A Theorem of Eliashberg and Thurston on Foliations and Contact Structures

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Introduction

The aim of these notes is to give a detailed proof of the following result due to Eliashberg and Thurston [ET]:

Theorem 0.0.1. *Let M be a closed oriented 3-manifold and let $\mathcal{F}$ be a cooriented C^2-smooth codimension-1 foliation on M. Assume that $(M, \mathcal{F})$ is not diffeomorphic to the product foliation on $S^2 \times S^1$. Then arbitrarily close to $\mathcal{F}$ in the C^0 topology there exist a positive and a negative C^∞ contact structure.*

(See Chapter 1 for the appropriate definitions.) In [ET] this theorem appears as one of several very interesting results in the theory of *confoliations*, which Eliashberg and Thurston have introduced by interpolating between the notions of *foliation* and *contact structure*.

The proof of Theorem 0.0.1 presented here follows very closely the plan described in [ET], and all the essential ideas which appear here are due to Eliashberg and Thurston. Our main point here is to single out this result among those established in [ET], and prove it from scratch in full detail. Only when dealing with the local perturbation procedure (Chapter 2) we have been forced to slightly modify the construction given in [ET] in order to ensure C^0-closeness, which we were unable to prove otherwise.

An aspect on which we will try to be rather precise during the proof of Theorem 0.0.1 is the regularity assumption required for the proof to work. In particular we will distinguish between the notions of C^k-smoothness for foliations and for plane fields, explaining the relations between them and making clear in the various statements to which notion we are referring. In Theorem 0.0.1 the foliation $\mathcal{F}$ is assumed to be C^2-smooth *in the sense of foliations.* Note that of course C^0-closeness refers to $\mathcal{F}$ as a plane distribution.

These notes originate from a seminar held in Pisa in November and December 1996 jointly by Riccardo Benedetti, Paolo Lisca and me. All the arguments which appear in these notes, if not directly taken from [ET], are the result of very many conversations among the three of us. It was only after the seminar was finished and all the various bits of the proof were in place that we decided that I would be in charge of writing the final version, but I want to emphasize that without their initiative and hard work this volume would never have existed. I would also like to thank, among the students who contributed to the seminar, Rosa Gini and Elena Giorgi.

I have had the good fortune of discussing several aspects of the proof directly with Prof. W. P. Thurston, and I warmly thank him for his generous help. I am also grateful to Prof. É. Ghys for explaining to me the proof of his Proposition 1.1.6, to E. Giroux for suggesting to clarify whether the regularity assumption on foliations is in the sense of foliations or in the sense of plane fields, and to P. Majer and V. M. Tortorelli for some helpful conversations about facts in analysis. Lastly, I would like to thank MSRI for financial support and for providing a stimulating atmosphere during the time I spent there.

Chapter 1

Definitions and plan of the proof

In the rest of this work M will always be a connected, closed and oriented 3-dimensional manifold with a fixed C^∞ structure (we know it is unique up to equivalence). Recall that a cooriented 2-plane distribution ξ on M can be defined by a global 1-form α, which is well-defined up to multiplication by positive scalar functions. We will say that ξ is C^k*-smooth in the sense of plane distributions* if α can be chosen to be C^k.

A plane distribution ξ defined by a form α is called a foliation if $\alpha \wedge (\,\mathrm{d}\alpha)$ is everywhere 0; we will also call foliation a locally trivial partition of M into immersed surfaces, the equivalence of these two definitions being the content of the Frobenius theorem. The foliation ξ will be called C^k*-smooth in the sense of foliations* if there exists on M a C^k atlas of charts $\phi_i : U_i \to \mathbb{R}^3$ such that for all i the plane field $(\phi_i)_*(\xi|_{U_i})$ is the constant horizontal one. The following holds true (see [CN]):

Proposition 1.0.2. *Let ξ be a foliation on M and let $k \geq 1$.*

1. *If ξ is C^k-smooth in the sense of foliations then it is C^{k-1}-smooth (but in general not C^k-smooth) in the sense of plane distributions;*

2. *If ξ is C^{k-1}-smooth in the sense of plane distributions then it is C^{k-1}-smooth (but in general not C^k-smooth) in the sense of foliations.*

A plane distribution ξ defined by a form α is called a contact structure if $\alpha \wedge (\,\mathrm{d}\alpha)$ is nowhere 0; the sign of the structure depends on whether this 3-form induces the orientation of M or not.

To understand the statement of Theorem 0.0.1 we are only left to define the C^0-topology: one way to do this is to consider a Riemannian metric on M and define the distance between two plane distributions as the norm of the difference between their unit positive normal vector fields. In practice we will always tacitly refer to some fixed distance inducing the topology.

As the notation suggests, there is an obvious notion of C^k-topology for every k, obtained for instance by comparing the derivatives of the unit normal vectors on some fixed finite atlas. Actually, several of the steps toward Theorem 0.0.1 also hold with respect to the C^k-topology. However, since the final statement only holds in general for C^0, we have decided to state also the partial results in C^0 terms. More comments on this point will be made later on.

Before we outline the plan of the proof we need to recall a few more definitions on foliations. We will be slightly informal about them, addressing the readers to the well-established references [HH] and [CN]. We will also cite some results which we will assume without proof, and establish a recent result of Ghys.

1.1 Definitions and classical results

The aim of this section is to state some results which will be used without proof, and recall the definitions required to understand these results. We will also state and prove a recent theorem due to Ghys, whose only written version appears in [ET].

From now on all manifolds will be oriented and all plane fields will be cooriented. It will be convenient for the sequel to fix (besides M) another connected and oriented 3-manifold N without boundary; note that we allow N to be non-compact. Various lemmas will be expressed in terms of N and, when summing up to get the proof of Theorem 0.0.1, N will play the rôle of some open portion of M.

Consider on N a foliation $\mathcal{F}$ which is C^k in the sense of foliations, a leaf $F \in \mathcal{F}$, a point $x \in F$ and a parametrization $p : \mathbb{R} \to N$ of a segment transversal to $\mathcal{F}$, with $p(0) = x$. Given a loop γ at x (assume by simplicity that γ is C^1-embedded) we can find a small annulus $G : S^1 \times \mathbb{R} \to N$ such that $G(1,t) = p(t)$, $G(z,0) = \gamma(z)$ and $G(\{z\} \times \mathbb{R})$ is transversal to $\mathcal{F}$ for all z. If we look at the foliation induced by $\mathcal{F}$ on $S^1 \times \mathbb{R}$ via G, for small $t \in \mathbb{R}$ there exists a function $[0,1] \ni s \mapsto \tau_t(s) \in \mathbb{R}$ such that $\tau_t(0) = t$ and for all t the set
$$\{(\mathrm{e}^{2\pi i s}, \tau_t(s)) : \ s \in [0,1]\}$$
is contained in a leaf. The map $t \mapsto \tau_t(1)$ is an increasing C^k diffeomorphism of a neighbourhood of 0 in $\mathbb{R}$ onto another such neighbourhood, and its germ, as an element of $\mathcal{G}^k(\mathbb{R}, 0)$, depends only on γ and p, is denoted by $h^{(p)}(\gamma)$ and called the *holonomy* of γ (in practice, one often omits to specify p). The

map $\gamma \mapsto h^{(p)}(\gamma)$ induces a representation of $\pi_1(F, x)$ into $\mathcal{G}^k(\mathbb{R}, 0)$, which is changed by a conjugation if the choice of p is changed. The holonomy is called trivial if it is the identity on all loops.

The derivative of $h^{(p)}(\gamma)$ in 0 will be called the *linear holonomy* of γ. Note that it only depends on the homotopy class of γ, and the condition that it be trivial (i.e. its value be 1) is independent of p.

A *minimal set* of a foliation on M is a closed union of leaves which is minimal (with respect to inclusion) among closed unions of leaves. It turns out that if some leaf is somewhere dense then every leaf everywhere is dense, and in this case the foliation is called minimal (because M is the only minimal set). Of course a compact leaf is a minimal set. A minimal set which is neither M nor a compact leaf is called *exceptional*. If X is an exceptional minimal set and $x \in X$, for every open segment σ transversal to $\mathcal{F}$ with $\sigma \ni x$, the set $X \cap \sigma$ is perfect and meager, therefore it is homeomorphic to the Cantor set.

We will call *Reeb component* a foliation on the solid torus $D^2 \times S^1$ homeomorphic to the following one: $D^2 \times \mathbb{R}$ is foliated so that the boundary is a leaf, and the other leaves are $\{(x, t + 1/(1 - |x|)) : x \in D^2\}$ where t varies in $\mathbb{R}$; then the action $(x, t) \mapsto (x, t + n)$ of $\mathbb{Z}$ on $D^2 \times \mathbb{R}$ is considered, and $D^2 \times S^1$ is given the induced foliation.

We are now ready to state some classical results which we will need.

Theorem 1.1.1 (Tischler, [T]) *If $\mathcal{F}$ is a foliation of M, $\mathcal{F}$ is C^2 in the sense of foliations and $\mathcal{F}$ has trivial holonomy then arbitrarily C^0-close to $\mathcal{F}$ there exists a fibration over S^1.*

Theorem 1.1.2 (Sacksteder, [HH]) *If $\mathcal{F}$ is a foliation of M and $\mathcal{F}$ is C^2 in the sense of foliations then $\mathcal{F}$ has finitely many exceptional minimal sets and each of them contains a curve with non-trivial linear holonomy.*

Theorem 1.1.3 (Novikov, [CN]) *If $\pi_2(M) \neq 0$ and $\mathcal{F}$ is a foliation on M which is C^2 in the sense of foliations then either $\mathcal{F}$ is the product foliation on $S^2 \times S^1$ or it has a Reeb component.*

We also state and prove the following:

Theorem 1.1.4 (Ghys, [ET]) *If $\mathcal{F}$ is a minimal foliation on M, $\mathcal{F}$ is C^2 in the sense of foliations and $\mathcal{F}$ has non-trivial holonomy, then $\mathcal{F}$ has non-trivial linear holonomy.*

The proof of this result exploits the same technology used to prove Theorem 1.1.2, namely pseudo-groups, whose definition we will now recall. The reader interested in understanding the plan of the proof of Theorem 0.0.1 might skip the rest of this section at first, because the notions introduced here are not used elsewhere in these notes.

Let X be a topological space. A *pseudogroup* $\mathcal{P}$ on X is a family

$$\{(\phi_i, D_i, R_i) : \ i \in I\}$$

where all D_i's and R_i's are open subsets of X and $\phi_i : D_i \to R_i$ is a homeomorphism, such that the following properties are verified:

1. The identity of X belongs to $\mathcal{P}$;
2. For all i, the triple (ϕ_i^{-1}, R_i, D_i) also belongs to $\mathcal{P}$;
3. If $R_j \cap D_i \neq \emptyset$ then
$$(\phi_i \circ \phi_j, \phi_j^{-1}(R_j \cap D_i), \phi_i(R_j \cap D_i))$$
belongs to $\mathcal{P}$.

We will say that $\mathcal{P}'$ is a *sub-pseudogroup* of $\mathcal{P}$ if it is a pseudogroup and every element of $\mathcal{P}'$ is the restriction of some element of $\mathcal{P}$.

Let $\mathcal{F}$ be a C^k foliation on M. We can cover M with (finitely many) open sets $\{U_i : \ i \in I\}$ such that for all i there exists a diffeomorphism

$$f_i : (U_i, \mathcal{F}|_{U_i}) \to (\mathbb{R}^3, \{\mathbb{R}^2 \times \{t\} : \ t \in \mathbb{R}\}),$$

and moreover if $U_i \cap U_j \neq \emptyset$ then there exists an open set V_{ij} which contains $U_i \cup U_j$ and a diffeomorphism

$$g_{ij} : (V_{ij}, \mathcal{F}|_{V_{ij}}) \to (\mathbb{R}^3, \{\mathbb{R}^2 \times \{t\} : \ t \in \mathbb{R}\}).$$

Let us define X as the disjoint union of the transversals $\sigma_i = f_i^{-1}(\{0\} \times \mathbb{R})$. It is quite easy to see that if $U_i \cap U_j \neq \emptyset$ then there exist open intervals $D_{ij} \subset \sigma_i$ and $R_{ij} \subset \sigma_j$ and a well-defined homeomorphism $\phi_{ij} : D_{ij} \to R_{ij}$ characterized by the property that x and $\phi_{ij}(x)$ belong to the same leaf of $\mathcal{F}|_{V_{ij}}$. Let $\mathcal{P}$ be the pseudogroup on X generated by triples $(\phi_{ij}, D_{ij}, R_{ij})$, and note that under the current assumptions each element of $\mathcal{P}$ is an increasing C^k diffeomorphism between open intervals. Any $\mathcal{P}$ obtained like this will be called a *holonomy pseudogroup*. The following properties are immediate:

1. The leaves of $\mathcal{F}$ correspond bijectively to the $\mathcal{P}$-orbits in X;
2. The previous bijection induces a bijection between minimal sets of $\mathcal{F}$ and closed $\mathcal{P}$-invariant subsets of X which are minimal among closed $\mathcal{P}$-invariant subsets;

3. M is minimal if and only if all the $\mathcal{P}$-orbits in X are dense;

4. A minimal set is exceptional if and only if the corresponding subset of X is a Cantor set;

5. If $x \in X$ corresponds to $L \in \mathcal{F}$, the holonomy group of L is the set of germs at x of elements f of $\mathcal{P}$ such that $f(x) = x$.

Note that the last property explains the terminology we have introduced, and justifies the following one: given a pseudogroup $\mathcal{P}$ of C^k diffeomorphisms between open intervals, we will say that $\mathcal{P}$ has *non-trivial holonomy* if there exist $f \in \mathcal{P}$ and x such that $f(x) = x$ and f is not the identity in a neighbourhood of x. If $k \geq 1$ we will say that $\mathcal{P}$ has *non-trivial linear holonomy* if there exist $f \in \mathcal{P}$ and x such that $f(x) = x$ and $f'(x) \neq 1$.

Given these definitions and remarks, it is quite evident that Sacksteder's Theorem 1.1.2 is a consequence of the following:

Proposition 1.1.5 ([HH]) *If $\mathcal{P}$ is a finitely generated pseudogroup of C^2 increasing diffeomorphisms between intervals and $\mathcal{P}$ has a minimal set which is a Cantor set, then $\mathcal{P}$ has non-trivial linear holonomy.*

Using this proposition it is also clear that Theorem 1.1.4 is a consequence of the following:

Proposition 1.1.6. *If $\mathcal{P}$ is a pseudogroup of increasing homeomorphisms between intervals and $\mathcal{P}$ has dense orbits then $\mathcal{P}$ has a finitely generated sub-pseudogroup which has a Cantor minimal set.*

Proof of 1.1.6. Let f be an element of $\mathcal{P}$ with a fixed point x such that f is not the identity in the neighbourhood of x. Up to reversing all orientations we can assume that f is not the identity on a right neighbourhood of x. Let us restrict f to some interval $(x - r, x + r)$ such that $f(x + r)$ is defined and $f(x + r) \neq x + r$. Up to replacing x by $\sup\{t > x : \; f(t) = t\}$ we can assume that $f(t) \neq t$ for $t > x$ in the domain of f. Up to replacing f by f^{-1} we can assume that $f(t) < t$ for $t > x$. Now let $y > x$ be some fixed element of the domain of f.

Since $\mathcal{P}$ has dense orbits there exists some element g of $\mathcal{P}$ such that $x < g(x) < y$. Now we choose some element $z > x$ in the domain of g close enough to x so that $z < g(x)$ and $g(z) < y$. The condition that $f(t) < t$ for $t > x$ easily implies that $\lim_n f^n(y) = x$; therefore, up to replacing f by some f^n, we can assume that $f(y) < z$.

Summing up, we have found $f, g \in \mathcal{P}$ and $x < z < y$ such that f is defined on some open interval containing $[x, y]$, g is defined on some open interval containing $[x, z]$, and $x = f(x) < f(y) < z < g(x) < g(z) < y$. Therefore the composition $h = g \circ f$ is defined on $[x, y]$. Let us define $\mathcal{Q}$ as the sub-pseudogroup of $\mathcal{P}$ generated by f and h. We claim that $\mathcal{Q}$ has a Cantor minimal set.

Let us define inductively $X_0 = [x, y]$, $X_{n+1} = f(X_n) \bigcup h(X_n)$ and $X = \cap_n X_n$. One easily sees that X is closed and non-empty, and moreover $f(X) = h(X) = X$, so X is $\mathcal{Q}$-invariant. Let $Y \subset X$ be a minimal closed $\mathcal{Q}$-invariant set. We claim that it is a perfect meager set, whence a Cantor set. We first remark that Y cannot be $\{x\}$ because $h(x) \neq x$. Moreover f has infinite orbits on $(x, y]$, so Y must be an infinite set, and hence (since it is a subset of $[x, y]$) it must have accumulation points. Now it is quite easy to see that the set of accumulation points and the boundary of a closed invariant set are both closed and invariant. The minimality of Y then implies that it has no isolated points and it has empty interior, whence the conclusion.

$\boxed{1.1.6}$

1.2 Plan of the proof

We will prove Theorem 0.0.1 only for positive contact structures; the negative case is handled by reversing the orientation of M (note that foliations are insensitive to orientation). We have now some remarks concerning regularity, to which, on the contrary, foliations are very much sensitive but contact structures are not. First of all, the following implies that it is sufficient to prove Theorem 0.0.1 with C^1 contact structures:

Remark 1.2.1. C^∞*-plane distributions are dense in the set of* C^1*-plane distributions with respect to the* C^1*-topology, and the contact condition is open in this topology. Moreover the* C^0*-topology is coarser than the* C^1*-topology.*

Our criterion in stating the partial results leading to Theorem 0.0.1 has been to always consider approximations in the C^0-topology, and to assume on the plane distributions and foliations the minimal regularity for which the result is true.

