, z_n$ define a Euclidean structure on T up to scaling.*

Proof of 4.5. Since $H_1(T) \cong \mathbb{Z}^2$, the first assertion follows from the analysis of pairs of commuting elements in $\mathrm{Aff}(\mathbb{C})$. For the second assertion one notes that $\mathrm{Im}(\rho)$ consists of translations if and only if $z_1, \ldots, z_n$ is a solution also of $\mathcal{M}_{\mathfrak{T}}$, and the results of [10] apply. $\boxed{4.5}$

Conjecture 4.6. *The space X is homeomorphic to T if and only if the common fixed point of $\mathrm{Im}(\rho)$ lies outside $D(Q)$, and in this case a similarity structure is naturally defined on T.*

As a supporting evidence for this conjecture, I note that in the example of Proposition 4.1 indeed w_0 belongs to $D(Q)$. In general, it should not be hard to see that if $w_0 \in D(Q)$ then X is non-Hausdorff. For the opposite implication, I believe that if $w_0 \notin D(Q)$ then X is obtained, if not exactly from $D(Q)$, from the preimage of $D(Q)$ under a finite covering of $\mathbb{C} \setminus \{w_0\}$ over itself. Of course, by Example 4.4, to get T one must not let the whole of $\mathrm{Im}(\rho)$ act on $D(Q)$. The right set of glueing maps is probably obtained by taking the holonomy of loops in T which lift to simple paths in ∂Q. I denote this (finite) subset of $\mathrm{Aff}(\mathbb{C})$ by $\rho(\partial Q)$.

To conclude this paragraph I note *en passant* that the construction of Q, D and ρ described above actually depends on the maximal tree in $G(\mathfrak{T})$ chosen at the beginning, but the relevant geometric properties do not depend on the tree.

3-dimensional case I have already noted that the compatibility and completeness equations of a triangulation $\mathfrak{T}$ of a 3-manifold M actually live on the boundary tori of M. Moreover the argument in [10] suggests that if there is no degeneration of topology on ∂M then everything goes well also in M. So it is natural to extend Conjecture 4.6 to the following one:

Conjecture 4.7. *The identification space corresponding to a partially flat solution $z_1, \ldots, z_n$ of $\mathcal{C}^*_{\mathfrak{T}}$ is homeomorphic to M if and only if:*

1. *On each boundary torus there is a fat triangle, and*

2. *On each boundary torus the holonomy either consists of translations or it has common fixed point outside the developed image of the fundamental domain.*

If this is the case then $z_1, \ldots, z_n$ define a hyperbolic structure on M.

It is also possible that condition (1) is automatic, or at least implied by condition (2). Several examples I have worked out seem to suggest that a totally flat torus forces everything to be flat.

I note that Conjecture 4.7 would have as a direct consequence the following fact stated in [10]:

Proposition 4.8. *If $z^{(0)}$ is a partially flat solution of $\{\mathcal{C}^*_{\mathfrak{T}}, \mathcal{M}_{\mathfrak{T}}\}$ then there exist a neighbourhood U of $z^{(0)}$ in $\mathbb{C}^n$ such that all partially flat solutions z of $\mathcal{C}_{\mathfrak{T}}$ lying in U define a hyperbolic structure on M.*

The proof of this result was omitted in [10] because the proof we had at that time was very complicated. I believe the right proof goes through Conjecture 4.7. The next sections explain the motivations for being interested in non-complete solutions near the complete one. Concerning the previous statement, note that any solution of $\mathcal{C}_{\mathfrak{T}}$ (even a non-partially flat one), close enough to one of $\mathcal{C}^*_{\mathfrak{T}}$, is automatically a solution also of $\mathcal{C}^*_{\mathfrak{T}}$. This fact will be tacitly used below.

5 Hyperbolic Dehn filling

In this section I will state Thurston's hyperbolic Dehn filling theorem and give an outline of the general strategy to prove it. Then I will quickly mention the contribution to the proof given in [9]. I will not attempt to explain here the many reasons why this theorem is considered to be of paramount importance in 3-dimensional topology. The reader is addressed to [9].