To outline the scheme of the proof, it is now convenient to introduce the notion of *positive confoliation.* This will be a plane distribution on M which is defined by a 1-form α with $\alpha \wedge (\,\mathrm{d}\alpha) \geq 0$ (formally, $\alpha \wedge (\,\mathrm{d}\alpha)$ should pointwise be a non-negative multiple of a volume form which induces the orientation).

The way the proof goes is now roughly speaking as follows: given a foliation, one first selects a curve whose holonomy satisfies a certain non-triviality condition, and one shows that the foliation can be approximated by a positive confoliation which is a contact structure near the curve. This first step will be called the *local approximation*. Next, using general considerations on foliations (which require C^2-regularity in the sense of foliations), one shows that the local approximation can be performed on sufficiently many curves so that the contact zone in the resulting confoliation can be accessed from any other point by an (injective) curve tangent to the confoliation (such a confoliation will be called *contact-to-be*). The last step of the proof then consists in showing that a positive contact-to-be confoliation can be approximated by a genuine positive foliation.

To give a precise statement of the three steps we still have to introduce a technical notion. A germ f of increasing homeomorphism of $\mathbb{R}$ at 0 is called *two-sided weakly-contracting* if there exist sequences $\{\zeta_n\}$ and $\{\zeta'_n\}$ with

$$\lim_{n\to\infty}\zeta_n = \lim_{n\to\infty}\zeta'_n = 0 \quad \text{and} \quad \zeta'_n < f(\zeta'_n) < 0 < f(\zeta_n) < \zeta_n \quad \text{for all } n \in \mathbb{N}.$$

Since this notion is invariant under conjugation, it makes sense to say that a curve contained in a leaf of a foliation has *two-sided weakly-contracting holonomy.*

We can now state the main three partial results which we will prove:

Proposition 1.2.2 (local approximation) *Let $\mathcal{F}$ be a foliation on N which is C^2-regular in the sense of foliations and let γ be a C^1 embedded curve in N tangent to $\mathcal{F}$. Assume that γ has two-sided weakly-contracting holonomy. Then there exist neighbourhoods U, Z, V of γ and a positive C^1 confoliation ξ on N such that:*

1. *ξ is arbitrarily close to $\mathcal{F}$ in the C^0 topology;*

2. *V is arbitrarily small;*

3. *$U \subset V$ and $Z \subset V$;*

4. *$\xi|_{M\setminus V} = \mathcal{F}|_{M\setminus V}$;*

5. *The contact zone of ξ is exactly Z;*

6. *There exists a diffeomorphisms $\psi : N \to N$ with*

$$\psi(V) = V, \quad \psi|_{M\setminus V} = \mathrm{id}, \quad \psi(U) = Z \quad \psi_*(\mathcal{F}|_{M\setminus U}) = \xi|_{M\setminus Z}.$$

Proposition 1.2.3 (contact-to-be approximation) *If $\mathcal{F}$ is a foliation on M, $\mathcal{F}$ is C^2-smooth in the sense of foliations and $(M, \mathcal{F})$ is not diffeomorphic to the product foliation on $S^2 \times S^1$ then arbitrarily close to $\mathcal{F}$ in the C^0-topology there exist C^1 contact-to-be positive confoliations.*

Proposition 1.2.4 (globalization) *If ξ is a C^1 contact-to-be positive confoliation then arbitrarily close to ξ in the C^0-topology there exist C^1 positive contact structures.*

To be completely formal, the proof of Proposition 1.2.3 will require not only Proposition 1.2.2 as it is stated above, but also the construction by which it is proved. We have preferred to proceed like this to avoid having an excessively long statement of 1.2.2: one should see the core of the proof of 1.2.3 (which appears in Section 3.3) as a direct continuation of the proof of 1.2.2.

As we already mentioned, it is mainly in the statement and proof of Proposition 1.2.2 that our exposition differs from the original source [ET]. Moreover we have refrained to state variants of the proposition which, under stronger assumptions on the holonomy, allow to perform the approximation more efficiently (that is, in a better class of regularity and with a much easier technical method).

We conclude this section by remarking that Theorem 0.0.1 is an immediate consequence of Propositions 1.2.3 and 1.2.4, together with the remarks made above. Propositions 1.2.2 and 1.2.4 are proved essentially from scratch, while Proposition 1.2.3 requires (together with Proposition 1.2.2) the classical results stated in the previous section and some general facts which will be established in Section 3.2.

Chapter 2

Local approximation

This chapter is devoted to the proof of Proposition 1.2.2. The argument first requires the choice of appropriate coordinates in the neighbourhood of a curve tangent to a foliation, with the property that to certain conditions on the holonomy of the curve there correspond global conditions on the 1-form defining the foliation in these coordinates. (Actually, we will only need to deal with the condition that the holonomy be two-sided weakly-contracting, but for the sake of completeness we will also mention other conditions.)

Once suitable coordinates have been chosen, a direct argument, based on a technical lemma on real functions of one variable, allows to complete the proof. Since the argument itself is rather long, it is perhaps useful to sketch here the geometric meaning of the constructions we will make.

First of all, one easily sees that a neighbourhood of the curve in exam can be parametrized as $S^1_x \times \mathbb{R}_y \times \mathbb{R}_z$ (where the subscript just means the name of the coordinate, and x is understood to be modulo $\mathbb{Z}$) in such a way that all the segments $\{x_0\} \times \mathbb{R}_y \times \{z_0\}$ are tangent to the foliation. The idea is now to consider a segment $\{x_0\} \times [-1,1]_y \times \{z_0\}$ and, starting at $y = 1$ with the tangent plane to the foliation and proceeding leftwards, monotonically rotate the plane while keeping it tangent to the segment. Doing this for all x_0, z_0 gives rise to a positive contact structure in $S^1_x \times (-1,1)_y \times \mathbb{R}_z$ which can be glued to the foliation along $S^1_x \times \{1\}_y \times \mathbb{R}_z$ producing a positive confoliation η on $S^1_x \times (-1,+\infty)_y \times \mathbb{R}_z$.

Now, using the condition on the holonomy, one shows (this is the key technical point) that the above procedure can be carried out in such a way that the holonomies of $S^1_x \times \{1\}_y \times \{0\}_z$ in $\mathcal{F}$ and of $S^1_x \times \{-1\}_y \times \{0\}_z$ in $(S^1_x \times \{-1\}_y \times \mathbb{R}_z, \eta)$ are conjugate to each other. The conjugating diffeomorphism is homotopic to the identity, and one can use this homotopy to extend η to $S^1_x \times [-2,+\infty)_y \times \mathbb{R}_z$ so that it is a foliation for $-2 \leq y \leq -1$ and it coincides with $\mathcal{F}$ for $y=-2$. Then one can further extend η to the whole of $S^1_x\times\mathbb{R}_y\times\mathbb{R}_z$ setting $\eta=\mathcal{F}$ for $y\leq -2$; this leads to a positive confoliation which is contact near $S^1_x\times\{0\}_y\times\{0\}_z$ and coincides with $\mathcal{F}$ sufficiently far from this curve; moreover, outside the contact zone, η is a foliation diffeomorphic to $\mathcal{F}$.

This construction is exactly as explained in [ET]. However we were not able to prove in general that the confoliation η thus constructed is close to $\mathcal{F}$ in the C^0 topology. One case when this is true is when the conjugating diffeomorphism is close to the identity in the C^1 topology, but this does not seem to be the case in general. (It is, however, when the holonomy is supposed to have derivative smaller than 1, a stronger condition than being two-sided weakly-contracting. As a matter of fact Eliashberg and Thurston treat the case of non-trivial linear holonomy first, and then they assert that the same techniques work for the case of two-sided weakly-contracting holonomy; since we were not able to prove the C^0-closeness, we have found it necessary to somewhat refine the argument.)

The way we achieve C^0-closeness is roughly speaking as follows: we first restrict to a horizontal slice $-r\leq z\leq r$ on which $\mathcal{F}$ is ε-close to the horizontal foliation, for some fixed small ε. Then we apply the above technique to obtain a confoliation η which differs from $\mathcal{F}$ only for $-r/2<z<r/2$. Such an η could be very far from being horizontal, but now we squeeze the segment $[-r/2,r/2]_z$ to some very short $[-\delta,\delta]_z$, in such a way that the resulting plane field is ε-close to the horizontal one. At the same time the intervals $[-r,-r/2]$ and $[r/2,r]$ are stretched respectively to $[-r,-\delta]$ and $[\delta,r]$; since the stretching factor is at most 2, the result is 2ε-close to the horizontal plane distribution. Therefore the new confoliation is 3ε-close to $\mathcal{F}$.

While performing this contruction one has to keep under control various other data which we have not mentioned here, in particular in view to the application to the proof of Proposition 1.2.3. This is why the statement of Proposition 1.2.2 (and its proof, given in Section 2.3) are rather long.

2.1 Local coordinates

In this section we will show how to choose coordinates in the neighbourhood of a curve tangent to a foliation in such a way that some special properties of the holonomy of the curve reflect into special properties of the form which defines the foliation in these coordinates. For the purpose of proving Proposition 1.2.2 we will actually only need to examine the property that the holonomy be two-sided weakly-contracting. However the same technique allows to obtain other results, and we will mention them.

First of all we note that if we have an embedded loop contained in a leaf of a cooriented codimension-1 foliation $\mathcal{F}$ of an oriented 3-manifold, then we can parametrize a neighbourhood of the curve as $S^1_x \times \mathbb{R}_y \times \mathbb{R}_z$ in such a way that $\mathcal{F}$ in these coordinates is defined by a 1-form $\mathrm{d}z + a(x,z)\,\mathrm{d}x$ with $a(x,0) = 0$. This implies that the relevant situation is essentially 2-dimensional, because the foliation is trivial in the y-direction. We are naturally lead to considering foliations $\mathcal{F}$ of $S^1_x \times \mathbb{R}_z$ which contain $S^1_x \times \{0\}_z$ as a leaf. The germ of such a foliation near $S^1_x \times \{0\}_z$ is determined by the holonomy h of the generator of $\pi_1(S^1)$, where h is a germ at 0 of diffeomorphism of $\mathbb{R}$. We will show in this section that given h one can reparametrize $S^1_x \times \mathbb{R}_z$ (in the same regularity class as $\mathcal{F}$) in such a way that the resulting equation $\mathrm{d}z + a(x,z)\,\mathrm{d}x$ of $\mathcal{F}$ satisfies certain properties whenever h does. In particular we will see that if h is two-sided weakly-contracting then there exist sequences $\{\zeta_n\}$ and $\{\zeta'_n\}$ with $\zeta'_n < 0 < \zeta_n$, $\lim_n \zeta_n = \lim_n \zeta'_n = 0$ and $a(x,\zeta'_n) < 0 < a(x,\zeta_n)$ for all n and $x \in S^1$.

The geometric idea of the choice of these coordinates goes as follows. Assume one starts with a C^k holonomy h. What we want is a "preferred" C^k foliation of $S^1_x \times \mathbb{R}_z$ with holonomy h. To get it we consider the universal cover $\mathbb{R}_x$ of S^1_x which is acted on by $\mathbb{Z}$, and we foliate $\mathbb{R}_x \times \mathbb{R}_z$ in such a way that each segment joining a point (n,z) with a point $(n+1, h(z))$ is contained in a leaf. This gives rise to a foliation of $S^1_x \times \mathbb{R}_z$ which has holonomy h, but is not even C^1. To obtain the desired foliation then we regularize the leaves of the foliation of $\mathbb{R}_x \times \mathbb{R}_z$ by taking the convolution with a symmetric bell function supported in $[-1/4, 1/4]$. It will take some efforts to prove that this indeed provides a C^k foliation on $S^1_x \times \mathbb{R}_z$, and to establish the desired properties.

Lemma 2.1.1. *Let $h \in \mathcal{G}^k(\mathbb{R}, 0)$ be a germ at 0 of a C^k diffeomorphism between intervals of $\mathbb{R}$ such that $h(0) = 0$. Assume that $h'(0) > 0$. Consider a germ of foliation $\mathcal{F}_h$ on $S^1_x \times \mathbb{R}_z$ near $S^1_x \times \{0\}$ obtained as sketched above, and consider an equation $\mathrm{d}z + a(x,z)\,\mathrm{d}x$ for $\mathcal{F}_h$. Then:*

1. *The function a is C^k-regular, hence the foliation $\mathcal{F}_h$ is C^k-regular in the sense of plane distributions (in particular, in the sense of foliations);*

2. *$S^1_x \times \{0\}$ is a leaf of $\mathcal{F}_h$ with holonomy h;*

3. *If $h(\zeta) < \zeta$ for some $\zeta > 0$ then $a(x,\zeta) > 0$ for all $x \in S^1$; similarly if $0 > h(\zeta') > \zeta'$ then $a(x,\zeta') < 0$.*

Before proving this result, for the reader's information, we state some more properties of this foliation $\mathcal{F}_h$ which could be established along the same lines:

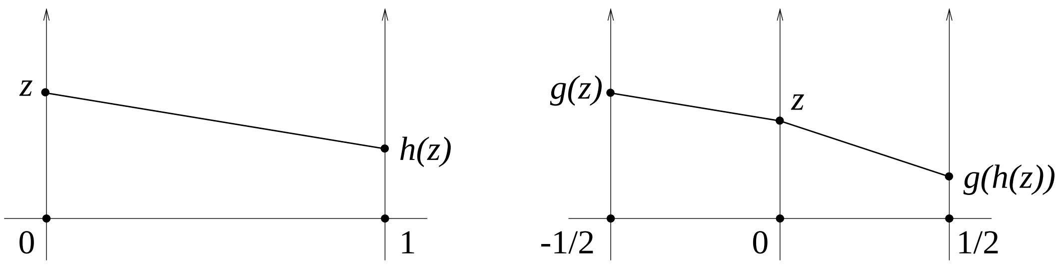

Figure 2.1: The unregularized foliation in the universal cover

4. If $h'(0) = 1$ (i.e. $S^1 \times \{0\}$ has trivial linear holonomy) then
$$\frac{\partial a}{\partial z}(x, 0) = 0 \qquad \text{for all } x;$$

5. If $h'(0) < 1$ (i.e. $S^1 \times \{0\}$ has contracting linear holonomy) then
$$\frac{\partial a}{\partial z}(x, 0) > 0 \qquad \text{for all } x;$$

6. If $h(z) < z$ for all $z > 0$ (i.e. $S^1 \times \{0\}$ has contracting holonomy on the positive side) then $a(x, z) > 0$ for $z > 0$ and all x;

7. If $h(z') - h(z) < z' - z$ for all $0 < z < z'$, then for each x the function $z \mapsto a(x, z)$ is strictly increasing for $z \geq 0$.

Proof of 2.1.1. We first need to formalize the construction of $\mathcal{F}_h$. By simplicity we extend h to a diffeomorphism of $\mathbb{R}$, so that the construction becomes global (but of course the relevant properties only have to be checked in the sense of germs).

We first examine the foliation $\mathcal{G}_h$ before taking the convolution. By definition, in the slice $[0, 1]_x \times \mathbb{R}_z$ of the universal cover, the leaves we see are as shown on the left-hand side of Fig. 2.1. This implies that in the slice $[-1/2, 1/2]_x \times \mathbb{R}_z$ we see what is shown on the right-hand side of the same figure, where g is the diffeomorphism of $\mathbb{R}$ given by
$$g(z) = \frac{1}{2}(h^{-1}(z) + z).$$
Let us define for $w \in \mathbb{R}$ a map $\ell_w : \mathbb{R} \to \mathbb{R}$ as
$$\ell_w(x) = \begin{cases} z + x(z - h^{-1}(z)) & \text{if } x \leq 0 \\ z + x(h(z) - z) & \text{if } x \geq 0 \end{cases}, \qquad \text{where } z = g^{-1}(w).$$
Note that the holonomy of $\mathcal{G}_h$ with basepoint $-1/2$ is given by $\tilde{h}(w) = \ell_w(1/2)$; one easily sees that $\tilde{h} = g \circ h \circ g^{-1}$, which is coherent with the fact that the holonomy of a curve is well-defined only up to conjugation.

Consider now the real functions

$$v(t) = \begin{cases} e^{\frac{1}{1-(4t)^2}} & \text{for } |t| < 1/4 \\ 0 & \text{otherwise} \end{cases}, \qquad u(t) = v(t) \left(\int_{-\infty}^{\infty} v(s) ds \right)^{-1}$$

and define

$$L_w(x) = \int_{-\infty}^{\infty} u(t-x)\ell_w(t)\, \mathrm{d}t.$$

If $w > w'$ we have $\ell_w(t) > \ell_{w'}(t)$ for all t, hence $L_w(x) > L_{w'}(x)$ for all x. This easily implies that by projecting to $S^1_x \times \mathbb{R}_z$ the sets $\{L_w(x) : \ |x| \leq 1/2\}$ we get a C^0-foliation $\mathcal{F}_h$.

Now note that for $|x| \geq 1/4$ the map ℓ_w has the form $t \mapsto at + b$ on $[x - 1/4, x + 1/4]$; therefore

$$\begin{aligned} L_w(x) &= \int_{x-1/4}^{x+1/4} u(t-x)(at+b)\, \mathrm{d}t = \int_{-1/4}^{1/4} u(\tau)(a\tau + ax + b)\, \mathrm{d}\tau \\ &= a \int_{-1/4}^{1/4} \tau u(\tau)\, \mathrm{d}\tau + (ax+b) \int_{-1/4}^{1/4} u(\tau)\, \mathrm{d}\tau = ax + b \end{aligned}$$

because u is an even function and its integral is 1. So $L_w(x) = \ell_w(x)$ for $|x| \geq 1/4$. This implies that the holonomy of $S^1_x \times \{0\}_z$ in $\mathcal{F}_h$ is still $\tilde{h}$ (i.e. h, up to conjugation). Moreover each individual leaf of $\mathcal{F}_h$ is C^∞ regular. To complete the proof of properties 1 and 2 we are left to prove that $\mathcal{F}_h$ is C^k regular.

To this end we compute the form $\mathrm{d}z + a(x,z)\,\mathrm{d}x$ which defines $\mathcal{F}_h$. Since $a(x,z)$ is the opposite of the slope of the leaf through (x,z) we have for x in a neighbourhood of $[-1/2, 1/2]$ and for all w that

$$a(x, L_w(x)) = -\frac{\mathrm{d}L_w}{\mathrm{d}x}(x) = \int_{-\infty}^{\infty} u'(t-x)\ell_w(t)\, \mathrm{d}t.$$

Now we define $\Psi_x(w) = L_w(x)$ and we claim that:

I. $(x,w) \mapsto \Psi_x(w)$ is C^k regular;

II. $\Psi'_x(0) \neq 0$ for all x.

Assume for a moment these claims to be proved. Then Ψ_x has a local inverse Φ_x near 0 and the map $(x,z) \mapsto \Phi_x(z)$ is C^k-regular because locally

$$\Big((x,z) \mapsto (x, \Phi_x(z))\Big) = \Big((x,w) \mapsto (x, \Psi_x(w))\Big)^{-1}$$

by the implicit function theorem. Moreover with easy computations

$$\begin{aligned} a(x,z) &= \int_{-\infty}^{0} u'(t-x)\Big(g^{-1}(\Phi_x(z)) + 2t(g^{-1}(\Phi_x(z)) - \Phi_x(z))\Big)\,\mathrm{d}t \\ &+ \int_{0}^{\infty} u'(t-x)\Big(g^{-1}(\Phi_x(z)) + 2t(\tilde{h}(\Phi_x(z)) - g^{-1}(\Phi_x(z)))\Big)\,\mathrm{d}t \end{aligned}$$

(where as above $\tilde{h} = g \circ h \circ g^{-1}$) which immediately implies that a is jointly C^k-regular (at least near $z=0$, which is sufficient for us).

Now we prove the claims. Claim I immediately follows from the formula:

$$\begin{aligned} \Psi_x(w) &= \int_{-\infty}^{0} u(t-x)\Big(g^{-1}(w) + 2t(g^{-1}(w) - w)\Big)\,\mathrm{d}t \\ &+ \int_{0}^{\infty} u(t-x)\Big(g^{-1}(w) + 2t(\tilde{h}(w) - g^{-1}(w))\Big)\,\mathrm{d}t. \end{aligned}$$

To prove claim II we conside the Taylor expansion $h(z) = cz + \mathrm{o}(z)$, which implies that

$$g(z) = \frac{1+c}{2c} z + \mathrm{o}(z), \qquad g(h(z)) = \frac{1+c}{2} z + \mathrm{o}(z).$$

Hence:

$$\Psi'_x(0) = \lim_{w\to 0} \frac{\Psi_x(w)}{w} = \lim_{z\to 0} \frac{\Psi_x(g(z))}{z} \frac{z}{g(z)} = \frac{2c}{1+c} \lim_{z\to 0} \frac{\Psi_x(g(z))}{z}.$$

Moreover:

$$\begin{aligned} &\lim_{z\to 0} \frac{\Psi_x(g(z))}{z} \\ &= \int_{-\infty}^{0} u(t-x) \lim_{z\to 0} \frac{z + 2t(z - g(z))}{z}\,\mathrm{d}t + \int_{0}^{\infty} u(t-x) \lim_{z\to 0} \frac{z + 2t(g(h(z)) - z)}{z}\,\mathrm{d}t \\ &= \int_{-\infty}^{0} u(t-x)\left(1 + 2t\left(1 - \frac{1+c}{2c}\right)\right)\mathrm{d}t + \int_{0}^{\infty} u(t-x)\left(1 + 2t\left(\frac{1+c}{2} - 1\right)\right)\mathrm{d}t \\ &= 1 + \frac{c-1}{c} \int_{-\infty}^{0} u(t-x)t\,\mathrm{d}t + (c-1)\int_{0}^{\infty} u(t-x)t\,\mathrm{d}t. \end{aligned}$$

Note that

$$\int_{-\infty}^{\infty} u(t-x)t\,\mathrm{d}t = x,$$

so we can rewrite $\frac{(1+c)}{2c}\Psi'_x(0)$ in the following two ways:

$$\begin{aligned}
\text{(A)} &= 1 + \frac{c-1}{c}\int_{-\infty}^{0} u(t-x)t\,\mathrm{d}t + (c-1)\left(x - \int_{-\infty}^{0} u(t-x)t\,\mathrm{d}t\right) \\
&= 1 + (c-1)x - \frac{(c-1)^2}{c}\int_{-\infty}^{0} u(t-x)t\,\mathrm{d}t \\
\text{(B)} &= 1 + \frac{c-1}{c}\left(x - \int_{0}^{\infty} u(t-x)t\,\mathrm{d}t\right) + (c-1)\int_{0}^{\infty} u(t-x)t\,\mathrm{d}t \\
&= 1 + \frac{c-1}{c}x + \frac{(c-1)^2}{c}\int_{0}^{\infty} u(t-x)t\,\mathrm{d}t.
\end{aligned}$$

In both (A) and (B) the third summand is strictly positive; moreover

- if $c \le 1$ then $1 + (c-1)x \ge 0$ for all $x \in [-1/2, 1/2]$ (recall that $c > 0$);
- if $c \ge 1$ then $1 + \frac{c-1}{c}x \ge 0$ for all $x \in [-1/2, 1/2]$.

This immediately implies claim II and hence properties 1 and 2 of the lemma.

We are left to prove property 3. We will only deal with the case $0 < h(\zeta) < \zeta$, the other one being analogous.

We first note that $h^{m+1}(\zeta) < h^m(\zeta)$ for all $m \in \mathbb{Z}$. We define $\zeta_0 = \zeta$ and $\zeta_1 = (g \circ h^{-1} \circ g^{-1})(\zeta_0) = \tilde{h}^{-1}(\zeta_0)$. In other words ζ_1 is defined by the equivalent relations

$$\ell_{\zeta_1}(1/2) = \zeta_0, \qquad L_{\zeta_1}(1/2) = \zeta_0.$$

We claim that:

(i) For all $x \in [-1/2, 1/2]$ there exists $w \in [\zeta_0, \zeta_1]$ such that $L_w(x) = \zeta_0$;

(ii) For all $w \in [\zeta_0, \zeta_1]$ and $x \in \mathbb{R}$, we have

$$\frac{\mathrm{d}L_w}{\mathrm{d}x}(x) < 0.$$

Note that (i) and (ii) immediately imply the conclusion because

$$a(x, L_w(x)) = -\frac{\mathrm{d}L_w}{\mathrm{d}x}(x).$$

To show (i) we will check that:

(i-a) $L_{\zeta_0}(x) \leq \zeta_0$ for all $x \in [-1/2, 1/2]$;

(i-b) $L_{\zeta_1}(x) \geq \zeta_0$ for all $x \in [-1/2, 1/2]$;

This is sufficient to establish (i) because we know that the map $w \mapsto L_w(x)$ is continuous and increasing for any fixed x.

Since L_{ζ_0} is obtained by taking the average of ℓ_{ζ_0}, to show (i-a) and (i-b) it is sufficient to prove that $\ell_{\zeta_0}(x) \leq \zeta_0$ and $\ell_{\zeta_1}(x) \geq \zeta_0$ for all x. Recall that ℓ_w is the union of the two line segments joining the points

$$(-1/2, w), \qquad (0, g^{-1}(w)), \qquad (1/2, (g \circ h \circ g^{-1})(w)).$$

Therefore to prove (i-a) and (i-b) one has to establish the following inequalities:

$$\begin{array}{ll} \text{(i-a-1)} & g^{-1}(\zeta_0) < \zeta_0; \\ \text{(i-a-2)} & (g \circ h \circ g^{-1})(\zeta_0) < \zeta_0; \\ \text{(i-b-1)} & \zeta_1 > \zeta_0; \\ \text{(i-b-2)} & g^{-1}(\zeta_1) > \zeta_0. \end{array}$$

Now

$$\begin{array}{rcl} \text{(i-a-1)} & \Leftrightarrow & \zeta_0 < g(\zeta_0) \\ & \Leftrightarrow & \zeta_0 < \frac{1}{2}(\zeta_0 + h^{-1}(\zeta_0) \\ & \Leftrightarrow & \zeta_0 < h^{-1}(\zeta_0) \\ & \Leftrightarrow & h(\zeta_0) < \zeta_0 \end{array}$$

and the last inequality is true. For (i-a-2) we note that $g \circ h$ is increasing, so from (i-a-1) we get

$$(g \circ h \circ g^{-1})(\zeta_0) < (g \circ h)(\zeta_0) = \frac{1}{2}(\zeta_0 + h(\zeta_0)) < \zeta_0.$$

The inequalities (i-b-1) and (i-b-2) are proved essentially in the same way.

Now we turn to point (ii). Let us recall that the convolution with u of a strictly decreasing function has negative derivative, So it will be sufficient to show that ℓ_w is strictly decreasing in x for $w \in [\zeta_0, \zeta_1]$. Recalling again the definition of ℓ_w one sees that it is sufficient to establish the following inequalities:

$$\begin{array}{ll} \text{(I)} & w > g^{-1}(w) \quad \text{for all } w \in [\zeta_0, \zeta_1], \\ \text{(II)} & g^{-1}(w) > (g \circ h \circ g^{-1})(w) \quad \text{for all } w \in [\zeta_0, \zeta_1]. \end{array}$$

We start with (II). From what we have shown above it easily follows that

$$h(\zeta_0) < g^{-1}(\zeta_0) < \zeta_0 < g^{-1}(\zeta_1) < h^{-1}(\zeta_0).$$

So (II) is implied by

$$z > (g \circ h)(z) \quad \text{for all} \quad z \in [h(\zeta_0), h^{-1}(\zeta_0)];$$

using the formula for g we see that this inequality is equivalent to

$$z > h(z) \quad \text{for all} \quad z \in [h(\zeta_0), h^{-1}(\zeta_0)].$$

Now note that h is a continuous monotonic function, so it maps $[h(\zeta_0), \zeta_0]$ to $[h^2(\zeta_0), h(\zeta_0)]$, and $[\zeta_0, h^{-1}(\zeta_0)]$ to $[h(\zeta_0), \zeta_0]$. This implies that $z > h(z)$ respectively for $z \in [h(\zeta_0), \zeta_0]$ and for $z \in [\zeta_0, h^{-1}(\zeta_0)]$, whence the desired inequality.

The proof of (I) is similar. Using the same formulas as above one shows that

$$\begin{aligned} \text{(I)} \quad &\Leftarrow \quad g(z) > z \quad \text{for all} \quad z \in [h(\zeta_0), h^{-1}(\zeta_0)] \\ &\Leftrightarrow \quad h^{-1}(z) > z \quad \text{for all} \quad z \in [h(\zeta_0), h^{-1}(\zeta_0)] \\ &\Leftrightarrow \quad z > h(z) \quad \text{for all} \quad z \in [h(\zeta_0), h^{-1}(\zeta_0)] \end{aligned}$$

and this was already established above.

$\boxed{2.1.1}$

Let us recall now in a formal way the following local stability result already mentioned above (see [HH]):

Proposition 2.1.2. *Let $\mathcal{F}$ be a foliation; if $\mathcal{F}$ is C^k-smooth in the sense of foliations and $F \in \mathcal{F}$ is compact then the germ of $\mathcal{F}$ in the neighbourhood of F is determined up to C^k diffeomorphism by the holonomy of F.*

This allows us to state the consequence of Lemma 2.1.1 which we will need in practice.

Corollary 2.1.3. *Let $\mathcal{F}$ be a codimension-1 cooriented foliation of an oriented 3-manifold N, assume that $\mathcal{F}$ is C^k in the sense of foliations, and let γ be a C^k embedded loop which lies in a leaf of $\mathcal{F}$. Suppose that the holonomy of γ is two-sided weakly-contracting. Then there exists a C^k parametrization $F : S^1_x \times \mathbb{R}_y \times \mathbb{R}_z \to N$ of a tubular neighbourhood of γ such that $F^*(\mathcal{F})$ is defined by a 1-form $\mathrm{d}z + a(x,z)\,\mathrm{d}x$ with the following property: there exist sequences $\{\zeta_n\}$ and $\{\zeta'_n\}$ with $\zeta'_n < 0 < \zeta_n$, $\lim_n \zeta_n = \lim_n \zeta'_n = 0$ and $a(x, \zeta'_n) < 0 < a(x, \zeta_n)$ for all n and $x \in S^1$. Moreover a is a C^k function.*

We have now a few more comments concerning regularity. Let Σ be a surface with a fixed C^∞ structure. We know that if $\mathcal{F}$ is a foliation which is C^k smooth in the sense of foliations then there exists on Σ a system of C^k charts $u_i : \mathbb{R}^2_{(x,z)} \to U_i$ such that $u_i^*(\mathcal{F}|_{U_i})$ is defined by the form $\mathrm{d}z$ (a C^∞, in particular C^k, form on $\mathbb{R}^2$). However such a parametrization in general exists only locally: we know that on Σ the foliation is defined by a form which is in general at most C^{k-1}, so if Σ' is another surface and $u : \Sigma' \to \Sigma$ is C^k then $u^*(\mathcal{F})$ is defined in general by a C^{k-1} form. Therefore Lemma 2.1.1 implies the non-obvious fact that near any curve tangent to $\mathcal{F}$ there exists a C^k chart in which $\mathcal{F}$ is defined by a C^k form. Note however that this gives no information about the form which defines $\mathcal{F}$ on Σ.

2.2 Analytic core of the proof

In this section we prove a result on real functions of one variable which will be the key technical tool for the local perturbation near a curve with two-sided weakly-contracting holonomy.

Lemma 2.2.1. *Let X be a compact topological space and let*

$$v : X \times [-1,1] \ni (x,z) \mapsto v_x(z) \in \mathbb{R}$$

be a continuous function such that:

1. *Each v_x is C^1 and $\sup\{|v_x'(z)| : (x,z) \in X \times [-1,1]\} < \infty$;*
2. *There exist sequences $\{\zeta_n\}$ and $\{\zeta_n'\}$ in $[-1,1]$ such that $\zeta_n' < 0 < \zeta_n$ for all n, $\lim_n \zeta_n = \lim_n \zeta_n' = 0$ and $v_x(\zeta_n') < 0 < v_x(\zeta_n)$ for all $x \in X$ and $n \in \mathbb{N}$.*

Then there exists a sequence $\{f_n\}$ of C^∞ diffeomorphisms of $[-1,1]$ such that:

(i) f_n is the identity outside an open interval U_n which contains 0;

(ii) $f_n'(z) \cdot v_x(z) < v_x(f_n(z))$ for all $x \in X$, $n \in \mathbb{N}$ and $z \in U_n$;

(iii) U_n shrinks to 0 as $n \to \infty$ (hence f_n converges to the identity in the C^0 topology).

Proof of 2.2.1. We just define U_n as (ζ_n', ζ_n), so the last property will be immediate. We fix a certain n and show how to construct the corresponding

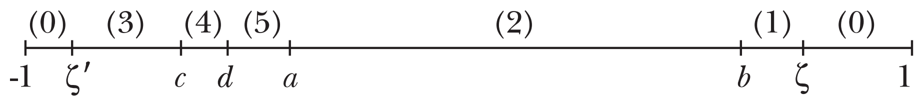

Figure 2.2: The subdivision of the interval

f_n, omitting from now on n from the notation. Let us define:

$$\begin{aligned} M &= \max\Big\{|v_x(z)| : \ (x,z) \in X \times [-1,1]\Big\}, \\ K &= \max\Big\{|v_x'(z)| : \ (x,z) \in X \times [-1,1]\Big\}. \end{aligned}$$

We also select points a, b with $\zeta' < a < 0 < b < \zeta$ such that $v_x(z)$ is non-zero on $[\zeta', a]$ and $[b, \zeta]$ (here we are using the continuity of v and the compactness of X) and we define:

$$\begin{aligned} T &= \min\{|v_x(z)| : \ x \in X, \ z \in [\zeta', a] \cup [b, \zeta]\}, \\ c &= (\zeta' + a)/2. \end{aligned}$$

Let us introduce the following constants:

$$\begin{aligned} \beta &= \frac{2K}{T} \cdot (\zeta - b)^2 \\ \alpha &= \left(1 - \frac{T}{2M}\right) \frac{(b-\zeta)^2}{\beta} \exp\Big(\beta(\zeta - b)^{-1}\Big) \\ \delta &= \frac{2K}{T} \cdot (c - \zeta')^2 \\ \gamma &= \left(\frac{2M}{T} - 1\right) \frac{(\zeta' - c)^2}{\delta} \exp\Big(\delta(c - \zeta')^{-1}\Big) \end{aligned}$$

(their motivation will be clear in the sequel).

We will subdivide the interval $[-1, 1]$ into various subintervals, and describe the function f separately on these subintervals by imposing some restrictions, and checking at each step that indeed a function with the required properties exists. Later we will verify that a function f satisfying the various restrictions verifies the inequality $f'(z) \cdot v_x(z) < v_x(f(z))$. The subdivision of $[-1, 1]$ is described in Fig. 2.2, where the point d is not determined a priori, it will be chosen in (c, a) during the construction of f, which goes as follows:

(0) On $[-1, 1] \setminus [\zeta', \zeta]$ we define f to be the identity.

(1) For $z \in [b, \zeta]$ we define $f(z) = z + \alpha \exp\Big(\beta(z - \zeta)^{-1}\Big)$.

(2) On $[a,b]$ we define f as a strictly increasing function such that $f(a) > b$ and $0 < f'(z) < T/M$ for all $z \in [a,b]$. To verify that there exists such an extension of the function already defined, we only need to note that from (1) we have $f(b) > b$ and

$$f'(b) = 1 - \frac{\alpha\beta}{(b-\zeta)^2}\exp\left(\beta(b-\zeta)^{-1}\right) = \frac{T}{2M} < \frac{T}{M}.$$

(3) On $[\zeta', c]$ we define $f(z) = z + \gamma\exp\left(\delta(\zeta'-z)^{-1}\right)$.