Let M be an orientable manifold bounded by tori $T_1, \ldots, T_k$, such that $\mathrm{Int}(M)$ is (complete) hyperbolic (by Theorem 1.2 one knows that all non-compact, finite-volume hyperbolic manifolds look like this). For all i, choose a basis λ_i, μ_i of $H_1(T_i)$, and denote by C the set of coprime pairs of integers, together with a symbol ∞. For $c_1, \ldots, c_k \in C$ denote by $M_{c_1 \cdots c_k}$ the manifold obtained from M as follows: if $c_i = \infty$, remove T_i; if $c_i = (p_i, q_i)$, then glue to M along T_i the solid torus $D^2 \times S^1$, with the meridian $S^1 \times \{*\}$ being glued to a curve homologous to $p_i\lambda_i + q_i\mu_i$. Such a manifold is said to be obtained from M by *Dehn filling*, and the filling coefficients $c_1, \ldots, c_k$ are known to determine it up to homeomorphism. Note that the set C of filling coefficients can be topologized as a subset of $S^2 = \mathbb{R}^2 \sqcup \{\infty\}$, and that $M = M_{\infty,\ldots,\infty}$.

Theorem 5.1. *Under the assumptions just stated there exists a neighbourhood U of $(\infty, \ldots, \infty)$ in C^k, such that for $(c_1, \ldots, c_k) \in U$ the manifold $M_{c_1 \ldots c_k}$ admits a complete finite-volume hyperbolic structure.*

Two proofs of this result are known, and both are based on the following general scheme:

1. A space Def(M) of "deformations" of the complete structure on M is considered. The elements of Def(M) are non-complete hyperbolic structures on M which, in a suitable case, are close to the complete structure. In both proofs Def(M) is actually a germ of algebraic variety.

2. The structure of Def(M) near the complete point is studied, and in particular it is shown that Def(M) has complex dimension exactly k there.

3. The completions of the elements of Def(M) are analyzed, and the informations on Def(M) itself are used to show that all the Dehn-filled manifolds of the statement arise as hyperbolic completions of deformed structures.

The proof originally due to Thurston [14] uses the fact that the hyperbolic structure on M determines a holonomy representation $\rho_0 : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$. The set Def$(M)$ is then defined as the space of all representations $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$. If $\pi_1(M)$ has a presentation with g generators and r relations then this space is a subset of $\mathrm{PSL}_2(\mathbb{C})^g$ defined by r algebraic equations.

Another proof sketched by Thurston and later developed by Neumann and Zagier [7] starts from a geodesic ideal triangulation of M and defines Def(M) as the set of solutions of the compatibility equations $\mathcal{C}_{\mathfrak{T}}$ only. A difficulty with this approach, however, is that it is not known whether the initial triangulation always exists or not (Conjecture 2.3). Using a triangulation which is only partially flat makes the situation harder to understand, because near a partially flat solution of $\mathcal{C}^*_{\mathfrak{T}}$ and $\mathcal{M}_{\mathfrak{T}}$ there will exist solutions of $\mathcal{C}_{\mathfrak{T}}$ involving moduli z_i with negative imaginary part. This corresponds geometrically to having "inverted" tetrahedra, *i.e.* tetrahedra which overlap with other tetrahedra, thus contributing in a negative fashion to the volume. It is not quite clear in this case how to formally define the identification space associated to such a solution, and [9] was precisely devoted to this matter. The next section explains in detail where the subtle points arise and which questions remain open.

6 Triangulations with inverted tetrahedra

The idea of deforming a partially flat triangulation, described and motivated in the previous section, raises the following general question.

Question 6.1. *Let $\mathfrak{T}$ be topological ideal triangulation of a compact oriented 3-manifold M bounded by tori, and let $z_1, \ldots, z_n$ be a solution of $\mathcal{C}^*_{\mathfrak{T}}$ with the z_i's lying in $\mathbb{C} \setminus \{0, 1\}$. Under what assumptions do $z_1, \ldots, z_n$ define a hyperbolic structure on M?*

I can describe the situation in a way which makes the difficulty more transparent. The moduli $z_1, \ldots, z_n$ define geometric tetrahedra in $\mathbb{H}^3$. Some of these tetrahedra are flat, and others are marked as being inverted. Then there are glueing instructions between the faces, and one wants to understand the space resulting from the glueing. An example in one dimension less is described in Fig. 4. This figure shows on the left a portion of triangulation with an almost flat triangle (the triangles have been pulled

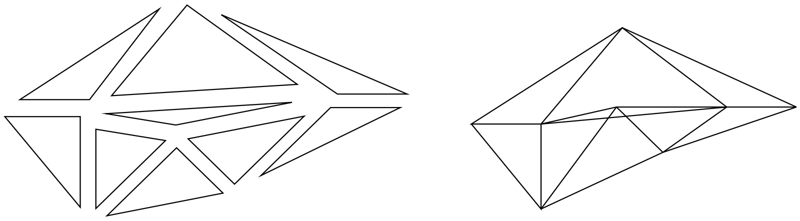

Figure 4: A triangulation with an inverted triangle in dimension 2

apart to be seen more clearly), and on the right a deformation of the same triangulation in which the previously almost flat triangle has become an inverted one. One sees quite clearly in the figure that the triangles overlap, and in particular the inverted triangle contributes negatively to the area, so it must be in some sense removed from the rest.