(4) We select a point $d \in (c,a)$ close enough to a so that an extension of f exists on $[c,d]$ such that $f'(z) > M/T$ for all $z \in [c,d]$ and $b < f(d) < f(a)$, where the value of f in a in the last formula comes from (2). To check that this is possible we need to note from (3) that

$$f'(c) = 1 + \frac{\gamma\delta}{(\zeta'-c)^2}\exp\left(\delta(\zeta'-c)^{-1}\right) = \frac{2M}{T} > \frac{M}{T}.$$

Since we know that $b < f(a)$ indeed it is sufficient to choose d close enough to c and f steep enough.

(5) On $[d,a]$ we choose any smooth increasing extension of what we already have, which exists because $f(d) < f(a)$.

In Fig. 2.3 we illustrate the construction of f.

Now let us verify the inequality $f'(z)\cdot v_x(z) < v_x(f(z))$ on (ζ',ζ). We will proceed separately on the various subintervals.

(1) On $[b,\zeta)$ we have $f(z) > z$. So, by the mean value theorem and the definition of K we have on this interval that

$$v(f(z)) \geq v(z) - K(f(z)-z).$$

It is therefore sufficient to prove that

$$\begin{aligned}
& f'(z)v(z) < v(z) - K(f(z)-z)\\
\Leftrightarrow\ & \left(1 - \frac{\alpha\beta}{(z-\zeta)^2}\exp\left(\beta(z-\zeta)^{-1}\right)\right)v(z) < v(z) - K\alpha\exp\left(\beta(z-\zeta)^{-1}\right)\\
\Leftrightarrow\ & -v(z)\frac{\alpha\beta}{(z-\zeta)^2} < -K\alpha\\
\Leftrightarrow\ & v(z) > \frac{K}{\beta}(z-\zeta)^2.
\end{aligned}$$

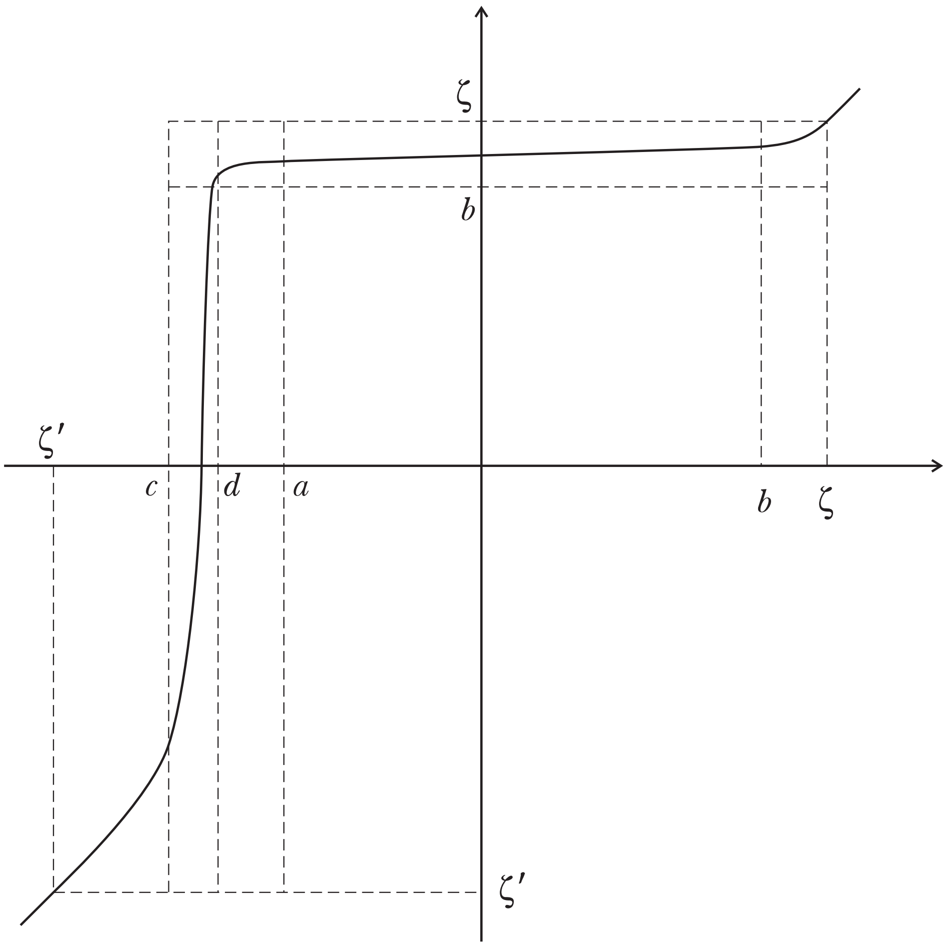

Figure 2.3: The diffeomorphism f

Now $\min_{z\in[b,\zeta]} v(z) \geq T$ while

$$\max_{z\in[b,\zeta]} \frac{K}{\beta}(z-\zeta)^2 = \frac{K}{\beta}(b-\zeta)^2 = \frac{T}{2},$$

so the inequality is verified in this case.

(2) For $z \in [a,b]$ we have $f(z) \in [b,\zeta]$ so $v(f(z)) \geq T$ while $0 < f'(z) < T/M$ and $|v(z)| \leq M$, whence

$$f'(z)v(z) \leq |f'(z)v(z)| < \frac{T}{M}M = T$$

and the conclusion holds also in this case.

(3) For $z \in (\zeta', c]$ we have $f(z) > z$ so by the choice of K and the mean value theorem we have

$$v(f(z)) > v(z) - K(f(z) - z)$$

and therefore it is sufficient to prove the following:

$$\begin{aligned}
& f'(z)v(z) < v(z) - K(f(z)-z) \\
\Leftrightarrow\ & \left(1 + \frac{\gamma\delta}{(\zeta'-z)^2}\exp\left(\delta(\zeta'-z)^{-1}\right)\right)v(z) < v(z) - K\gamma\exp\left(\delta(\zeta'-z)^{-1}\right) \\
\Leftrightarrow\ & v(z)\frac{\gamma\delta}{(\zeta'-z)^2} < -K\gamma \\
\Leftrightarrow\ & v(z) < -\frac{K}{\delta}(\zeta'-z)^2.
\end{aligned}$$

Now $\max_{z\in[\zeta',c]} v(z) \leq -T$ while

$$\min_{z\in[\zeta',c]} -\frac{K}{\delta}(\zeta'-z)^2 = -\frac{K}{\delta}(\zeta'-c)^2 = -\frac{T}{2},$$

so the inequality holds also in this case.

(4) On $[c,d]$ we have $v(z) \leq -T$, $f'(z) > M/T$ and $v(f(t)) \geq -M$, whence the desired relation.

(5) On $[d,a]$ we have $v(z) < 0$, $f'(z) > 0$ and $v(f(z)) > 0$, whence the conclusion.

This completes the proof.

$\boxed{2.2.1}$

2.3 Approximation near a curve

In this section we will carry out in full detail the argument sketched at the beginning of the chapter, proving Proposition 1.2.2. Let us remark again here that not only the statement of 1.2.2 is required in Section 3.3, but also the geometric construction which proves it, as explained in the rest of this section.

Proof of 1.2.2. Let us consider a neighbourhood W of the curve γ parametrized as in Corollary 2.1.3 by $S^1_x \times \mathbb{R}_y \times \mathbb{R}_z$. Recall that the parametrization is C^2-smooth and orientation-preserving, so a C^1-smooth positive confoliation in the coordinates gives rise to a C^1-smooth positive confoliation on W. From now on we will operate only within W, making sure that we modify $\mathcal{F}$ only on a compact subset of W. Moreover, by the remark just made, we are allowed to omit mentioning the parametrization. As announced, we will need to modify the plane field twice, so we handle the approximation parameter with care. First of all we choose some distance on the set of plane fields which induces the C^0 topology.

Fix $\varepsilon > 0$ and select $\tilde{\varepsilon} > 0$ such that if ξ, ξ' are plane distributions on N which coincide outside $S^1_x \times [-1,1]_y \times [-1,1]_z$ and there are defined by forms

$$\mathrm{d}z + X\,\mathrm{d}x + Y\,\mathrm{d}y \qquad \mathrm{d}z + X'\,\mathrm{d}x + Y'\,\mathrm{d}y$$

with $\max\{|X - X'| + |Y - Y'|\} < 11 \cdot \tilde{\varepsilon}$ then ξ and ξ' are ε-close on N. (The factor 11 is of course motivated by the proof. We also assume that $\tilde{\varepsilon} < 1$.)

We also fix $1 \geq p > 0$ and look for confoliations which coincide with $\mathcal{F}$ outside $S^1_x \times [-p,p]_y \times [-p,p]_z$. This will allow to satisfy property 2 of the statement.

Recall that $\mathcal{F}$ is defined by $\alpha(x,y,z) = \mathrm{d}z + a(x,z)\,\mathrm{d}x$ and $a(x,0) = 0$. So there exists $r > 0$ such that $|a(x,z)| < \tilde{\varepsilon}$ for $|z| < r$. We also assume $r < \min\{1/2, \tilde{\varepsilon} \cdot p\}$. Now Lemma 2.2.1 applies to $\{a(x,\cdot) :\ x \in S^1\}$. Therefore there exists a diffeomorphism f of $\mathbb{R}_z$ which is the identity outside an interval $[-\rho_1, \rho_2]$ with $0 < \rho_1, \rho_2 < r/2$ (here we are using property (iii) of Lemma 2.2.1) and $f'(z) \cdot a(x,z) < a(x, f(z))$ on $(-\rho_1, \rho_2)$. Now consider a C^1 non-decreasing real function u such that $u \equiv 0$ on a neighbourhood of $(-\infty, -p/2]$ and $u \equiv 1$ on a neighbourhood of $[-p/4, +\infty)$, as described in Fig. 2.4. We also define $U = S^1_x \times (-p/4, p/2)_y \times (-\rho_1, \rho_2)_z$ (this will be the U of the statement) and $T = S^1_x \times \mathbb{R}_y \times \mathbb{R}_z \setminus U$. We consider the following diffeomorphism F of T onto itself:

$$F(x,y,z) = \begin{cases} (x, y, f_y(z)) & \text{if } y \in [-p/2, -p/4] \\ (x,y,z) & \text{otherwise,} \end{cases}$$

where $f_y(z) = u(y) \cdot f(z) + (1 - u(y)) \cdot z$. Note that T is not open in $S^1 \times \mathbb{R}^2$,

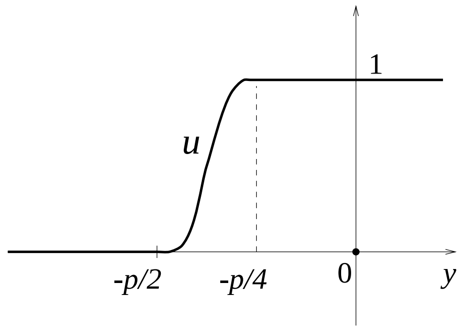

Figure 2.4: The function u

but one easily sees that F extends to a diffeomorphism of $S^1 \times \mathbb{R}^2$ onto itself, so it makes sense to call F a diffeomorphism.

Consider now on T the foliation $F_*(\mathcal{F}|_T)$, defined by the form $F_*(\alpha) = (F^{-1})^*(\alpha)$. Near $y = -p/4$ we will have

$$\begin{aligned}(F^{-1})^*(\alpha)(x,y,z) &= (f^{-1})'(z)\,\mathrm{d}z + a(x, f^{-1}(z))\,\mathrm{d}x \\ &= \frac{\mathrm{d}z}{f'(f^{-1}(z))} + a(x, f^{-1}(z))\,\mathrm{d}z.\end{aligned}$$

Moreover the coefficient of $\mathrm{d}z$ in $F_*(\alpha)$ is never 0, so $F_*(\mathcal{F}|_T)$ can be defined by a form

$$\beta(x,y,z) = \mathrm{d}z + b(x,y,z)\,\mathrm{d}x + c(x,y,z)\,\mathrm{d}y$$

with the following relations near $y = -p/4$:

$$\begin{cases} c(x,y,z) = 0 \\ b(x,y,z) = f'(f^{-1}(z)) \cdot a(x, f^{-1}(z)). \end{cases}$$

Now we extend c to be 0 on U, getting of course a C^1 function (one has to check that not only near $y = -p/4$, but also near the rest of ∂U the function c is null).

We are now going to extend also b. Recall first that for $z \in (-\rho_1, \rho_2)$ by assumption we have for all x

$$(*) \qquad f'(f^{-1}(z)) \cdot a(x, f^{-1}(z)) < a(x,z);$$

moreover the left-hand side equals $b(x, -p/4, z)$ and the right-hand side equals $b(x, p/2, z)$. Let us now consider a real C^1 function θ with

$$\begin{aligned} &\theta \equiv 1 \quad \text{on} \quad (-\infty, -p/4], \\ &\theta \equiv 0 \quad \text{on} \quad [p/2, +\infty, -p/4), \\ &\theta'(y) < 0 \quad \text{for all } y \in (-p/4, p/2). \end{aligned}$$

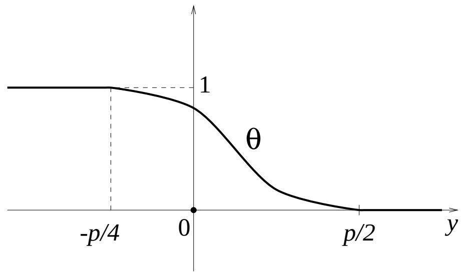

Figure 2.5: The function θ

See also Fig. 2.5. We extend b to U as

$$b(x, y, z) = \theta(y) \cdot b(x, -p/4, z) + (1 - \theta(y)) \cdot b(x, p/2, z).$$

It is now immediate to check that such a b is indeed globally C^1 on $S^1 \times \mathbb{R}^2$. Moreover by equation $(*)$ and the remarks following it we have that $\partial b/\partial y$ is positive on U.

We can now define a global form

$$\beta(x, y, z) = \mathrm{d}z + b(x, y, z)\,\mathrm{d}x + c(x, y, z)\,\mathrm{d}y,$$

and it follows from the above choices that β is a positive confoliation whose contact zone is exactly U. However β need not be close to $\mathcal{F}$. Now we consider the open set $V = S^1_x \times (-p, p)_y \times (-r, r)_z$ (this will be the V of the statement); we will define in the rest of the proof a diffeomorphism ϕ of M onto itself such that $\phi|_{M\setminus V} = \mathrm{id}$, $\phi(\gamma) = \gamma$ and $\phi^*(\beta)$ is ε-close to $\mathcal{F}$. Let us remark soon that it is then sufficient to define $Z = \phi^{-1}(U)$, $\xi = \phi^*(\beta)$ and $\psi = \phi^{-1} \circ F$ to get all the conclusions of the statement.

To define ϕ, let us denote by m the maximum of $|b| + |c|$ on $\overline{V}$, and let us choose constants τ_1, τ_2 such that

$$\frac{\rho_1}{\tau_1} = \frac{\rho_2}{\tau_2} > \frac{m}{\tilde{\varepsilon}}.$$

We can now choose a C^1 diffeomorphism k of $[-r, r]_z$ onto itself such that k is the identity near $z = \pm r$, it maps $[-\tau_1, \tau_2]$ linearly onto $[-\rho_1, \rho_2]$ and $k' > 1/2$ everywhere (this is possible because $\rho_1, \rho_2 < r/2$). See also Fig. 2.6. Let us also consider a C^1 function g such that $g(y) = 0$ for $|y| \geq p$, and $g(y) = 1$ for $|y| \leq p/2$ as in Fig. 2.7. Of course we can also assume that $|g'(y)| < 4/p$ for all y. Now we define:

$$\begin{aligned} k_y(z) &= g(y) \cdot k(z) + (1 - g(y)) \cdot z \\ \phi(x, y, z) &= (x, y, k_y(z)). \end{aligned}$$

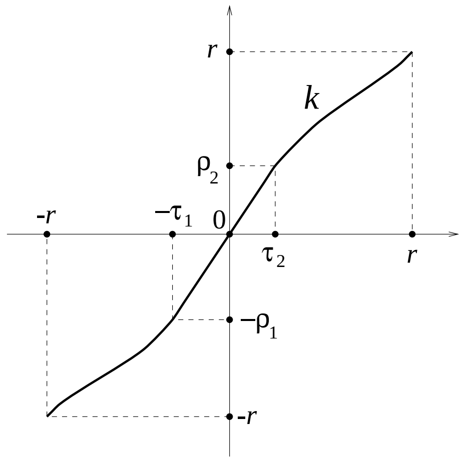

Figure 2.6: The function k

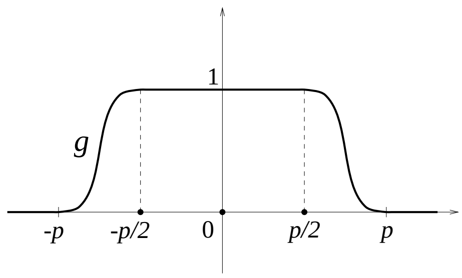

Figure 2.7: The function g

Let us note immediately that

$$\left|\frac{\partial k_y}{\partial y}(z)\right| < \frac{4}{p} \cdot r < 4\tilde{\varepsilon}$$

because $|k(z) - z| < r$. Now:

$$(\phi^*)(x, y, z) = \frac{\partial k_y}{\partial y}(z)\,\mathrm{d}y + k'_y(z)\,\mathrm{d}z + b(x, y, k_y(z))\,\mathrm{d}x + c(x, y, k_y(z))\,\mathrm{d}y$$

and hence we can define the confoliation also by means of the form

$$\mathrm{d}z + \frac{b(x, y, k_y(z))}{k'_y(z)}\,\mathrm{d}x + \frac{(\partial k_y/\partial y)(z) + c(x, y, k_y(z))}{k'_y(z)}\,\mathrm{d}y.$$

Now we have

$$\left|\frac{(\partial k_y/\partial y)}{k'_y}\right| < 8\tilde{\varepsilon}$$

everywhere. Moreover for $(y, z) \in [-p/2, p/2] \times [-\tau_1, \tau_2]$ we have

$$k'_y(z) = k'(z) = \frac{\rho_1}{\tau_1} = \frac{\rho_2}{\tau_2} > \frac{m}{\tilde{\varepsilon}}$$

and hence also

$$\left|\frac{b}{k'_y}\right| + \left|\frac{c}{k'_y}\right| < \tilde{\varepsilon}.$$

For all other values of (y, z) we have $c = 0$ and $b = a$; moreover $k'_y > 1/2$ and $|a| < \tilde{\varepsilon}$, so

$$\left|\frac{b}{k'_y}\right| + \left|\frac{c}{k'_y}\right| < 2\tilde{\varepsilon}.$$

In particular the last inequality holds everywhere, and $\phi^*(\beta)$ can be defined by a form $\mathrm{d}z + \tilde{b}\,\mathrm{d}x + \tilde{c}\,\mathrm{d}y$ such that $|\tilde{b}| + |\tilde{c}| < 10\tilde{\varepsilon}$ on $S^1_x \times [-p, p]_y \times [-r, r]_z$ and the form coincides with α outside this set. By the initial choices of $\tilde{\varepsilon}$ and r we conclude that $\phi^*(\beta)$ is ε-close to $\mathcal{F}$.

$\boxed{1.2.2}$

Chapter 3

Contact-to-be confoliations

In this chapter we will prove that under the assumption of C^2 regularity a foliation (with the exception of the product foliation on $S^2 \times S^1$) can be approximated by a contact-to-be confoliation. The argument is roughly as follows: using general facts about foliations (most of which require C^2 regularity and are stated in Section 1.2) one first reduces to the case where there exist finitely many curves such that on one hand the local approximation of Chapter 2 can be performed near each of them, and on the other hand every leaf of the foliation meets any open neighbourhood of the union of the curves. Once this setting has been achieved, the conclusion is a direct geometric argument (actually, a continuation of the proof of Proposition 1.2.2).

3.1 Technical preliminaries on closed leaves

In this section we will give elementary proofs of some facts which we will need for the proof of Proposition 1.2.3. Even if the statements could be given more in general, for simplicity we will restrict to cooriented codimension 1 foliations $\mathcal{F}$ of oriented closed 3-manifolds M, as in the rest of these notes. Let us remark that for a leaf of such a foliation it is equivalent to be *closed* (or, equivalently, compact) as a subset of M and to be *closed* as an abstract surface. Our first task is to examine the local aspect of a foliation near such a leaf.

Lemma 3.1.1. *If $k \geq 2$, $\mathcal{F}$ is C^k in the sense of foliations and $F \in \mathcal{F}$ is a closed leaf then there exists a C^k parametrization $\phi : F \times \mathbb{R} \to U$ of a tubular neighbourhood U of F in M such that if a closed leaf of $\mathcal{F}$ intersects U then it is contained in U and it has the form $\phi(F \times \{t\})$.*

Proof of 3.1.1. For the sake of simplicity we will only prove a one-sided version of this fact, where U is a positive semi-neighbourhood and $\phi : F \times [0, \infty) \to U$; the two-sided version is established by the same technique.

So, let us assume that any positive semi-neighbourhood of F intersects some other closed leaf. We will then provide a parametrization $\phi : F \times [0, 1] \to \overline{U}$ of a closed positive semi-neighbourhood of F such that $\phi(F \times \{1\})$ also is a leaf of $\mathcal{F}$, and all other closed leaves in U have the form $\phi(F \times \{t\})$. This is of course sufficient to conclude. We first concentrate on finding a ϕ with the first property (i.e. $\phi(F \times \{1\})$ leaf). We will need a result in one dimension less.

Sublemma 3.1.2. *Let v be a nowhere-vanishing vector field on $S^1 \times [0, 1]$ transversal to all the segments $\{x\} \times [0, 1]$, and let $\mathcal{G}$ be the set of orbits of v. (Note that no assumption is made on $v|_{S^1 \times \{1\}}$, so some elements of $\mathcal{G}$ could be points.) Assume that $S^1 \times \{0\}$ belongs to $\mathcal{G}$. Let $G \in \mathcal{G}$ be closed and assume that its projection on S^1 is surjective. Then this projection is also injective.*

Proof of 3.1.2. Let $(x_0, t_0) \in G$ with t_0 minimal. If $t_0 = 1$ then of course the projection is injective, so we assume $t_0 < 1$. Let $x_0 = \exp(i\, s_0)$. Consider the path $\gamma : s \mapsto \exp(i\, s)$ in S^1. By the transversality assumption, starting at (x_0, t_0), we can find a lifting $\tilde{\gamma} = (\gamma, \tau)$ of γ defined on some interval $[s_1, s_2]$ which contains s_0. Since $t_0 < 1$ we have $s_1 < s_0 < s_2$, and the property that G surjects onto S^1 implies that $s_2 - s_1 \geq 2\pi$. Choose $\overline{s}$ such that $s_1 \leq \overline{s} \leq s_0 \leq \overline{s} + 2\pi \leq s_2$ and let $t_1 = \tau(\overline{s})$ and $t_2 = \tau(\overline{s} + 2\pi)$. If $t_1 = t_2$ then $G = \tilde{\gamma}([\overline{s}, \overline{s} + 2\pi])$ and the conclusion easily follows. If $t_1 < t_2$ then $\tilde{\gamma}$ extends to a map defined on $(-\infty, \overline{s} + 2\pi]$ and $\tau(s_0 - 2\pi) < \tau(s_0) = t_0$, which contradicts the minimality of t_0. Similarly the inequality $t_1 > t_2$ leads to a contradiction, and the sublemma is proved.

$\boxed{3.1.2}$

Now let us go back to the proof of Lemma 3.1.1 and select any parametrization $\phi : F \times [0, 1] \to \overline{U}$ of a closed positive semi-neighbourhood, and let us consider the induced partition $\mathcal{F}' = \phi^*(\mathcal{F})$. Note that $\mathcal{F}'$ is a foliation on $S^1 \times [0, 1)$, but there is no control on the local behaviour on $S^1 \times \{1\}$. However if $\mathcal{F}$ is defined by a form α then $\mathcal{F}'$ is defined (in an obvious way which we will not spell out) by $\phi^*(\alpha)$. Since $F \times \{0\}$ is a leaf of $\mathcal{F}'$, up to restricting ϕ and rescaling we can assume that $\mathcal{F}'$ (or, more precisely, the form which defines it) is transversal to all the segments $\{x\} \times [0, 1]$. This implies that given a path $\gamma : [0, a] \to F$ and $0 \leq t_0 < 1$, for some $0 < \tilde{a} \leq a$ there exists a lifting $\tilde{\gamma} : [0, \tilde{a}] \to F \times [0, 1]$ tangent to $\mathcal{F}'$ and starting at $(\gamma(0), t_0)$.

Now we fix on F a metric such that $\mathrm{diam}(F) < 1$. We claim that for some $\varepsilon > 0$ the following property holds:

(P) If $\gamma : [0,1] \to F$ has length at most 1 then for $0 \le t_0 \le \varepsilon$ the lifting of γ to a path tangen to $\mathcal{F}'$ starting from the point $(\gamma(0), t_0)$ is globally defined on $[0,1]$ and takes values in $F \times [0, 1/2)$.

To see this, assume that $\mathcal{F}'$ is defined by the form $\mathrm{d}t + \omega_t$. Since $\omega_0 = 0$ and $\partial\omega_t/\partial t$ is continuous, and hence bounded, on $F \times [0,1]$, there exists δ such that $\|\omega_t\| \le \delta \cdot t$ on $F \times [0,1]$ (here $\|\cdot\|$ is defined using the metric of F, and we are using the fact that $k \ge 2$, so the form which defines $\mathcal{F}$ is at least C^1). Now let $0 < \varepsilon < 1/2 \cdot \exp(-\delta)$. Take γ, t_0 as in property (P) and note that as long as $\tilde{\gamma}(s) = (\gamma(s), \tau(s))$ is defined we have

$$|\tau'(s)| = |-\omega_{\tau(s)}(\gamma'(s))| \le \|\omega_{\tau(s)}\| \cdot \|\gamma'(s)\| \le \|\omega_{\tau(s)}\| \le \delta \cdot \tau(s)$$

(we have assumed, without loss of generality, that γ is parametrized by arc length). Therefore

$$\tau(s) \le t_0 \exp(\delta s).$$

Since $t_0 \exp(\delta s) < \varepsilon \exp(\delta) < 1/2$ the estimate holds for all s and allows to conclude. Property (P) is established.

Now let F' be a closed leaf of $\mathcal{F}$ which intersects $\phi(F \times [0, \varepsilon))$ in a point $\phi(x_0, t_0)$. Let us denote by F'' the connected component of $F' \cap \phi(F \times [0,\varepsilon))$ which contains $\phi(x_0, t_0)$. For $(x_1, t_1) \in \phi^{-1}(F'')$ consider a smooth immersed path $\tilde{\gamma}_1 = (\gamma_1, \tau_1)$ which joins (x_0, t_0) to (x_1, t_1) in $\phi^{-1}(F'')$. Since $\operatorname{diam}(F) < 1$ we can find a path γ_0 in F such that γ_0 has length less than 1 and $\gamma_0 \cdot \gamma_1^{-1}$ is an immersed circle (note that γ_1 necessarily is an immersion, i.e. $\gamma_1' \neq 0$, because $\mathcal{F}'$ is transversal to the segments $\{x\} \times [0,1]$). Consider the immersion $\psi : S^1 \times [0,1] \to F \times [0,1]$ given by $(\gamma_0 \cdot \gamma_1^{-1}, \mathrm{id})$. Then $\mathcal{G} = \psi^*(\mathcal{F}')$ and $G = \psi^{-1}(\phi^{-1}(F''))$ satisfy the hypotheses of Sublemma 3.1.2, because by (P) the path γ_0 can be lifted to some (γ_0, τ_0) defined on $[0,1]$. Then Sublemma 3.1.2 implies that $\tau_0(1) = \tau_1(1)$; but $\tau_1(1) = t_1$ by assumption and $\tau_0(1) < 1/2$ by (P). We have therefore proved that $\phi^{-1}(F'') \subset F \times [0, 1/2)$. This easily implies that actually $F' \subset \phi(F \times [0, 1/2))$.

In particular the projection of $\phi^{-1}(F')$ to F is a local homeomorphism of a compact Hausdorff space to a connected one, so it is a covering map. Moreover Sublemma 3.1.2 implies that the projection is a bijection, so F' is parametrized as

$$F \ni x \mapsto \phi(x, \tau(x)) \in F'$$

where of course τ is as regular as $\mathcal{F}$. Then we define

$$f : F \times [0,1] \ni (x,t) \mapsto (x, t \cdot \tau(x)) \in F \times [0,1]$$

and we replace ϕ by $\phi \circ f$. (Note that in the two-sided version of the lemma this is the only point where the proof must actually be slightly changed: one only needs to modify f so that it is the identity near $F \times \{0\}$.)

Having obtained a C^k parametrization $\phi : F \times [0,1] \to \overline{U}$ such that $\phi(F \times \{1\})$ is a leaf of $\mathcal{F}$, we can now concentrate on $F \times [0,1]$ with a C^k foliation transversal to the segments $\{x\} \times [0,1]$ in which $F \times \{0\}$ and $F \times \{1\}$ are leaves. We will denote such a foliation by $\mathcal{F}$ again. Let $U_1, \ldots, U_n$ be an open covering of F such that:

1. Each U_i is diffeomorphic to an open ball;
2. Each $U_i \cap U_j$ is either empty or diffeomorphic to a ball;
3. For $j = 2, \ldots, n-1$ there exists an index $i(j) \in \{1, \ldots, j-1\}$ such that $U_{i(j)} \cap U_j \neq \emptyset$.

Property 1 implies that we can choose a C^k diffeomorphism $\alpha_i : \pi^{-1}(U_i) \to U_i \times [0,1]$ (where $\pi : F \times [0,1] \to F$ is the projection) such that $(\alpha_i)_*(\mathcal{F})$ is the horizontal foliation. Property 2 implies that there exists a C^k diffeomorphism $\beta_{i,j}$ of $[0,1]$ such that

$$(\alpha_i \circ \alpha_j^{-1})(x,t) = (x, \beta_{i,j}(t)) \qquad \text{for } x \in U_i \cap U_j.$$

By property 3 we have $U_1 \cap U_2 \neq \emptyset$. Let us define $\tilde{\alpha}_2 = (\mathrm{id}, \beta_{1,2}) \circ \alpha_2$; then

$$\alpha_1 \circ \tilde{\alpha}_2^{-1} = \alpha_1 \circ \alpha_2^{-1} \circ (\mathrm{id}, \beta_{1,2}^{-1}) = (\mathrm{id}, \mathrm{id}).$$

So up to replacing α_2 with $\tilde{\alpha}_2$ we can assume that $\beta_{1,2} = \mathrm{id}$. Recursively we can assume that $\beta_{i(j),j} = \mathrm{id}$ for $j = 2, \ldots, n$. Let us note that by Sublemma 3.1.2 the projection of every closed leaf onto F is a bijection. Now fix $\overline{x} \in U_1$. We claim that if $\overline{t}$ is such that the leaf through $(\overline{x}, \overline{t})$ is compact then $\alpha_i(\tilde{x}, \tilde{t}) = (\tilde{x}, \overline{t})$ for all $(\tilde{x}, \tilde{t})$ in the leaf. For $i = 1$ this follows from the injectivity of the projection onto F, and the condition that $\beta_{i(j),j} = \mathrm{id}$ allows to conclude quite easily.

Let us now choose a partition of unity $u_1, \ldots, u_n$ relative to $U_1, \ldots, U_n$. We define

$$f(x,t) = \left(x, \sum_{i=1}^{n} u_i(x) \tau_i(x,t) \right)$$

where τ_i is such that $\alpha_i(x,t) = (x, \tau_i(x,t))$. Of course f is C^k-regular. Moreover for fixed x the second coordinate is a convex combinations of increasing diffeomorphisms of $[0,1]$, so it also is a diffeomorphism of $[0,1]$. This implies that f is invertible and its differential is upper triangular and invertible, so f is a C^k diffeomorphism. Moreover it follows from what we have proved above that if F' is a compact leaf through $(\overline{x}, \overline{t})$ then $f(F') = F \times \{\overline{t}\}$, whence the conclusion.

$\boxed{3.1.1}$

The next result proves that up to approximation one can stretch the holonomy of a closed leaf of a foliation to be trivial up to any order. We will state this result with an arbitrary C^k regularity, but we will give a complete proof only for the C^2 case, which is the one we are actually interested in. The general case requires some more computations, and we will give a hint toward it.

Lemma 3.1.3. *Let $\mathcal{F}$ be a foliation on M, assume $\mathcal{F}$ is C^k in the sense of foliations and let $F \in \mathcal{F}$ be a compact leaf. Then arbitrarily C^0-close to $\mathcal{F}$ there exists a foliation $\mathcal{F}'$ such that:*

1. *$F \in \mathcal{F}'$;*
2. *$\mathcal{F}'$ differs from $\mathcal{F}$ only on an arbitrarily small neighbourhood of F;*
3. *Each curve on F has holonomy trivial up to order k;*
4. *$\mathcal{F}'|_{M\setminus F}$ is C^k diffeomorphic to $\mathcal{F}|_{M\setminus F}$ (in particular, $\mathcal{F}'$ is homeomorphic to $\mathcal{F}$).*

Proof of 3.1.3. As announced we give a complete proof only for the case $k = 2$. It is of course sufficient to refer to the case where the manifold is $F_x \times \mathbb{R}_z$ and the foliation is defined by a form $\alpha(z) = \mathrm{d}z + \eta(z)$ where $\eta(z)$ is a 1-form on F and $\eta(0) = 0$ (so that $F \times \{0\}$ is a leaf). We will first assume that α itself is C^2 (we know that in general it is only C^1) and find a new foliation also defined by a C^2 form. The case where the foliation is more regular than the form which defines it is proved essentially by the same argument, as we will see at the end of the proof. Since we will need to change the foliation twice we handle the approximation parameter with some care. So we fix it and call it $\varepsilon > 0$.

We first choose a norm for 1-forms on F and we select $\tilde{\varepsilon} > 0$ such that if two foliations on $F \times \mathbb{R}$ are defined by forms $\mathrm{d}z + \theta_1$ and $\mathrm{d}z + \theta_2$ such that $\theta_1(z) = \theta_2(z)$ for $|z| \geq 1$ and $\|\theta_1(z) - \theta_2(z)\| \leq 3\tilde{\varepsilon}$ everywhere then the foliations are ε-close.

Now choose some small $r > 0$ such that $\|\eta(z)\| < \tilde{\varepsilon}$ for $|z| \leq 2r$. Since the final foliation will differ from $\mathcal{F}$ only for $|z| < 2r$, the arbitrariness of r allows to achieve property 2. We choose $r < 1/2$.

Let us now consider a homeomorphism f of $\mathbb{R}$ of class C^∞ such that:

1. $f(0) = 0$;
2. $f(z) = \mathrm{sgn}(z) \cdot \exp(-1/z^2)$ for small $z \neq 0$, where $\mathrm{sgn}(z) = z/|z|$;
3. $f(z) = z$ for $|z| \geq r$.

Note that the inverse of f is only continuous. (Recall that we are referring here only to the case $k = 2$; for bigger k one needs to choose the function f with a higher vanishing "order"; for $k = \infty$, and hence for all k, one could for instance choose $f(z) = \mathrm{sgn}(z) \cdot \exp(-\exp(1/z^2))$; computations are more complicated but the proof is otherwise identical.) We define a homeomorphism ϕ of $F \times \mathbb{R}$ as $\phi(x, z) = (x, f(z))$, and the foliation $\mathcal{G} = \phi^*(\mathcal{F})$ which is a priori only C^0 regular. Now note that ϕ is C^∞ and $\phi^*(\alpha)(x, z) = f'(z)\,\mathrm{d}z + \eta(f(z))$, so $\mathcal{G}$ can be defined by the form

$$\beta(z) = \mathrm{d}z + \theta(z), \qquad \theta(z) = \begin{cases} 0 & \text{if } z = 0 \\ \dfrac{\eta(z)}{f'(z)} & \text{otherwise.} \end{cases}$$

It is quite obvious that $F \times \{0\} \in \mathcal{G}$ and that $\mathcal{G} \setminus (F \times \{0\})$ is C^2 regular and diffeomorphic to $\mathcal{F} \setminus (F \times \{0\})$. So to prove that $\mathcal{G}$ is a C^2 foliation it is sufficient to prove that all partial derivatives up to order 2 of θ extend continuously to $F \times \{0\}$. We will actually show that all these derivatives vanish, which also implies quite easily that the holonomy of $F \times \{0\}$ in $\mathcal{G}$ is zero up to order 2.