The subtle point which arises while abstractly glueing genuine tetrahedra with inverted ones is that it is not clear at all how to formally define the resulting space. If one thinks of the construction dynamically, one finds himself in the situation of having to remove the inverted tetrahedra from regions which do not quite exist yet. This difficulty was by-passed in [9] by considering only the solutions of $\mathcal{C}^*_{\mathfrak{T}}$ which lie near a special solution of $\mathcal{C}^*_{\mathfrak{T}}$ and $\mathcal{M}_{\mathfrak{T}}$, namely one which arises by subdividing an Epstein-Penner decomposition and inserting flat tetrahedra as described in Fig. 2. The following seems however very reasonable to me:

Conjecture 6.2. *The set of moduli $z_1, \ldots, z_n$ which define a hyperbolic structure on M is open in the space of solutions of $\mathcal{C}^*_{\mathfrak{T}}$. In particular, all solutions near a complete partially flat solution are hyperbolic.*

In the rest of this section I will sketch a method for proving the 2-dimensional analogue of the second part of this conjecture. Extending the method to the 3-dimensional case might require a considerable technical effort, I do not attempt to do this here. The basic idea is as follows. I construct a fundamental domain for the space to be constructed by considering the image of the developing map defined in Section 3 and discarding the points which contribute negatively to the area. Then I glue the edges of this domain using the holonomy also defined in Section 3.

Let me first remark that the construction of Q, D and ρ, which I have described for a partially flat solution $z_1, \ldots, z_n$ of $\mathcal{C}^*_{\mathfrak{T}}$, makes sense also if the z_i's belong to $\mathbb{C} \setminus \{0,1\}$. Here $\mathfrak{T}$ is a triangulation of the torus with n triangles. Now I assume that the solution z lies arbitrarily close to a partially flat one $z^{(0)}$ of $\{\mathcal{C}^*_{\mathfrak{T}}, \mathcal{M}_{\mathfrak{T}}\}$, and I add subscripts z and $z^{(0)}$ to D and ρ to emphasize the solution which I am considering (of course Q depends only on $\mathfrak{T}$ and the tree chosen in $G(\mathfrak{T})$, not on the solution).

Theorem 3.2 implies that T is obtained from $D_{z^{(0)}}(Q)$ under the action of $\rho_{z^{(0)}}(\partial Q)$, which consists of translations. This is no longer true at z, as one sees for instance in Fig. 5: here the shadowed triangle is contained in $D_z(Q)$ but it must be removed to get the right fundamental domain for $\rho_z(\partial Q)$.

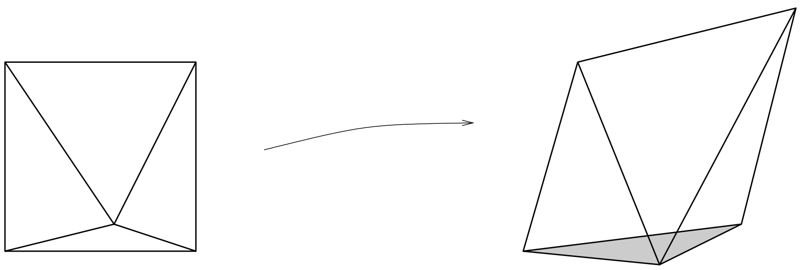

Figure 5: The right geometric fundamental domain may be smaller than the image of the developing map

I conclude by formalizing the construction of the "right" fundamental domain, and stating what I think should be true. For $w \in D_z(Q)$ I define

$$\begin{aligned} \varepsilon_z(w) &= \sum_{i=1}^{n} \sum_{x\in\mathrm{Int}(\Delta_i),\ D(x)=w} \mathrm{sgn}(\Im(z_i)) \\ F_z &= \overline{\{w \in D_z(Q):\ \varepsilon_z(w) > 0\}}. \end{aligned}$$

Conjecture 6.3. *Let z be a solution of $\mathcal{C}_{\mathfrak{T}}$ sufficiently close to a partially flat solution of $\{\mathcal{C}^*_{\mathfrak{T}}, \mathcal{M}_{\mathfrak{T}}\}$. Then F_z is a polygon, the elements of $\rho_z(\partial Q)$ define a set of edge-pairings on F_z, and the space resulting from these pairings is homeomorphic to T. In particular, z defines a similarity structure on T.*

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Dipartimento di Matematica
Università di Pisa
Via F. Buonarroti, 2
I-56127 Pisa, Italy
petronio@dm.unipi.it