In the computations which follow we will always omit to express explicitly the dependence on the point x of F. Whenever h is a function of z, by writing that a form ω which depends on z is $\mathrm{o}(h(z))$ or $\mathrm{O}(h(z))$ we will mean respectively that

$$\lim_{z \to 0} \frac{\|\omega(z)\|}{h(z)} = 0 \qquad \text{or} \qquad \|\omega(z)\| \leq C \cdot h(z)$$

uniformly for $x \in F$. All the estimates we will make hold because of the compactness assumption on F. By simplicity in all the computations which follow we will refer to $z > 0$; for $z < 0$ one only has to multiply the formulae by -1 in appropriate places. For small $z > 0$ we have

$$\theta(z) = \frac{1}{2} z^3 \mathrm{e}^{1/z^2} \eta(\mathrm{e}^{-1/z^2}).$$

Since $\eta(0) = 0$ and η is C^1 we have $\eta(z) = \mathrm{O}(z)$ so

$$\theta(z) = \frac{1}{2} z^3 \mathrm{e}^{1/z^2} \mathrm{O}(\mathrm{e}^{-1/z^2}) = \frac{1}{2} z^3 \mathrm{O}(1) = \mathrm{o}(1).$$

In a similar way one shows that the differential of θ on F vanishes for $z = 0$. For the first derivative in direction z we have

$$\frac{\partial \theta}{\partial z}(z) = \left(\frac{3}{2} z^2 - 1\right) \mathrm{e}^{1/z^2} \eta(\mathrm{e}^{-1/z^2}) + \frac{\partial \eta}{\partial z}(\mathrm{e}^{-1/z^2}).$$

Since η is C^1 we have

$$\begin{aligned} \frac{\partial \eta}{\partial z}(z) &= \frac{\partial \eta}{\partial z}(0) + \mathrm{o}(1) \\ \eta(z) &= z \frac{\partial \eta}{\partial z}(0) + \mathrm{o}(z) \end{aligned}$$

and therefore

$$\begin{aligned}\frac{\partial\theta}{\partial z}(z) &= (\mathrm{o}(1)-1)\mathrm{e}^{1/z^2}\left(\mathrm{e}^{-1/z^2}\frac{\partial\eta}{\partial z}(0)+\mathrm{o}(\mathrm{e}^{-1/z^2})\right)+\frac{\partial\eta}{\partial z}(0)+\mathrm{o}(1)\\ &= (\mathrm{o}(1)-1)\left(\frac{\partial\eta}{\partial z}(0)+\mathrm{o}(1)\right)+\frac{\partial\eta}{\partial z}(0)+\mathrm{o}(1)=\mathrm{o}(1).\end{aligned}$$

So also the first derivative in the z direction vanishes, and the same argument works for the mixed second derivatives. We are left to deal with the second derivative in the z direction. Since η is C^2 we have

$$\frac{\partial^2\eta}{\partial z^2}(z)=\mathrm{O}(1);$$

moreover

$$\begin{aligned}\frac{\partial^2\theta}{\partial z^2}(z) &= \left(3z-\frac{3}{z}+\frac{2}{z^3}\right)\mathrm{e}^{1/z^2}\eta\left(\mathrm{e}^{-1/z^2}\right)\\ &\quad+\left(\frac{3}{z}-\frac{2}{z^3}\right)\frac{\partial\eta}{\partial z}\left(\mathrm{e}^{-1/z^2}\right)+\frac{2}{z^3}\mathrm{e}^{-1/z^2}\frac{\partial^2\eta}{z^2}\left(\mathrm{e}^{-1/z^2}\right)\end{aligned}$$

and hence (taking into account also estimates already used above)

$$\begin{aligned}\frac{\partial^2\theta}{\partial z^2}(z) &= \left(3z-\frac{3}{z}+\frac{2}{z^3}\right)\mathrm{e}^{1/z^2}\left(\frac{\partial\eta}{\partial z}(0)\mathrm{e}^{-1/z^2}+\mathrm{O}\left(\left(\mathrm{e}^{-1/z^2}\right)^2\right)\right)\\ &\quad+\left(\frac{3}{z}-\frac{2}{z^3}\right)\left(\frac{\partial\eta}{\partial z}(0)+\mathrm{O}\left(\mathrm{e}^{-1/z^2}\right)\right)+\frac{2}{z^3}\mathrm{e}^{-1/z^2}\mathrm{O}(1)\\ &= \left(\mathrm{o}(1)-\left(\frac{3}{z}-\frac{2}{z^3}\right)\right)\left(\frac{\partial\eta}{\partial z}(0)+\mathrm{O}\left(\mathrm{e}^{-1/z^2}\right)\right)\\ &\quad+\left(\frac{3}{z}-\frac{2}{z^3}\right)\left(\frac{\partial\eta}{\partial z}(0)+\mathrm{O}\left(\mathrm{e}^{-1/z^2}\right)\right)+\mathrm{o}(1)\\ &= \mathrm{o}(1).\end{aligned}$$

This completes the first part of the proof, namely we have found a foliation $\mathcal{G}$ which only differs from $\mathcal{F}$ on $F\times[-r,r]$ and satisfies properties 1, 3 and 4 of the statement. To conclude it is now sufficient to find a diffeomorphism ψ of $F\times\mathbb{R}$ supported in $F\times[-2r,2r]$ such that $\psi^*(\mathcal{G})$ is ε-close to $\mathcal{F}$. For the choice of r refer to the beginning of the proof.

Recall that $\mathcal{G}$ is defined by a form $\beta=\mathrm{d}z+\theta(z)$, and define

$$K=\max_{F\times[-1,1]}\|\theta\|.$$

Since $\theta(0)=0$ and θ is continuous we can find $\rho>0$ such that $\|\theta(z)\|<\tilde{\varepsilon}$ for $|z|\le\rho$. We also assume $\rho<r$. Let us define

$$\tau=\min\{\tilde{\varepsilon}(r-\rho)/K,r/2,\rho,r-\rho\}$$

and consider a diffeomorphism g of $\mathbb{R}$ such that:

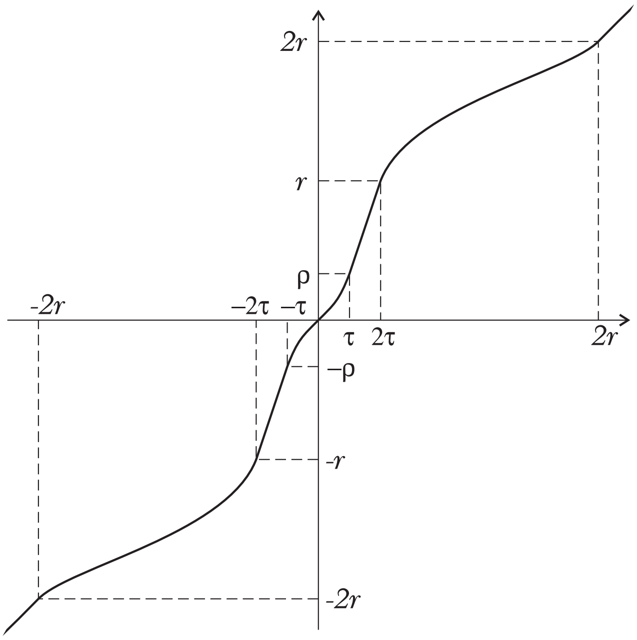

Figure 3.1: The function g

1. g is the identity outside $[-2r, 2r]$;

2. g maps $[2\tau, 2r]$ to $[r, 2r]$ and $[-2r, -2\tau]$ to $[-2r, -r]$, and $g'(z) \geq 1/2$ on these intervals;

3. g maps $[\tau, 2\tau]$ affinely to $[\rho, r]$ and $[-2\tau, \tau]$ affinely to $[-r, -\rho]$;

4. g maps $[-\tau, \tau]$ to $[-\rho, \rho]$ and $g'(z) > 1$ for $|z| \leq \tau$.

(Existence of such a g follows quite easily from our choice of τ; see also Fig. 3.1.)

Now we define $\psi(x, z) = (x, g(z))$ and $\mathcal{F}' = \psi^*(\mathcal{G})$. Since

$$(\psi^*(\beta))(z) = g'(z)\,\mathrm{d}z + \theta(g(z))$$

the foliation $\mathcal{F}$ can also be defined by the form

$$\mathrm{d}z + \vartheta(z), \qquad \vartheta(z) = \frac{\theta(g(z))}{g'(z)}.$$

Now:

1. For $|z| \geq 2r$ we have $\vartheta(z) = \eta(z)$;

2. For $2\tau \leq |z| \leq r$ we have $r \leq |g(z)| \leq 2r$ so (by the above construction) $\theta(g(z)) = \eta(g(z))$ and (by the choice of r at the beginning of the proof) $\|\eta(g(z))\| \leq \tilde{\varepsilon}$; moreover $g'(z) \geq 1/2$, hence $\|\vartheta(z)\| \leq 2\tilde{\varepsilon}$;

3. For $\tau \leq |z| \leq 2\tau$ we have $g'(z) = (r-\rho)/\tau = K/\tilde{\varepsilon}$ so by the definition of K we have $\|\vartheta(z)\| \leq \tilde{\varepsilon}$;

4. For $|z| \leq \tau$ we have $|g(z)| \leq \rho$ so $\|\theta(g(z))\| \leq \tilde{\varepsilon}$ and $g'(z) \geq 1$, so $\|\vartheta(z)\| \leq \tilde{\varepsilon}$.

Therefore:

$$\vartheta(z) = \eta(z) \text{ for } |z| \geq 2r, \text{ where } 2r < 1;$$
$$\|\vartheta(z)\| \leq 2\tilde{\varepsilon},\ \|\eta(z)\| \leq \tilde{\varepsilon} \text{ for } |z| \leq 2r.$$

The choice of $\tilde{\varepsilon}$ now immediately implies that $\mathcal{F}'$ is ε-close to $\mathcal{F}$, whence the conclusion.

So far we have proved that this stretching technique by means of the function f allows to establish the lemma in the case where the foliation is C^k (actually, C^2 in our proof) *in the sense of plane distributions*. Now we will deal with the case of a foliation which is C^2 in the sense of foliations. Again we can assume that the initial foliation $\mathcal{F}$ is defined by a form $\alpha(z) = \mathrm{d}z + \eta(z)$ on $F \times \mathbb{R}$, where the coordinate system is C^2 but the form α is only C^1 in the coordinates. As above we define $\mathcal{G} = \phi^*(\mathcal{F})$ with $\phi = (\mathrm{id}, f)$ for the same f. We must prove that $\mathcal{G}$ is C^2 in the sense of foliations and that the holonomy of $F \times \{0\}$ in $\mathcal{G}$ is trivial up to order 2. The argument which allows to achieve C^0-closeness is the same as above, and we will not repeat it.

The condition that $\mathcal{F}$ be C^2 in the sense of foliations can be expressed as follows: let $\mathbb{R}^2_{(x,y)}$ be a local coordinate system on a disc D of F. Define a function

$$u : \mathbb{R}^2_{(x,y)} \times \mathbb{R}_z \to \mathbb{R}_z$$

with the property that $(x, y, u(x, y, z))$ and $(0, 0, z)$ belong to the same leaf of the foliation of $D \times \mathbb{R}$ obtained by restricting $\mathcal{F}$. Then the condition is that u should be a C^2 function.

Now we can define the corresponding function w for the foliation $\mathcal{G}$. Since the leaf of $\mathcal{G}$ through $(0, 0, z)$ is the preimage under ϕ of the leaf of $\mathcal{F}$ through $(0, 0, f(z)) = \phi(0, 0, z)$, we deduce that

$$w(x, y, z) = f^{-1}(u(x, y, f(z))).$$

So we have to prove that w is a C^2 function. Actually we will also prove that

$$(*) \qquad \frac{\partial w}{\partial z}(x, y, 0) = 1, \qquad \frac{\partial^2 w}{\partial z^2}(x, y, 0) = 0 \qquad \text{for all } (x, y) \in \mathbb{R}^2;$$

this implies quite easily the conclusion that the holonomy of $F \times \{0\}$ in $\mathcal{G}$ is trivial up to order 2, because the holonomy of any loop is the composition of finitely many functions of the form $z \mapsto w(x_0, y_0, z)$.

To deal with the regularity of w we first note that on the set $z \neq 0$ the function certainly is C^2, because u is C^2 and f is a diffeomorphism. Now let us recall that w is the solution of a system of linear partial differential equations in which only the first derivatives in x and y appear, and the coefficients are C^1 by the computations carried out in the first part of the proof. This implies immediately that the function w is C^2 in the horizontal coordinates (x, y). So the point is essentially to prove that the first and second derivative of w in the direction z at (x, y, z) converge to the appropriate limit given by $(*)$ as z tends to 0. (Actually, one also has to deal with the mixed second derivatives, but their existence and continuity follows by plugging in the system of partial differential equations solved by w the continuity of the first derivative in z.)

We will carry out explicitly only the computation of the first derivative in z; the second derivative requires a very long but otherwise trivial calculation. Alternatively one could use more conceptual approaches based on differential equations, but we will not mention them.

Now, for the computation, we will refer as above to $z > 0$. Using the explicit formula for f^{-1} (and omitting (x, y) from the notation) we obtain that:

$$\begin{aligned}
w(z) &= \left(-\log u\left(\mathrm{e}^{-1/z^2}\right)\right)^{-1/2} \\
\Rightarrow\ w'(z) &= -\frac{1}{2}\left(-\log u\left(\mathrm{e}^{-1/z^2}\right)\right)^{-3/2} \cdot \frac{-1}{u\left(\mathrm{e}^{-1/z^2}\right)} \cdot u'\left(\mathrm{e}^{-1/z^2}\right) \cdot \frac{2}{z^3} \cdot \mathrm{e}^{-1/z^2} \\
&= \underbrace{\frac{1}{z^3}\left(-\log u\left(\mathrm{e}^{-1/z^2}\right)\right)^{-3/2}}_{(1)} \cdot \underbrace{\frac{\mathrm{e}^{-1/z^2} \cdot u'\left(\mathrm{e}^{-1/z^2}\right)}{u\left(\mathrm{e}^{-1/z^2}\right)}}_{(2)}.
\end{aligned}$$

Now using the fact that $u(z) = u'(0) \cdot z + \mathrm{o}(z)$ it is quite easy to see that the factor (2) in the previous formula tends to 1 as z tends to 0. As for (1) one shows that

$$\begin{aligned}
\log\left(u'(0) \cdot \mathrm{e}^{-1/z^2} + \mathrm{o}\left(\mathrm{e}^{-1/z^2}\right)\right) &= \log\left(u'(0) \cdot \mathrm{e}^{-1/z^2}\right) + \mathrm{o}\left(\frac{1}{z^2}\right) \\
&= \log(u'(0)) - \frac{1}{z^2} + \mathrm{o}\left(\frac{1}{z^2}\right) = -\frac{1}{z^2} + \mathrm{o}\left(\frac{1}{z^2}\right)
\end{aligned}$$

from which it follows immediately that also (2) tends to 1 as z tends to 0. This concludes at last the proof.

$\boxed{3.1.3}$

3.2 General facts on foliations

We prove in this section two facts concerning cooriented foliations on compact oriented 3-manifolds, which could be stated and proved (with the very same techniques) for more general situations. Both results rely on the preliminary lemmas established in the previous section; recall that we have given a complete proof of Lemma 3.1.3 only in the C^2 case; as far as the proof of Theorem 0.0.1 is concerned we could confine ourselves to C^2 regularity also in this section, but for the reader's information we have preferred to give more general statements under this respect.

Lemma 3.2.1. *Any foliation $\mathcal{F}$ of M which is C^k-smooth in the sense of foliations and is different from the product foliation on $S^2 \times S^1$ can be C^0 approximated by C^k foliations in which there are only finitely many closed leaves.*

Proof of 3.2.1. We will first prove the easier fact that up to C^0-approximation we can assume that only finitely many closed leaves are not isolated. Later we will use Lemma 3.1.3 to conclude.

For the first step, let us fix an approximation parameter ε. Let us first remark that if $\mathcal{G}$ is a foliation on $F \times \mathbb{R}$ such that $F \times \{0\} \in \mathcal{G}$ then there exists $\delta > 0$ such that $\mathcal{F}|_{F\times[-\delta,\delta]}$ is ε-close to the product foliation. Moreover if t_0 is an accumulation point of the set $\{t : \ F \times \{t\} \in \mathcal{G}\}$ then $F \times \{t_0\}$ is a leaf and its holonomy is trivial up to order k.

Using this remark and Lemma 3.1.1, up to rescaling the constant δ, for every closed leaf F of $\mathcal{F}$ we can find an open tubular neighbourhood U_F of F and a parametrization $\phi_F : F \times [-1, 1] \to \overline{U_F}$ such that $\mathcal{F}$ on $\overline{U_F}$ is ε-close to $(\phi_F)_*(\mathrm{d}z)$, and all compact leaves which intersect $\overline{U_F}$ have the form $\phi_F(F \times \{t\})$. Now it is a well-known fact [HH] that the union of all compact leaves is a closed subset of M. Hence we can find $F_1, \ldots, F_p$ such that every compact leaf is contained in some U_j (where $U_j = U_{F_j}$ and $\phi_j = \phi_{F_j}$). We will now modify $\mathcal{F}$ inside the U_j's which contain which contain more than one non-isolated closed leaf. By simplicity we assume that this is the case for all j's. We start with U_1 and define:

$$\begin{aligned} A_1 &= \Big\{t \in [-1, 1] : \ \exists \text{ sequence } t_n \to t \text{ with } \phi_1(F_1 \times \{t_n\}) \in \mathcal{F} \ \forall n\Big\} \\ m_1 &= \min(A_1) \\ M_1 &= \max(A_1). \end{aligned}$$

Note that $m_1 < M_1$. Moreover the leaves $\phi_1(F_1 \times \{m_1\})$ and $\phi_1(F_1 \times \{M_1\})$ have trivial holonomy up to order k. Let us note that, by Reeb's stability theorem, F_1 is not S^2, otherwise $\mathcal{F}$ is the product foliation on $S^2 \times S^1$. Now let us select a C^k diffeomorphism h of $\mathbb{R}$ which is the identity outside $[m_1, M_1]$ and such that $h(t) \neq t$ for $t \in (m_1, M_1)$. Note that h can be chosen to be arbitrarily C^0 close to the identity. Now we fix a presentation:

$$\left\langle \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g : \prod_{i=1}^{g} [\lambda_i, \mu_i] = 1 \right\rangle$$

of $\pi_1(F_1)$ and we consider the foliation $\mathcal{F}_h$ on M defined as follows:

1. $\mathcal{F}_h$ differs from $\mathcal{F}$ only on $\phi_1(F_1 \times [m_1, M_1])$;

2. On $F_1 \times [m_1, M_1]$, via ϕ_1, the set $F_1 \times \{m_1\}$ is a leaf and the holonomy of each curve λ_i and μ_i, viewed as a curve in $F_1 \times \{m_1\}$, is h.

The condition that in $(\phi_1)^*(\mathcal{F})$ the leaves $F_1 \times \{m_1\}$ and $F_1 \times \{M_1\}$ have trivial holonomy implies that $\mathcal{F}_h$ is C^k regular. The choice of h implies that there are no closed leaves of $\mathcal{F}_h$ in $F_1 \times (m_1, M_1)$. Moreover, as h tends to the identity in the C_0 topology, $\mathcal{F}_h$ tends to the horizontal foliation on $F_1 \times [m_1, M_1]$. By assumption $\mathcal{F}$ is ε-close to this foliation, so for suitable h we have that $\mathcal{F}_h$ is ε-close to $\mathcal{F}$. Note also that all closed leaves of $\mathcal{F}_h$ on U_1 are isolated.

We can now proceed in a similar way for $U_2, \ldots, U_p$. We only need to exclude from the definition of the A_j's the compact leaves which we have already got rid of in the previous steps. Note that the various modifications take place on disjoint sets, so the final approximation parameter remains ε.

Having reduced to the case where only finitely many closed leaves are non-isolated, we show now how to conclude. To be completely formal about the approximation parameter one should be rather careful, as we have been in the proofs of Lemma 3.1.3 and Proposition 1.2.2. Here we will be slightly more sketchy.

It is of course sufficient to examine the case of a foliation $\mathcal{G}$ on $F \times \mathbb{R}$ in which 0 is the only non-isolated closed leaf, and all closed leaves have the form $F \times \{t\}$. Let us choose $t_1 \leq 0 \leq t_2$ such that $t_1 < t_2$ and $F \times \{t_1\}$, $F \times \{t_2\}$ are leaves of $\mathcal{G}$. Now we use Lemma 3.1.3 to find an approximating foliation $\mathcal{G}'$ homeomorphic to $\mathcal{G}$ in which $F \times \{t_1\}$ and $F \times \{t_2\}$ have trivial holonomy. Now, using a map h exactly as we did above, we can modify $\mathcal{G}'$ into another $\mathcal{G}''$ which coincides with $\mathcal{G}'$ outside $F \times (t_1, t_2)$, so that $\mathcal{G}''$ contains no compact leaf in $F \times (t_1, t_2)$. In particular all closed leaves of $\mathcal{G}''$ are isolated.

If t_1 and t_2 are initially chosen to be small enough then near $[t_1, t_2]$ the foliation $\mathcal{G}$ is close to the horizontal one, and so is $\mathcal{G}'$ if the approximation parameter in Lemma 3.1.3 is chosen small enough. If the function h is close enough to the identity then $\mathcal{G}''$ also is close to the horizontal foliation where it differs from $\mathcal{G}$, whence the conclusion.

$\boxed{3.2.1}$

Lemma 3.2.2. *If $\mathcal{F}$ is a foliation on M, $\mathcal{F}$ is C^k-smooth in the sense of foliations and $\mathcal{F}$ is not diffeomorphic to the product foliation on $S^2 \times S^1$ then $\mathcal{F}$ can be C^0 approximated by a C^k foliation with finitely many closed leaves each having a curve with two-sided weakly-contracting holonomy.*

Proof of 3.2.2. According to Lemma 3.2.1 we can assume that $\mathcal{F}$ has finitely many closed leaves. So it is sufficient (using Lemma 3.1.1) to concentrate on $F \times \mathbb{R}$ where F is a closed oriented surface, the foliation $\mathcal{F}$ is defined by a form $\mathrm{d}z + \eta(z)$ and the only compact leaf is $F \times \{0\}$. According to Lemma 3.1.3 we can also assume that every curve in $F \times \{0\}$ has holonomy trivial up to order k.

Let us fix now a distance on the set of plane distributions and the approximation parameter ε. We choose a norm for 1-forms on F and select $\tilde{\varepsilon} > 0$ such that if β is a 1-form on F depending on $z \in \mathbb{R}$ and $\|\beta(z)\| < 4\tilde{\varepsilon}$ for $|z| \leq 1$ then $\mathrm{d}z + \beta(z)$ is ε-close to $\mathrm{d}z$ on $F \times [-1, 1]$. Choose $r < 1/2$ such that $\|\eta(z)\| < \tilde{\varepsilon}$ for $|z| \leq 2r$, and consider a diffeomorphism $f_+ : [r, \infty) \to [0, \infty)$ such that $f_+(z) = z - r$ near $z = r$, $f_+(z) = z$ on $[2r, \infty)$ and $f'_+(z) \geq 1/3$ for all z (of course such an f_+ exists). Choose $f_-(-\infty, -r] \to (-\infty, 0]$ in a similar way and consider on $F \times ((-\infty, -r] \cup [r, \infty))$ the foliation $\mathcal{F}'$ given by $f_+^*(\mathcal{F}) \cup f_-^*(\mathcal{F})$. Our choices easily imply that $\mathcal{F}'$ (where it is defined) is ε-close to $\mathcal{F}$.

Now let us consider a presentation

$$\left\langle \gamma_1, \ldots, \gamma_{2g} : \prod_{j=1}^{g} [\gamma_{2j-1}, \gamma_{2j}] = 1 \right\rangle$$

of the fundamental group of F. Since $\mathcal{F}$ is not a product on the positive side there exists some j such that the holonomy of $\gamma_j \times \{0\}$ is non-trivial on the positive side. Up to reversing all the γ_k's (which does not affect the presentation of the fundamental group) we can assume that the holonomy of $\gamma_j \times \{0\}$ is weakly contracting on the positive side, namely that there exists a sequence $\{\zeta_n\}$ such that $\zeta_n > 0$, $\lim_n \zeta_n = 0$ and $h(\gamma_j \times \{0\})(\zeta_n) < \zeta_n$. Similarly we can find i such that $h(\gamma_i \times \{0\})$ is non-trivial on the negative side. Now, $h(\gamma_i \times \{0\})$ is necessarily either weakly contracting or weakly expanding (obvious definition) on the negative side.

Assume first that $h(\gamma_i \times \{0\})$ in $\mathcal{F}$ is weakly contracting on the negative side. In this case we select a C^k diffeomorphism u of $[-r, r]$ such that:

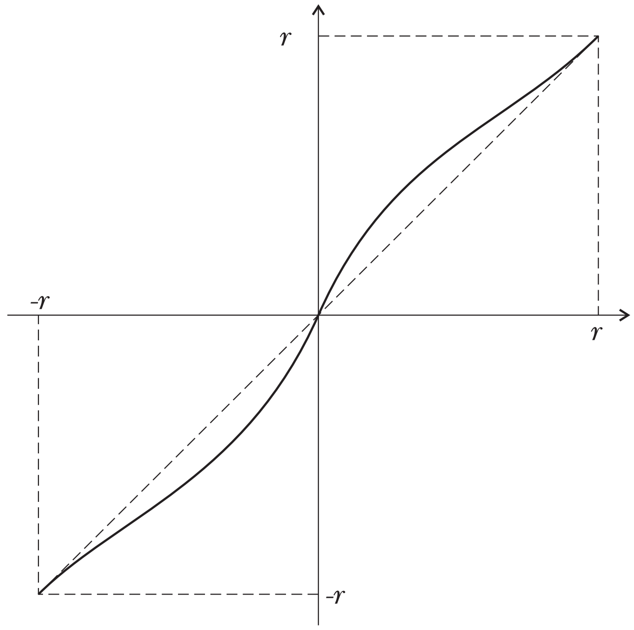

Figure 3.2: The function u

1. u agrees with the identity up to order k at $\pm r$;

2. $u(0) = 0$;

3. $u(z) < z$ for $z < 0$ and $u(z) > z$ for $z > 0$.

(See also Fig. 3.2.) Now we can extend the foliation $\mathcal{F}'$ to $F \times [-r, r]$ so that $F \times \{0\}$ is a leaf and the holonomy of all the curves $\gamma_k \times \{0\}$ should be u (this makes sense because if we set $h(\gamma_k) = u$ for all k then we have a well-defined representation of the fundamental group). Let us denote this foliation by $\mathcal{F}''$. We note that if u is chosen close enough to the identity then $\mathcal{F}''$ is $\tilde{\varepsilon}$-close to the horizontal foliation on $F \times [-r, r]$, so $\mathcal{F}''$ is ε-close to $\mathcal{F}$. Moreover $\mathcal{F}''$ has exactly three closed leaves at the levels 0, r and $-r$. For some small $a > 0$ we have that $\mathcal{F}''|_{F\times[r,r+a)}$ is diffeomorphic to $\mathcal{F}|_{F\times[0,a)}$, and this, with the choice of u, implies quite easily that $\gamma_j \times \{r\}$ has two-sided weakly-contracting holonomy in $\mathcal{F}''$. In a similar way $\gamma_i \times \{-r\}$ has two-sided weakly-contracting holonomy, and the same is true for all the curves $\gamma_k^{-1} \times \{0\}$.

If $h(\gamma_i \times \{0\})$ in $\mathcal{F}$ is weakly expanding on the negative side then we can proceed as above using a diffeomorphism v as shown in Fig. 3.3. The resulting foliation in this case has only two compact leaves at the levels $+r$ and $-r$, and the holonomies of the curves $\gamma_j \times \{r\}$ and $\gamma_i^{-1} \times \{-r\}$ on these leaves are two-sided weakly-contracting.

$\boxed{3.2.2}$

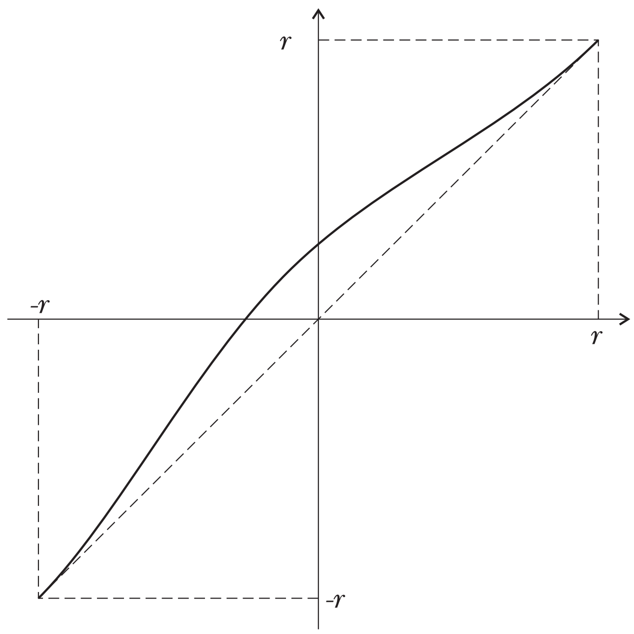

Figure 3.3: The function v

3.3 Contact-to-be approximation

This section is devoted to the proof of Proposition 1.2.3. Let us first be slightly more formal on the definition. Given a plane distribution ξ on M we will say that a curve $\gamma : [0,1] \to M$ is ξ-horizontal if γ is at least C^1-regular and $\gamma'(t)$ belongs to $\xi(\gamma(t))$ for all t. The *contact zone* of a confoliation ξ defined by a 1-form α is the region $Z(\xi) = \{x : (\alpha \wedge (\mathrm{d}\alpha))(x) \neq 0\}$. A confoliation ξ on M is called *contact-to-be* if every point of M can be joined to a point of $Z(\xi)$ by a ξ-horizontal *embedded* curve. (The assumption that the curve should be embedded, i.e. injective as a map from $[0,1]$ to M, is not included in [ET], but we have found convenient to add it for the sequel.)

Once again we note that in the argument which we are going to provide now to prove Proposition 1.2.3 we are going to refer to the proof of Proposition 1.2.2, not only to its statement.

Proof of 1.2.3. As already announced, the assumption of C^2 regularity is only needed to be able to apply the various general results of Section 1.1, not for the geometric argument itself.

Consider first the case where $\mathcal{F}$ has trivial holonomy. Then by Theorem 1.1.1 we can approximate $\mathcal{F}$ by a fibration over S^1, where the fibre is some oriented surface Σ. If $\Sigma = S^2$ then necessarily the fibration is trivial, whence $M = S^2 \times S^1$. Therefore Theorem 1.1.3 applies to $(M, \mathcal{F})$: since Reeb components have non-trivial holonomy then the foliation must be the product one on $S^2 \times S^1$, and this is specifically ruled out by the assumptions.

From the above it follows that if $\mathcal{F}$ has trivial holonomy then it can be approximated by a foliation $\mathcal{F}'$ which contains a region diffeomorphic to $\Sigma \times \mathbb{R}$ with the product foliation, where Σ is a non-simply connected surface. Now, proceeding as in the proof of Lemma 3.2.2, we can select a non-trivial diffeomorphism h of $\mathbb{R}$ which is close to the identity and coincides with the identity outside $[0,1]$; then we choose generators $\gamma_1, \ldots, \gamma_{2g}$ of $\pi_1(\Sigma)$ and we replace $\mathcal{F}'$ on $\Sigma \times \mathbb{R}$ by a foliation $\mathcal{F}''$ in which $\Sigma \times \{0\}$ is a leaf and the holonomy of each curve $\gamma_j \times \{0\}$ is h. Such an $\mathcal{F}''$ is close to $\mathcal{F}$ and has non-trivial holonomy.

The previous remarks imply that we can restrict to foliations with non-trivial holonomy. Assume now that $\mathcal{F}$ is minimal. Then, according to Theorem 1.1.4, there exists a curve γ with non-trivial linear holonomy. Up to replacing the curve by its inverse we can assume that the holonomy has derivative smaller than 1, whence it is two-sided weakly-contracting. Let us apply Proposition 1.2.2 to the curve γ in the foliation $\mathcal{F}$, and let us call ξ the resulting foliation. By property 1 such a ξ is an approximation of $\mathcal{F}$. Now we claim that ξ is a contact-to-be confoliation. To this end we will refer to the way ξ has been constructed, and we will freely use the notation introduced in Section 1.2.2. Let us first recall that ξ is the image under a diffeomorphism ϕ of some other form β; this form, under a suitable parametrization of a tubular neighbourhood of γ as $S^1_x \times \mathbb{R}_y \times \mathbb{R}_z$, differs from $\mathcal{F}$ only on $S^1_x \times (-p/2, p/2)_y \times (-\rho_1, \rho_2)_z$, and the contact zone is precisely $U = S^1_x \times (-p/4, p/2)_y \times (-\rho_1, \rho_2)_z$. Moreover if we define Y as $S^1_x \times (-p/2, -p/4)_y \times (-\rho_1, \rho_2)_z$ then there exists a diffeomorphism F of M such that F is the identity on $M \setminus \overline{Y \cup U}$ and $\beta = F_*(\mathcal{F})$ on this set.

Now we know that the foliation is minimal, so every leaf intersects U. This implies that given a point of M there exists an embedded $\mathcal{F}$-horizontal path a which joins the given point to U. Up to isotopy we can assume that a intersects the region $S^1_x \times [-p/2, p/2]_y \times [-2\rho_1, 2\rho_2]_z$ only in segments parallel to the y-axis. Let t_0 be a maximal time such that $a([0,t_0]) \subset M \setminus U$ (it is a key point that a could meet $\overline{U}$ before time t_0: it could contain a segment $\{x_0\} \times [-p/4, p/2] \times \{-\rho_1\}$ or $\{x_0\} \times [-p/4, p/2] \times \{\rho_2\}$). Consider the path $\tilde{a} = F \circ a$. Then $\tilde{a}$ is β-horizontal. Moreover (since this must be the case for a) we have that $\tilde{a}$ reaches one of the vertical walls $S^1_x \times \{-p/4\}_y \times (-\rho_1, \rho_2)_z$ or $S^1_x \times \{p/2\}_y \times (-\rho_1, \rho_2)_z$ transversely at time t_0, pointing towards U. Then we can slightly extend $\tilde{a}$ to another path which is still embedded and β-horizontal, and allows to reach U from the point initially given. (Actually, if the starting point is in Y one has to apply the construction starting from $F^{-1}(y)$, as the reader will easily see.)

Now we turn to the case where $\mathcal{F}$ is not minimal. According to Proposition 3.2.2 we can assume that $\mathcal{F}$ has finitely many closed leaves, each with a curve having two-sided weakly-contracting holonomy. Let us denote these curves by $\gamma_1, \ldots, \gamma_l$. By Theorem 1.1.2 there are finitely many exceptional minimal sets of $\mathcal{F}$, each with a curve with non-trivial linear holonomy. Up to replacing the curves by their inverses we can assume that these holonomies have derivative less than 1, whence they are two-sided weakly-contracting. Let us denote these curves by $\gamma_{l+1}, \ldots, \gamma_k$.

Since the curves $\{\gamma_j\}$ are pairwise disjoint, we can find open neighbourhoods $\{N_j\}$ with pairwise disjoint closures, and apply Proposition 1.2.2 to each $(N_j, \mathcal{F}|_{N_j})$, obtaining confoliations ξ_j. Then we can define a confoliation ξ which coincides with ξ_j on N_j and with $\mathcal{F}$ elsewhere. Such a confoliation will be close to $\mathcal{F}$ if each ξ_j is, and again we only have to prove that ξ is a contact-to-be confoliation.

This is proved essentially by the same argument as above. One first notes that it sufficient to refer to the confoliation β which is diffeomorphic to ξ under the stretching function ϕ. Next one notes that each point of the manifold can be joined to an element of $U_1 \cup \ldots \cup U_k$ by an $\mathcal{F}$-horizontal path (this is because of the dynamical properties of the foliation and the choice of the curves $\gamma_1, \ldots, \gamma_k$). Such a path can also be supposed to be embedded and to cross nicely the regions near which the approximation of $\mathcal{F}$ has taken place. To conclude one again restricts to a maximal subpath in $M \setminus (U_1 \cup \ldots \cup U_k)$ and applies the very same argument as above.

$\boxed{1.2.3}$

Chapter 4

Diffusion

In this chapter (once again, expanding ideas explained in [ET]) we will prove Proposition 1.2.4, namely that a contact-to-be positive confoliation can be approximated by a positive contact structure.

The geometric meaning of the construction we will present is as follows. Near any path tangent to a plane distribution we can choose coordinates (t, u, v), where t is the coordinate of the path, such that the distribution is defined in these coordinates by a form $\mathrm{d}v + a(t, u, v)\,\mathrm{d}u$. Then the plane distribution is a positive contact structure near the path if $\partial a/\partial t > 0$, which means that the tangent plane is rotating in the positive direction along the path with respect to this choice of coordinates. Similarly the distribution is a foliation if $\partial a/\partial t = 0$.

Now suppose that we have a path parametrized as above in a positive confoliation, such that the initial point is a foliation point and the final point is a contact point. This means that in the above coordinates the distribution rotates positively along the path for t close to 1, it does not rotate for $t = 0$, and it never rotates negatively. Then we can modify the plane distribution by slowing down the decay in the rotation angle as we proceed leftwards from 1 to 0, so that the rotation takes place for all $t \in (0, 1]$.

This is essentially the method which allows to extend the contact zone of a confoliation in a neighbourhood of any path which joins the foliation zone with the contact zone. To globalize this procedure one has to cover the manifold by finitely many such neighbourhoods and proceed successively for each of them, being slightly careful because, at each step, the procedure for one path has to be applied to the confoliation which arises from the previous step. To deal with this technicality we have a few preliminaries.

4.1 Parametrization of curve neighbourhoods

In this section we will provide a special type of parametrization of a curve tangent to a plane distribution and show that this parametrization is somehow "stable". Namely, if the plane distribution is slightly modified then also the parametrization can be modified accordingly, and the neighbourhood of the curve it covers is approximately the same.

Lemma 4.1.1. *Let ξ be a cooriented plane distribution on the closed oriented 3-manifold M. Let $\gamma : [0,1]_t \to M$ be a ξ-horizontal embedded path. Then there exists an orientation-preserving embedding $\psi : [0,1]_t \times \mathbb{R}_u \times \mathbb{R}_v$ such that:*

1. $\gamma(t) = \psi(t,0,0)$*;*

2. $\psi^*(\xi)$ *is defined by a form* $\mathrm{d}v + a(t,u,v)\,\mathrm{d}u$*.*

Proof of 4.1.1. Start with any ψ satisfying the first property. By a change of coordinates of the form

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\vartheta(t)) & -\sin(\vartheta(t)) \\ 0 & \sin(\vartheta(t)) & \cos(\vartheta(t)) \end{pmatrix} \cdot \begin{pmatrix} t \\ u \\ v \end{pmatrix}$$

we can further assume that ξ has equation $\mathrm{d}v$ on $[0,1] \times \{0\} \times \{0\}$. So, up to restricting to a smaller neighbourhood and then rescaling, we can assume that ξ has equation $\mathrm{d}v + A(t,u,v)\,\mathrm{d}u + B(t,u,v)\,\mathrm{d}t$. For $(u,v) \in \mathbb{R}^2$ we consider the Cauchy problem

$$\begin{cases} w_{(u,v)}(0) = v \\ w'_{(u,v)}(t) = -B(t,u,w_{(u,v)}(t)) \end{cases}$$

which translates the condition that the curve $t \mapsto (t,u,w_{(u,v)}(t))$ should start at $(0,u,v)$ and be ξ-horizontal.

For $(u,v) = (0,0)$ we have the solution $w_{(u,v)} \equiv 0$ defined on the whole $[0,1]_t$. Therefore the solution will exist defined on $[0,1]_t$ for all (u,v) in some neighbourhood of $(0,0)$, which, up to rescaling, we can assume to be the whole of $\mathbb{R}^2$. Now, since

$$\frac{\partial w_{(u,v)}(0)}{\partial v} = 1$$

we can also assume that the map

$$\phi : (t,u,v) \mapsto (t,u,w_{(u,v)}(t))$$

is a change of coordinates. It is now clear that $\phi^*(\xi)$ is defined by a form as required.

$\boxed{4.1.1}$

Before proceeding let us note that in the first coordinates chosen during the previous proof we have that ξ has equation $\mathrm{d}v$ along γ, but this is false in general in the final coordinates. This can be seen directly from the analytic formula, or one could remark that the previous construction applied when ξ is a positive contact structure necessarily leads to an a such that $\partial a/\partial t > 0$.

Lemma 4.1.2. *Let ξ be a plane distribution on $[0,1]_t \times \mathbb{R}^2_{(u,v)}$ defined by a differential 1-form $\mathrm{d}v + a(t,u,v)\,\mathrm{d}u$. Fix some distance which induces the C^0-topology on plane distributions. Let X be a compact set containing $[0,1]_t \times \{(0,0)\}$. Then there exists $\varepsilon > 0$ such that if $\tilde{\xi}$ is another plane distribution within distance ε from ξ, then there exists an orientation-preserving embedding*

$$\varphi : [0,1]_{\tilde{t}} \times \mathbb{R}^2_{(\tilde{u},\tilde{v})} \to [0,1]_t \times \mathbb{R}^2_{(u,v)}$$

such that:

1. *$\varphi(t,0,0) = (t,0,0)$ and $\varphi(\{t\} \times \mathbb{R}^2) \subset \{t\} \times \mathbb{R}^2$ for all $t \in [0,1]$;*
2. *The image of φ contains X;*
3. *$\varphi^*(\tilde{\xi})$ is defined by a form $\mathrm{d}\tilde{v} + \tilde{a}(\tilde{t},\tilde{u},\tilde{v})\,\mathrm{d}\tilde{u}$;*

Proof of 4.1.2. If $\tilde{\xi}$ is close enough to ξ then (at least in a neighbourhood of X, which we can take to be big enough for our later purposes) $\tilde{\xi}$ is defined by a 1-form

$$\mathrm{d}v + (a(t,u,v) + h(t,u,v))\,\mathrm{d}u + k(t,u,v)\,\mathrm{d}t$$

where h and k are small functions. Let $r > 0$ be such that $X \subset [0,1] \times \overline{B(0,r)}$. Now note that the Cauchy problem

$$\begin{cases} w_{(u,v)}(0) = v \\ w'_{(u,v)}(t) = -k(t,u,w_{(u,v)}(t)) \end{cases}$$

has a unique solution defined on $[0,1]_t$ for $(u,v) \in \overline{B(0,r)}$ when $k = 0$. So the same is true when k is small enough. Moreover for $t_0 \in [0,1]$ and $(u_0,v_0) \in \overline{B(0,r)}$ the problem

$$\begin{cases} w(t_0) = v_0 \\ w'(t) = -k(t,u_0,w(t)) \end{cases}$$

has a solution with $(u_0, w(t)) \in B(0,2r)$ for all $t \in [0,1]$ when $k = 0$. Again, this happens for sufficiently small k and implies that the image of the map

$$[0,1] \times B(0,2r) \ni (t,u,v) \mapsto (t,u,w_{(u,v)}(t)) \in [0,1] \times \mathbb{R}^2$$

contains $[0,1] \times \overline{B(0,r)}$, whence the conclusion.

$\boxed{4.1.2}$

4.2 Extension of the contact zone

As mentioned at the beginning of this chapter, we first show how to extend the contact zone of a confoliation in the neighbourhood of a path which joins a foliation point to a contact point. Later we will show how to iterate this construction to approximate any contact-to-be confoliation by a contact structure.

Lemma 4.2.1. *Let ξ be a C^1 distribution of tangent 2-planes on a neighbourhood of $[0,1]_t \times [-1,1]_u \times [-1,1]_v$; suppose ξ is defined by a form $\mathrm{d}v + a(t,u,v)\,\mathrm{d}u$. Assume that ξ is a positive confoliation everywhere, and it is a contact structure on $\{1\} \times [-1,1]^2$. Then arbitrarily C^0-close to ξ there exists a positive C^1 confoliation $\tilde{\xi}$ such that $\tilde{\xi}$ coincides with ξ outside $(0,1) \times (-1,1)^2$, and it is a positive contact structure on this set.*

Proof of 4.2.1. We know that $\partial a/\partial t \geq 0$ everywhere and $\partial a/\partial t > 0$ for $t = 1$, and we want to replace a by a C^1 function $\tilde{a}$ which is C^0 close to a, coincides with a outside $(0,1) \times (-1,1)^2$ and satisfies $\partial a/\partial t > 0$ on $(0,1) \times (-1,1)^2$. We will actually define $\tilde{a}$ only on $[0,1] \times [-1,1]^2$, making sure that $\tilde{a}$ coincides with a up to first order on the boundary of this set.

We choose $t_0 < 1$ such that $(\partial a/\partial t)(t,u,v) > 0$ for $(t,u,v) \in [t_0,1] \times [-1,1]^2$, and define

$$K = \min_{[t_0,1]\times[-1,1]^2} \frac{\partial a}{\partial t}.$$

Consider a function $\rho : [0,1] \to \mathbb{R}$ such that:

1. ρ is null up to first order for $t = 0$ and $t = 1$;
2. ρ is positive on $(0, t_0)$;
3. ρ is negative on $(t_0, 1)$;
4. ρ attains its only minimum $-K/2$ at $(1 + t_0)/2$;

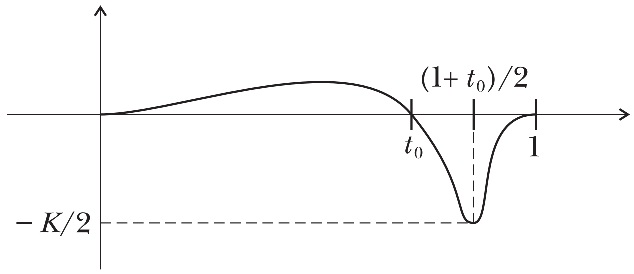

Figure 4.1: The function ρ

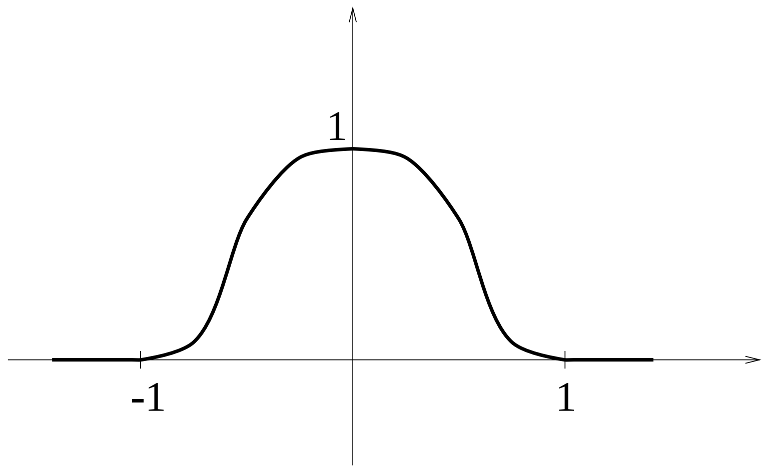

Figure 4.2: The function λ

5. The integral of ρ between 0 and 1 is null.

(See also Fig. 4.1; existence is obvious.)

Now we define

$$\eta(t) = \int_0^t \rho(s)\,\mathrm{d}s$$

and we consider a bell function λ supported in $[-1,1]$ as shown in Fig. 4.2. For $0 < k \leq 1$ we define

$$\tilde{a}_k(t,u,v) = a(t,u,v) + k\lambda(u)\lambda(v)\eta(t).$$

Then for $0 < t < t_0$ and $-1 < u, v < 1$ we have

$$\frac{\partial \tilde{a}_k}{\partial t}(t,u,v) = \frac{\partial a}{\partial t}(t,u,v) + k\lambda(u)\lambda(v)\rho(t) > \frac{\partial a}{\partial t}(t,u,v) \geq 0$$

whereas for $t \geq t_0$ we have

$$\frac{\partial \tilde{a}_k}{\partial t}(t,u,v) \geq \frac{\partial a}{\partial t}(t,u,v) - |\rho(t)| \geq K - K/2 > 0.$$

Moreover it is quite easy to check that $\tilde{a}_k$ and all its first derivatives coincide with those of a on the boundary of $[0,1] \times [-1,1]^2$. To conclude we only have to choose k small enough to have $\tilde{a}_k$ sufficiently close to a.

$\boxed{4.2.1}$

Proof of 1.2.4. Let us recall from the statement that ξ is a positive contact-to-be cooriented confoliation on a closed oriented 3-manifold M, and we want to approximate ξ by a positive contact structure.

For any $p \in M$ consider an embedded ξ-horizontal path $\tilde{\gamma}_p : [1/3, 2/3] \to M$ such that $p = \tilde{\gamma}_p(1/3)$ and $\tilde{\gamma}_p(2/3)$ belongs to the contact zone $Z(\xi)$ of ξ. Extend $\tilde{\gamma}_p$ to an embedded ξ-horizontal path $\gamma_p : [0,1] \to M$ such that $\gamma_p([2/3,1])$ is contained in $Z(\xi)$. Select a parametrization $\psi_p : [0,1]_t \times \mathbb{R}_u \times \mathbb{R}_v \to M$ of a neighbourhood of γ_p as in Lemma 4.1.1. Then rescale (u,v) so that $\psi_p(\{1\} \times \mathbb{R}^2) \subset Z(\xi)$. Now

$$\{Z(\xi)\} \cup \Big\{\psi_p\Big((0,1)_t \times (-1,1)_u \times (-1,1)_v\Big) :\ p \in M\Big\}$$

is an open covering of M, so it admits a finite subcovering. Therefore it is sufficient to prove by induction on $k \in \mathbb{N}$ the following statement:

- Let ξ be a positive C^1 confoliation on M, let $\varepsilon > 0$ and assume that there exist orientation-preserving embeddings $\psi_1, \ldots, \psi_k$ with

  $$\psi_j : [0,1]_t \times \mathbb{R}^2_{(u,v)} \to M$$

  such that

  1. $\psi_j^*(\xi)$ is defined by $\mathrm{d}v + a_j(t,u,v)\,\mathrm{d}u$;
  2. ξ is a contact structure on $\psi_j(\{1\} \times \mathbb{R}^2)$;
  3. M is covered by $Z(\xi)$ and $\psi_j((0,1)_t \times (-1,1)_u \times (-1,1)_v)$ for $j = 1, \ldots, k$.

  Then ε-close to ξ there exists a C^1 positive contact structure.

For $k = 0$ we can take ξ itself. So we assume the statement to be proved for a certain $k \geq 0$, and we prove it for $k+1$. For $i = 1, \ldots, k$, according to Lemma 4.1.2, there exists a positive constant ε_i such that for every ε_i-approximation $\tilde{\xi}$ of ξ there exists an embedding $\tilde{\psi}_i : [0,1] \times \mathbb{R}^2 \to M$ such that

$$\tilde{\psi}_i([0,1] \times \mathbb{R}^2) \supset \psi_i([0,1] \times [-1,1]^2),$$

$\tilde{\psi}_i^*(\tilde{\xi})$ is defined by a form $\mathrm{d}v + \tilde{a}_i(t,u,v)\,\mathrm{d}u$ and $\tilde{\psi}_i(\{1\} \times \mathbb{R}^2) \subset \psi_i(\{1\} \times \mathbb{R}^2)$.

Now let $\delta = \min\{\varepsilon/2, \varepsilon_1, \ldots, \varepsilon_k\}$ and use Lemma 4.2.1 to find a δ-approximation $\tilde{\xi}$ of ξ which differs from ξ only on $\psi_{k+1}((0,1) \times (-1,1)^2)$ and is a contact structure on this set. Consider then the $\tilde{\psi}_i$'s which exist by the choice of δ, and rescale each $\tilde{\psi}_i$ so that

$$\tilde{\psi}_i([0,1] \times (-1,1)^2) \supset \psi_i([0,1] \times [-1,1]^2),$$

whence in particular

$$\tilde{\psi}_i([0,1] \times (-1,1)^2) \supset \psi_i([0,1] \times (-1,1)^2).$$

This implies that $\tilde{\xi}, \tilde{\psi}_1, \ldots, \tilde{\psi}_k$ satisfy the inductive conditions: to check the second one recall that

$$\tilde{\psi}_i(\{1\} \times \mathbb{R}^2) \subset \psi_i(\{1\} \times \mathbb{R}^2) \subset Z(\xi) \subset Z(\tilde{\xi});$$

the other properties are obvious. Then we use the inductive assumption to find a contact structure which $\varepsilon/2$-approximates $\tilde{\xi}$, and the conclusion follows immediately.

$\boxed{1.2.4}$

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